Neutral subspaces of pairs of symmetric/skewsymmetric real matrices

Leiba Rodman
Peter Semrl

Follow this and additional works at: http://repository.uwyo.edu/ela

Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.1485

This Article is brought to you for free and open access by Wyoming Scholars Repository. It has been accepted for inclusion in Electronic Journal of Linear Algebra by an authorized editor of Wyoming Scholars Repository. For more information, please contact scholcom@uwyo.edu.
Abstract. Let $A$ and $B$ be $n \times n$ real matrices with $A$ symmetric and $B$ skewsymmetric. Obviously, every simultaneously neutral subspace for the pair $(A, B)$ is neutral for each Hermitian matrix $X$ of the form $X = \mu A + i\lambda B$, where $\mu$ and $\lambda$ are arbitrary real numbers. It is well-known that the dimension of each neutral subspace of $X$ is at most $\text{In}(X) + \text{In}_0(X)$, and similarly, the dimension of each neutral subspace of $X$ is at most $\text{In}_-(X) + \text{In}_0(X)$. These simple observations yield that the maximal possible dimension of an $(A, B)$-neutral subspace is no larger than
\[
\min\{\min\{\text{In}_+(\mu A + i\lambda B) + \text{In}_0(\mu A + i\lambda B), \text{In}_-(\mu A + i\lambda B) + \text{In}_0(\mu A + i\lambda B)\}\},
\]
where the outer minimum is taken over all pairs of real numbers $(\lambda, \mu)$. In this paper, it is proven that the maximal possible dimension of an $(A, B)$-neutral subspace actually coincides with the above expression.

Key words. Symmetric matrix, Skewsymmetric matrix, Hermitian matrix, Inertia, Neutral subspace.

AMS subject classifications. 15A21, 15A22, 15B57.

1. Introduction and main result. Let $F$ be the field of real numbers $\mathbb{R}$, or the field of complex numbers $\mathbb{C}$. Denote by $F^{m \times n}$ the set of $m \times n$ matrices with entries in $F$, and let $(x, y)$ be the standard inner product in $F^n$ (short for $F^{n \times 1}$).

Let $A, B \in \mathbb{R}^{n \times n}$, where $A$ is symmetric and $B$ is skewsymmetric. A subspace $\mathcal{M} \subseteq \mathbb{R}^n$ is called simultaneously neutral for $A$ and $B$, or $(A, B)$-neutral, if
\[
(Ax, y) = 0, \quad (Bx, y) = 0 \quad \text{for all } x, y \in \mathcal{M}.
\]
Simultaneously neutral subspaces for a pair of real symmetric/skewsymmetric matrices, as well as those for a pair of complex hermitian matrices, play a key role in the theory of algebraic Riccati equations (see e.g. [7] and references therein), and in symmetric factorizations of matrix polynomials and rational matrix functions with
certain symmetries [2, 3, 5, 6, 10]; in the latter application, the \((A, B)\)-neutral subspaces have the additional property that they are \(B^{-1}A\)-invariant (and \(B\) is assumed to be invertible). In this paper, we focus on the following problem: **Find the maximal possible dimension of \((A, B)\)-neutral subspaces for symmetric/skewsymmetric pairs of real matrices.** We will describe this dimension in terms of inertia of complex hermitian matrices; we denote by

\[
\text{In}(A) = (\text{In}_+(A), \text{In}_-(A), \text{In}_0(A))
\]

the inertia of a hermitian matrix \(A \in \mathbb{C}^{n \times n}\). Thus, \(\text{In}_+(A), \text{In}_-(A),\) and \(\text{In}_0(A)\) stand for the number of positive, negative, and zero eigenvalues of \(A\), respectively, counted with multiplicities.

The following observation will be useful:

**Lemma 1.1.** Let \(A, B \in \mathbb{R}^{n \times n}\), \(A = A^T\), \(B = -B^T\). Then \(A + iB\) and \(A - iB\) are similar, and in particular

\[
\text{In}(A + iB) = \text{In}(A - iB).
\] (1.1)

**Proof.** Observe that \(x + iy \in \mathbb{C}^n\), where \(x, y \in \mathbb{R}^n\), is an eigenvector of \(A + iB\) corresponding to the eigenvalue \(t \in \mathbb{R}\) if and only if \(y + ix\) is an eigenvector of \(A - iB\) corresponding to the same eigenvalue \(t\). Clearly, the set of vectors \(x_1 + iy_1, \ldots, x_p + iy_p\) is linearly independent if and only if the set \(y_1 + ix_1, \ldots, y_p + ix_p\) is linearly independent. Hence, \(A + iB\) and \(A - iB\) have the same eigenvalues with the same multiplicities. \(\square\)

We now state our main result:

**Theorem 1.2.** Let \(A\) be symmetric, \(B\) skewsymmetric, \(A, B \in \mathbb{R}^{n \times n}\). Then the maximal dimension of an \((A, B)\)-neutral subspace \(M \subseteq \mathbb{R}^n\) coincides with

\[
\min\{\min\{\text{In}_+(\mu A + i\lambda B) + \text{In}_0(\mu A + i\lambda B), \text{In}_-(\mu A + i\lambda B) + \text{In}_0(\mu A + i\lambda B)\}\},
\]

where the outer minimum is taken over all pairs of real numbers \((\lambda, \mu)\).

Thus, the maximal dimension of an \((A, B)\)-neutral subspace is described in terms of inertia of suitable combinations of \(A\) and \(B\). Analogues of Theorem 1.2 in the context of pairs of complex or quaternionic hermitian matrices \(A\) and \(B\), where \(\mu A + i\lambda B\) of Theorem 1.2 is replaced by \(\mu A + \lambda B\), have been obtained in [9, 11]. We mention in passing that an analogue of Theorem 1.2 for pairs of real symmetric matrices fails, see [11] for more details.
Remark 1.3.

(1) Note that the inner minimum in (1.2) is attained at some nonzero \((\lambda_0, \mu_0)\); indeed, for \(\lambda = \mu = 0\), (1.2) takes value \(n\). Since

\[
\begin{align*}
\mathbb{I}_n + (tX) + \mathbb{I}_0(tX) &= \mathbb{I}_n + X^*, \quad t > 0,
\end{align*}
\]

and

\[
\begin{align*}
\mathbb{I}_n + (X) + \mathbb{I}_0(X) &= \mathbb{I}_n - (X) + \mathbb{I}_0(-X), \quad X \in \mathbb{C}^{n \times n}, \quad X = X^*,
\end{align*}
\]

we have that (1.2) is equal to

\[
\begin{align*}
\min_{0 \leq \alpha < 2\pi} \{ \mathbb{I}_n + ((\cos \alpha)A + i(\sin \alpha)B) + \mathbb{I}_0((\cos \alpha)A + i(\sin \alpha)B) \}.
\end{align*}
\]

(2) Note that (1.2) is also equal to

\[
\begin{align*}
\min_{t \in \mathbb{R}} \left\{ \min \{ \mathbb{I}_n(A + itB) + \mathbb{I}_0(A + itB) \} \right\},
\end{align*}
\]

as well as to the formula analogous to (1.4) with the roles of \(A\) and \(B\) interchanged. To verify that, one needs to observe that by the continuity of the spectrum there exists a real \(M > 0\) such that

\[
\begin{align*}
\mathbb{I}_n + itB) + \mathbb{I}_0(iB) &\geq \mathbb{I}_n + \left( \frac{1}{t} A + iB \right) + \mathbb{I}_0 + \left( \frac{1}{t} A + iB \right) \\
&= \mathbb{I}_n(A + itB) + \mathbb{I}_0(A + itB)
\end{align*}
\]

for all real numbers \(t > M\).

(3) It follows from (1.1) that (1.3) is actually equal to

\[
\begin{align*}
\min_{0 \leq \alpha \leq \pi} \{ \mathbb{I}_n((\cos \alpha)A + i(\sin \alpha)B) + \mathbb{I}_0((\cos \alpha)A + i(\sin \alpha)B) \}.
\end{align*}
\]

The rest of the paper is devoted to the proof of Theorem 1.2. Preliminary results, including the canonical form for pairs of real symmetric/skewsymmetric matrices, are stated and sometimes proved in Sections 2 - 4. The proof of Theorem 1.2 itself is given in Sections 5 and 6.

We fix some notation. By \(e_1, \ldots, e_n\) we denote the elements of the standard basis of \(\mathbb{F}^n\), and by \(\text{span}(x_1, \ldots, x_p)\) the linear span of vectors \(x_1, \ldots, x_p\). The symbol \#\(G\) stands for the cardinality of the set \(G\). We denote by \(I_k\) and \(0_k\) the \(k \times k\) identity and zero matrices, respectively.
2. Preliminaries on inertia of Hermitian matrices. If \( X \in \mathbb{C}^{n \times n} \) is Hermitian, a subspace \( M \subseteq \mathbb{C}^n \) is said to be \( X \)-neutral if \( (Xx, y) = 0 \) for all \( x, y \in M \), or equivalently \( (Xx, x) = 0 \) for all \( x \in M \).

**Proposition 2.1.** Let \( X \in \mathbb{C}^{n \times n} \) be hermitian. Then an \( X \)-neutral subspace \( M \subseteq \mathbb{C}^n \) is maximal, in the sense that no subspace properly containing \( M \) is \( X \)-neutral, if and only if

\[
\dim(M) = \min\{ \text{In}^+ (X) + \text{In}_0 (X), \ \text{In}^- (X) + \text{In}_0 (X) \}.
\]

Proposition 2.1 is standard; see for example [4, Section 2.3], where it is proved under the additional assumption that \( X \) is invertible.

**Lemma 2.2.** Let \( X \) be Hermitian matrix which is block partitioned as follows:

\[
(2.1) \quad X = \begin{bmatrix}
0_k & 0 & X_1 \\
X_1^* & X_0 & X_2 \\
X_3^* & X_2^* & X_3
\end{bmatrix}, \quad \text{or} \quad X = \begin{bmatrix}
X_3 & X_2 & X_1 \\
X_2^* & X_0 & 0 \\
X_1^* & 0 & 0_k
\end{bmatrix},
\]

where the block \( X_1 \) is \( k \times k \) and invertible. Then

\[
(2.2) \quad \text{In}_0 (X) = \text{In}_0 (X_0), \quad \text{In}_\pm (X) = k + \text{In}_\pm (X_0).
\]

**Proof.** Say \( X \) is given by the first formula in (2.1). Replacing \( X \) with \( SXS^* \), where

\[
S = \begin{bmatrix}
I_k & 0 & 0 \\
-X_2X_1^{-1} & I & 0 \\
-\frac{1}{2}X_3X_1^{-1} & 0 & I
\end{bmatrix},
\]

we may assume \( X_2 = 0, X_3 = 0 \). It is easy to see that

\[
\text{In}_\pm \begin{bmatrix}
0 & X_1 \\
X_1^* & 0
\end{bmatrix} = k.
\]

Now (2.2) is obvious. \( \square \)

3. **Properties of \( \Phi_\alpha(A, B) \).** In this section, we let \( A, B \in \mathbb{R}^{n \times n} \), where \( A = A^T \), \( B = -B^T \).

For convenience, denote

\[
\Phi_\alpha(A, B) := \text{In}_+ ((\cos \alpha)A + i(\sin \alpha)B) + \text{In}_0 ((\cos \alpha)A + i(\sin \alpha)B), \quad 0 \leq \alpha < 2\pi.
\]
We list some elementary properties of the quantity $\Phi_\alpha(A, B)$.

**Lemma 3.1.** (a) If $Q$ is any finite subset of $[0, 2\pi)$, then

$$\min_{0 \leq \alpha < 2\pi} (\Phi_\alpha(A, B)) = \min_{0 \leq \alpha < 2\pi, \alpha \notin Q} (\Phi_\alpha(A, B)).$$

(b) Assume

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}.$$ 

If

$$(3.1) \quad \Phi_\alpha(A_1, B_1) = \Phi_{\alpha'}(A_1, B_1)$$

for all $\alpha, \alpha' \in [0, 2\pi) \setminus Q$, where $Q$ is a finite (or empty) set, then

$$(3.2) \quad \min_{0 \leq \alpha < 2\pi} (\Phi_\alpha(A, B)) = \min_{0 \leq \alpha < 2\pi} (\Phi_\alpha(A_1, B_1)) + \min_{0 \leq \alpha < 2\pi} (\Phi_\alpha(A_2, B_2)).$$

Note that (3.2) is generally not valid without additional hypotheses on $A_j$ and $B_j$ (such as (3.1)).

**Proof.** Proof of (a). Let $\alpha_0 \in [0, 2\pi)$ be such that

$$(3.3) \quad \min_{0 \leq \alpha < 2\pi} (\Phi_\alpha(A, B)) = \Phi_{\alpha_0}(A, B).$$

Continuity of eigenvalues of a Hermitian matrix $X$ (as functions of the entries of $X$; it is assumed that the eigenvalues are arranged in the nondecreasing order) implies that

$$(3.4) \quad \text{In}_+((\cos \alpha_0)A + i(\sin \alpha_0)B) + \text{In}_0((\cos \alpha_0)A + i(\sin \alpha_0)B) \geq \text{In}_+((\cos \beta)A + i(\sin \beta)B) + \text{In}_0((\cos \beta)A + i(\sin \beta)B)$$

for all values of $\beta \in [0, 2\pi)$ sufficiently close to $\alpha_0$. However, (3.3) implies that the strict inequality is impossible in (3.4). Thus,

$$\text{In}_+((\cos \beta)A + i(\sin \beta)B) + \text{In}_0((\cos \beta)A + i(\sin \beta)B) = \min_{0 \leq \alpha < 2\pi} (\Phi_\alpha(A, B))$$

for all $\beta$ sufficiently close to $\alpha_0$. We see that the minimum $\min_{0 \leq \alpha < 2\pi}(\Phi_\alpha(A, B))$ is attained on a set that contains a nondegenerate interval. The statement (a) is now clear.
Proof of (b). We obviously have

\[ \Phi_\alpha(A, B) = \Phi_\alpha(A_1, B_1) + \Phi_\alpha(A_2, B_2), \quad \forall \alpha \in [0, 2\pi). \]

So (the first equality follows from part (a)):

\[ \min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A, B) = \min_{0 \leq \alpha < 2\pi, \alpha \notin Q} (\Phi_\alpha(A_1, B_1) + \Phi_\alpha(A_2, B_2)) \]

which by (3.1) is equal to

\[ \Phi_\alpha'(A_1, B_1) + \min_{0 \leq \alpha < 2\pi, \alpha \notin Q} \Phi_\alpha(A_2, B_2), \]

where \( \alpha' \in [0, 2\pi) \setminus Q \) is fixed. By part (a) we have

\[ \min_{0 \leq \alpha < 2\pi, \alpha \notin Q} \Phi_\alpha(A_2, B_2) = \min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A_2, B_2), \]

and we are done. \( \square \)

Remark 3.2. The result of Lemma 3.1 (with essentially the same proof) remains valid if the interval \([0, 2\pi)\) is replaced by any nondegenerate subinterval, with or without one of both endpoints, of \([0, 2\pi)\).

Lemma 3.3. Assume that \( A \) and \( B \) have the following block form

\[
A = \begin{bmatrix}
0_k & 0 & A_1 \\
0 & A_0 & A_2 \\
A_1^T & A_2^T & A_3
\end{bmatrix}, \quad B = \begin{bmatrix}
0_k & 0 & B_1 \\
0 & B_0 & B_2 \\
-B_1^T & -B_2^T & B_3
\end{bmatrix},
\]

where the blocks \( A_1 \) and \( B_1 \) are \( k \times k \). Assume furthermore that \((\cos \alpha)A_1 + i(\sin \alpha)B_1\) is invertible for all but finitely many values \( \alpha \in [0, 2\pi) \). Then

\[ (3.5) \quad \min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A, B) = k + \min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A_0, B_0). \]

Proof. Let

\[ Q = \{ \alpha \in [0, 2\pi) : (\cos \alpha)A_1 + i(\sin \alpha)B_1 \text{ is not invertible} \}. \]

By Lemma 3.1(a) we may replace the interval \([0, 2\pi)\) with \([0, 2\pi) \setminus Q\) in (3.5). By Lemma 2.2, \( \Phi_\alpha(A, B) = k + \Phi_\alpha(A_0, B_0) \) for \( \alpha \in [0, 2\pi) \setminus Q \), and (3.5) follows. \( \square \)
4. Canonical form. We present here the known canonical form of real symmetric/skewsymmetric matrix pencils

\[ A + \lambda B, \quad A, B \in \mathbb{R}^{n \times n}, \quad A = A^T, \quad -B = B^T \]

under R-congruence:

\[ A + \lambda B \mapsto S^T A S + \lambda S^T B S, \quad S \in \mathbb{R}^{n \times n} \text{ is invertible.} \]

(See e.g. [8] and references there.) The following notation will be used:

\[ \Xi_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \]

\[ F_q \text{ is the } q \times q \text{ real symmetric matrix with 1’s in positions } (1, q), (2, q-1), \ldots, (q, 1) \]

and zeros elsewhere;

\[ G_q = \begin{bmatrix} F_{q-1} & 0_{(q-1) \times 1} \\ 0_{1 \times (q-1)} & 0_1 \end{bmatrix}, \]

a \( q \times q \) real symmetric matrix, and we take \( G_1 = 0 \); we denote by \( J_{2m}(a \pm ib) \), where \( a \) and \( b \) are real and \( b > 0 \), the \( 2m \times 2m \) almost upper triangular real Jordan block of size \( 2m \times 2m \) having eigenvalues \( a \pm ib \).

It will be convenient to list the elementary blocks first:

- \((ss0)\): a square size zero matrix.

- \((ss1)\):

\[ G_{2c+1} + \lambda \begin{bmatrix} 0 & 0 & F_c \\ 0 & 0 & 0 \\ -F_c & 0 & 0 \end{bmatrix}. \]

- \((ss2)\):

\[ F_k + \lambda \begin{bmatrix} 0_1 & 0 & 0 \\ 0 & 0 & F_{k-1} \\ 0 & -F_{k-1} & 0 \end{bmatrix}, \quad k \text{ odd.} \]

- \((ss3)\):

\[ F_k + \lambda \begin{bmatrix} 0_1 & 0 & 0 & 0 \\ 0 & 0 & F_{k-2} & 0 \\ 0 & 0 & 0_1 & 0 \\ 0 & -F_{k-2} & 0 & 0 \end{bmatrix}, \quad k \text{ even.} \]
(sss4) \[ G_\ell + \lambda \begin{bmatrix} 0 & F_{\ell/2} \\ -F_{\ell/2} & 0 \end{bmatrix}, \quad \ell \text{ even.} \]

(sss5) \[ \begin{bmatrix} 0 & G_{\ell/2} \\ G_{\ell/2} & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & F_{\ell/2} \\ -F_{\ell/2} & 0 \end{bmatrix}, \quad \ell \text{ even and } \ell/2 \text{ odd.} \]

(sss6) \[ \begin{bmatrix} 0 & \gamma F_{\ell/2} + G_{\ell/2} \\ \gamma F_{\ell/2} + G_{\ell/2} & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & F_{\ell/2} \\ -F_{\ell/2} & 0 \end{bmatrix}, \quad \ell \text{ even}, \quad \gamma \in \mathbb{R} \setminus \{0\}. \]

(sss7) \[
\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & \nu \Xi_{2m+1}^m \\
0 & 0 & \cdots & 0 & -\nu \Xi_{2m+1}^m & -I_2 \\
0 & 0 & \cdots & \nu \Xi_{2m+1}^m & -I_2 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
(-1)^{m-1} \nu \Xi_{2m+1}^m & -I_2 & 0 & \cdots & 0 & 0 \\
\end{bmatrix} + \lambda \begin{bmatrix}
0 & 0 & \cdots & 0 & \Xi_{2m}^m \\
0 & 0 & \cdots & -\Xi_{2m}^m & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & (-1)^{m-2} \Xi_{2m}^m & \cdots & 0 & 0 \\
(-1)^{m-1} \Xi_{2m}^m & 0 & \cdots & 0 & 0 \\
\end{bmatrix}, \quad \nu > 0.
\]

The pencil in (sss7) is $2m \times 2m$, where $m$ is a positive integer. We denote the pencil in (sss7) by \[ \Omega_{2m}(\nu) + \lambda \tilde{\Omega}_{2m}. \]

Note that the matrices $\Omega_{2m}(\nu)$ and $\tilde{\Omega}_{2m}$ are symmetric and skewsymmetric, respectively, for every $m$ (and every real $\nu$).

(sss8) \[ \begin{bmatrix} 0 & J_{2m}(a \pm ib)^T \\ J_{2m}(a \pm ib) & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & I_{2m} \\ -I_{2m} & 0 \end{bmatrix}, \]

where $a, b > 0$. The matrix pencil here is $4m \times 4m$. 


Neutral Subspaces of Pairs of Symmetric/Skewsymmetric Real Matrices

**Theorem 4.1.** Let \( A + \lambda B \) be a real symmetric/skewsymmetric matrix pencil. Then \( A + \lambda B \) is \( R \)-congruent to a real symmetric/skewsymmetric pencil of the form

\[
(A_0 + \lambda B_0) \oplus \bigoplus_{j=1}^{r} \delta_j \begin{pmatrix} F_{k_j} + \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -F_{k_j-1} \frac{1}{2} \end{pmatrix}
\]

(4.1)

\[
\oplus \bigoplus_{t=1}^{p} G_t + \lambda \begin{pmatrix} 0 & F_{t_1/2} \\ -F_{t_1/2} & 0 \end{pmatrix} \oplus \bigoplus_{u=1}^{q} \zeta_u (\Omega_{2m_u}(\nu_u) + \lambda \bar{\Omega}_{2m_u}).
\]

(4.2)

Here, \( A_0 + \lambda B_0 \) is a direct sum of blocks of types (sss0), (sss1), (sss3), (sss5), (sss6), and (sss8) in which several blocks of the same type and of different and/or the same sizes may be present, and the \( k_j \)'s are odd positive integers, the \( \ell_t \)'s are even positive integers, the \( \nu_u \)'s are positive real numbers, \( \delta_j, \eta_t, \zeta_u \) are signs \( \pm 1 \), and the \( m_u \)'s are positive integers.

The blocks in (4.1) and (4.2) are uniquely determined by \( A + \lambda B \) up to a permutation of blocks.

Theorem 4.1 is found in many sources; see, for example, [8] for a detailed proof.

**5. Proof of Theorem 1.2: particular case.** In this section, we prove the following particular case of Theorem 1.2:

**Theorem 5.1.** Let \( A = A^T \in \mathbb{R}^{m \times m}, B = -B^T \in \mathbb{R}^{m \times m} \) be of the form

\[
A = \bigoplus_{j=1}^{q} \kappa_j (-\nu_j I_2) \oplus I_t, \quad B = \bigoplus_{j=1}^{q} \tau_j \Xi_2 \oplus 0_t,
\]

where \( t \) is a nonnegative integer, \( \nu_j \) are positive numbers, \( \kappa_j \) are signs \( \pm 1 \), and if \( \nu_{j_1} = \nu_{j_2} \) then \( \kappa_{j_1} = \kappa_{j_2} \). Then there exists an \( (A, B) \)-neutral subspace of dimension \( \min_{0 \leq a < 2^m} \Phi_a(A, B) \).

We will need preliminary results.

**Lemma 5.2.** Let

\[
A = \bigoplus_{j=1}^{q} \kappa_j (-\nu_j I_2) \oplus I_{t_1} \oplus -I_{t_2} \in \mathbb{R}^{m \times m}, \quad B = \bigoplus_{j=1}^{q} \tau_j \Xi_2 \oplus 0_{t_1 + t_2} \in \mathbb{R}^{m \times m},
\]

where \( t_1, t_2 \) are nonnegative integers, \( \nu_j \) are positive numbers, \( \kappa_j \) and \( \tau_j \) are signs \( \pm 1 \), and if \( \nu_{t_1} = \nu_{t_2} \) then \( \kappa_{t_1} = \kappa_{t_2} \). Let

\[
\rho_+(A, B) := \min_{v \in \mathbb{R}} \{ \text{Im}(A + viB) + \text{Im}(A + viB) \}.
\]

Then there exists an \( A \)-nonnegative \( B \)-neutral subspace \( \mathcal{M} \) of \( \mathbb{R}^{m \times m} \) of dimension \( \rho_+(A, B) \).
Recall that a subspace $M$ is called $A$-nonnegative if $\langle Ax, x \rangle \geq 0$ for all $x \in M$.

Before the proof of the lemma, it will be convenient to consider an example first.

**Example 5.3.** Let $A_0 = \nu' I_2 \oplus -\nu'' I_2$, $B_0 = \tau' \Xi_2 \oplus \tau'' \Xi_2$, where $\nu' > \nu'' > 0$ and $\tau', \tau'' = \pm 1$. It is easy to see that $\rho_+(A_0, B_0) = 2$. Then there exists an $A_0$-nonnegative $B_0$-neutral subspace of dimension two, for example,

$$\text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \pm 1 \end{bmatrix} \right\}$$

where the sign $\pm 1$ is taken $+1$ if $\tau' \neq \tau''$ and $-1$ if $\tau' = \tau''$.

**Proof of Lemma 5.2.** Without loss of generality, we assume that the $\nu_j$ are arranged in the nondecreasing order:

$$\nu_1 \leq \nu_2 \leq \cdots \leq \nu_q.$$

Let $\kappa = \kappa_1$, and separate the blocks in (5.1) according to the signs:

$$\kappa_j = \kappa \quad \text{for} \quad j = 1, 2, \ldots, p_1;$$

$$\kappa_j = -\kappa \quad \text{for} \quad j = p_1 + 1, p_1 + 2, \ldots, p_2;$$

$$\kappa_j = \kappa \quad \text{for} \quad j = p_2 + 1, p_2 + 2, \ldots, p_3;$$

and so on, and finally

$$\kappa_j = \pm \kappa \quad \text{for} \quad j = p_{s-1} + 1, p_{s-1} + 2, \ldots, p_s.$$

Here $1 \leq p_1 < p_2 < \cdots < p_s = q$. By the hypotheses of Lemma 5.2, $\nu_{p_\ell} < \nu_{p_{\ell+1}}$ for $\ell = 1, 2, \ldots, s - 1$.

In view of Lemma 1.1 and Remark 3.2, we have

$$\rho_+(A, B) = \min_{v \in \Omega} \{ \text{In}_+(A + viB) + \text{In}_0(A + viB) \},$$

where

$$\Omega := \{ v : v > 0 \quad \text{and} \quad v \notin \{ \nu_1, \ldots, \nu_q \} \}.$$
Neutral Subspaces of Pairs of Symmetric/Skewsymmetric Real Matrices

and since $A + viB$ is invertible for $v \in \Omega$, we also have

$$\rho_+(A, B) = \min_{v \in \Omega} \{ \ln_+(A + viB) \}.$$  

Letting

$$A' = \oplus_{j=1}^q \kappa_j(-\nu_j I_2), \quad B = \oplus_{j=1}^q \tau_j \Xi_2,$$

we clearly obtain

$$\rho_+(A', B') + t_1 = \rho_+(A, B).$$

On the other hand, if $\mathcal{M}'$ is an $A'$-nonnegative $B'$-neutral subspace of dimension $\rho_+(A', B')$, then

$$\begin{bmatrix} \mathcal{M} \\ \mathbb{R}^{t_1} \\ 0_{t_2} \end{bmatrix}$$

is an $A$-nonnegative $B$-neutral subspace of dimension $\rho_+(A, B) + t_1$. So, using induction on the size of matrices $A$ and $B$, we may (and do) assume that $t_1 = t_2 = 0$.

Observe that for $\tau = \pm 1$ and $\nu > 0$, we have

$$\ln_+(\tau(-\nu I_2) \pm iv \Xi_2) = \begin{cases} 0 & \text{if } 0 \leq v < \nu \text{ and } \tau = 1, \\ 1 & \text{if } v > \nu \text{ and } \tau = \pm 1, \\ 2 & \text{if } 0 \leq v < \nu \text{ and } \tau = -1. \end{cases}$$

Thus, for $v \in \Omega$ we have

$$\ln_+(A + ivB) = 2\# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1 \}$$

$$+ \# \{ j = 1, 2, \ldots, q : \nu_j < v \}$$

$$= q + \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1 \}$$

$$- \# \{ j = 1, 2, \ldots, q : \nu_j < v \text{ and } \kappa_j = 1 \}. $$

Therefore,

$$\rho_+(A, B) = q + \min_{v \in \Omega} \{ \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1 \}$$

$$- \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = 1 \} \}. $$

In particular, $\rho_+(A, B) \leq q$. We now consider several cases.

**Case (a):** Assume $\rho_+(A, B) = q$. Then in view of (5.3),

$$\# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1 \} \geq \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = 1 \}.$$
for all \( v \in \Omega \). So, rearranging blocks in \( A \) and \( B \) (this amounts to a simultaneous row and column permutation in \( A \) and \( B \)), we can bring \( A \) and \( B \) to the following form:

\[
A'' = \bigoplus_{j=1}^{q'} (\nu_{j,1} I_2 \oplus -\nu_{j,2} I_2) \bigoplus \bigoplus_{j=1}^{\nu_{q'}} \mu_j I_2,
\]

\[
B'' = \bigoplus_{j=1}^{\rho} (\tau_{j,1} \Xi_2 \oplus \tau_{j,2} \Xi_2) \bigoplus \bigoplus_{j=1}^{\nu_{q}} \gamma_j \Xi_2,
\]

where \( \nu_{j,1} > \nu_{j,2} > 0 \) for \( j = 1, 2, \ldots, q' \); \( \mu_j > 0 \) for \( j = 1, 2, \ldots, q' \); \( \tau_{j,1}, \tau_{j,2} \) and \( \gamma_j \) are signs \( \pm 1 \); \( 2q' + \nu_{q}' = q \). Clearly, every pair \( \mu_j I_2, \gamma_j \Xi_2 \) produces a one-dimensional \( \mu_j I_2 \)-nonnegative \( \gamma_j \Xi_2 \)-neutral subspace, for example \( \text{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), and every pair \( \nu_{j,1} I_2 \oplus -\nu_{j,2} I_2, \tau_{j,1} \Xi_2 \oplus \tau_{j,2} \Xi_2 \) produces a two-dimensional \((\nu_{j,1} I_2 \oplus -\nu_{j,2} I_2)\)-nonnegative \((\tau_{j,1} \Xi_2 \oplus \tau_{j,2} \Xi_2)\)-neutral subspace in view of Example 5.3. Putting all these subspaces together we obtain an \( A \)-nonnegative \( B \)-neutral subspace of the requisite dimension \( q \).

**Case (b):** Assume \( \rho_+(A, B) < q \) and \( \kappa_{p_+} = 1 \). Let

\[
A' = \bigoplus_{j=1}^{q'-1} \kappa_j (-\nu_{j} I_2), \quad B = \bigoplus_{j=1}^{\nu_{q}'-1} \tau_j \Xi_2.
\]

Using formula analogous to (5.3) for the pair \( A', B' \), we have

\[
\rho_+(A', B') = q - 1 + \min_{v \in \Omega} \{ \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1 \} - \# \{ j = 1, 2, \ldots, q - 1 : \nu_j > v \text{ and } \kappa_j = 1 \} \}
\]

which is equal to

\[
q - 1 + \min_{v \in \Omega} \min_{v < \nu_{q}} \{ \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1 \} - \# \{ j = 1, 2, \ldots, q - 1 : \nu_j > v \text{ and } \kappa_j = 1 \} \}
\]

\[
= q - 1 + \min_{v \in \Omega} \min_{v < \nu_{q}} \{ \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1 \} - \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = 1 \} + 1, 0 \}
\]

\[
= q + \min_{v \in \Omega} \min_{v < \nu_{q}} \{ \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1 \} - \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = 1 \} \} - 1.
\]

In turn, this is equal to \( \rho_+(A, B) \) in view of the formula (5.3) and our assumption \( \rho_+(A, B) < q \). Using the induction hypothesis, we find \( A' \)-nonnegative \( B' \)-neutral subspace \( M' \) of dimension \( \rho_+(A, B) \). Then

\[
\begin{bmatrix} M' \\ 0 \end{bmatrix} \subseteq \mathbb{R}^{m}
\]
is an $A$-nonnegative $B$-neutral subspace of dimension $\rho_+(A, B)$.

**Case (c):** Assume $\rho_+(A, B) < q$ and $\kappa_{p^*} = -1$.

Define the matrices $A'_j, B'_j$, $j = 1, 2, \ldots, q$, as follows: $A'_j$ is obtained from $A$ by replacing the block $\kappa_j(-\nu_j I_2)$ with $\kappa_j(-\nu_j)$ (leaving all other blocks in $A$ intact), and $B'_j$ is obtained from $B$ by replacing the block $\tau_j \Xi_2$ with zero (leaving all other blocks in $B$ intact). Thus, $A'_j, B'_j \in \mathbb{R}^{(m-1) \times (m-1)}$. Since $A'_j$, resp. $B'_j$, is obtained from $A$, resp. $B$, by removing the $2(j-1)+1$th row and column, the interlacing inequalities for eigenvalues of principal submatrices of Hermitian matrices yield

$$
\begin{align*}
\operatorname{In}_+ (A + viB) - 1 &\leq \operatorname{In}_+ (A'_j + ivB'_j) \leq \operatorname{In}_+ (A + viB), \quad v \in \Omega, \quad j = 1, 2, \ldots, q,
\end{align*}
$$

and therefore

$$
\rho_+(A, B) - 1 \leq \rho(A'_j, B'_j) \leq \rho_+(A, B), \quad j = 1, 2, \ldots, q.
$$

On the other hand, a computation using (5.2) shows that for $j_0 = 1, 2, \ldots, q$, and for $v \in \Omega$:

$$
\begin{align*}
\operatorname{In}_+ (A'_j + ivB'_j) &= \# \{ j = 1, 2, \ldots, q : j \neq j_0, \nu_j > v \text{ and } \kappa_j = -1 \} \\
&\quad + q - 1 + \chi_{j_0} - \# \{ j = 1, 2, \ldots, q : j \neq j_0, \nu_j > v \text{ and } \kappa_j = 1 \},
\end{align*}
$$

where $\chi_{j_0} = 1$ if $\kappa_{j_0} = -1$ and $\chi_{j_0} = 0$ if $\kappa_{j_0} = 1$. Thus,

$$
\rho_+(A'_j, B'_j) = q
$$

$$
+ \min_{v \in \Omega} \{ -1 + \chi_{j_0} + \# \{ j = 1, 2, \ldots, q : j \neq j_0, \nu_j > v \text{ and } \kappa_j = -1 \} \\
- \# \{ j = 1, 2, \ldots, q : j \neq j_0, \nu_j > v \text{ and } \kappa_j = 1 \} \}
$$

If there is $j_0$ such that

$$
w := \rho_+(A'_j, B'_j) = \rho_+(A, B),
$$

then we can use induction on the size $m \times m$ of the matrices $A$ and $B$ to show that there exists a $w$-dimensional $A'_j$-nonnegative $B'_j$-neutral subspace $\mathcal{M}_{j_0}$. Let $x_1, \ldots, x_w \in \mathbb{R}^{m-1}$ be a basis for $\mathcal{M}_{j_0}$, and write

$$
\begin{align*}
x_{\gamma} &= \begin{bmatrix}
x_{\gamma,1} \\
x_{\gamma,2} \\
\vdots \\
x_{\gamma,m-1}
\end{bmatrix}, \quad \gamma = 1, 2, \ldots, w.
\end{align*}
$$
Let
\[ \hat{x}_1, \ldots, \hat{x}_w \in \mathbb{R}^m \]
be obtained from \( x_1, \ldots, x_w \), respectively, by inserting a zero between \( x_{\gamma, 2(j_0 - 1)} \) and \( x_{\gamma, 2(j_0 - 1) + 1} \), \( \gamma = 1, 2, \ldots, w \). Then the subspace
\[ \hat{M}_{j_0} := \text{span} \{ \hat{x}_1, \ldots, \hat{x}_w \} \]
is \( w \)-dimensional and \( A \)-nonnegative and \( B \)-neutral.

It remains therefore to consider the situation when
\[ \rho_+(A'_{j_0}, B'_{j_0}) < \rho_+(A, B) \quad \forall \quad j_0 = 1, 2, \ldots, q, \]
(in this case, necessarily
\[ \rho_+(A'_{j_0}, B'_{j_0}) + 1 = \rho_+(A, B) \quad \forall \quad j_0 = 1, 2, \ldots, q, \]
in other words,
\begin{equation}
(5.4) \quad \min_{v \in \Omega} \left\{ \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1 \} - \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = 1 \} \right\}
= 1 + \min_{v \in \Omega} \left\{ -1 + \chi_{j_0} + \# \{ j = 1, 2, \ldots, q : j \neq j_0, \nu_j > v \text{ and } \kappa_j = -1 \} - \# \{ j = 1, 2, \ldots, q : j \neq j_0, \nu_j > v \text{ and } \kappa_j = 1 \} \right\}
\end{equation}
holds for \( j_0 = 1, 2, \ldots, q \). Thus, we assume that (5.4) holds. As we will see, this leads to a contradiction.

Consider the function
\[ f(v) = \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1 \} - \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = 1 \}, \]
where \( v \in \Omega \). We have
\[ \rho_+(A, B) = \min_{v \in \Omega} f(v) + q. \]
Select points \( \lambda_0, \ldots, \lambda_s \) so that
\[ 0 < \lambda_0 < \nu_1, \quad \nu_{p_1} < \lambda_1 < \nu_{p_1 + 1}, \ldots, \nu_{p_{s-1}} < \lambda_{s-1} < \nu_{p_{s-1} + 1}, \quad \nu_{p_s} < \lambda_s. \]
Clearly, at least one of the points \( \lambda_j, j = 0, 1, \ldots, s \), is a point of (global) minimum for \( f \). Since \( f(\lambda_s) + q = q > \rho_+(A, B) \), the point \( \lambda_s \) is not a point of minimum. Also, it follows from our assumption \( \kappa_q = -1 \) that
\[ f(\lambda_s) < f(\lambda_{s-1}), \quad f(\lambda_{s-1}) > f(\lambda_{s-2}), \quad f(\lambda_{s-2}) < f(\lambda_{s-3}), \]
and so on. So, only the points \( \lambda_{s-2}, \lambda_{s-4}, \lambda_{s-6}, \ldots \) can be points of (global) minimum of \( f \).

Suppose \( s \) is odd; then \( \kappa_1 = -1, \chi_1 = 1 \), and \( \lambda_0 \) is not a point of minimum for \( f \). The right hand side of (5.4) with \( j_0 = 1 \) takes the form

\[
1 + \min_{v \in \Omega} \{ \# \{ j = 1, 2, \ldots, q : j \neq 1, \nu_j > v \text{ and } \kappa_j = -1 \} - \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = 1 \} \}.
\]

Clearly the minimum is achieved at one of the points \( \lambda_0, \ldots, \lambda_s \). Thus,

\[
1 + \min_{v \in \Omega} \{ \# \{ j = 1, 2, \ldots, q : j \neq 1, \nu_j > v \text{ and } \kappa_j = -1 \} - \# \{ j = 1, 2, \ldots, q : j \neq j_0, \nu_j > v \text{ and } \kappa_j = 1 \} \} = 1 + \min \{ f(\lambda_0) - 1, f(\lambda_1), \ldots, f(\lambda_s) \} = 1 + \min \{ f(\lambda_0), f(\lambda_1), \ldots, f(\lambda_s) \}
\]

(because \( \lambda_0 \) is not a point of minimum for \( f \)), which is one more than the left hand side of (5.4), a contradiction with (5.4).

Thus, suppose \( s \) is even. Then \( \kappa_1 = 1 \). In this case, we select \( j_0 \) so that \( \kappa_{j_0} = 1 \), \( \chi_{j_0} = 0 \). The right hand side of (5.4) takes the form

\[
(5.5) \quad \min_{v \in \Omega} \{ \# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1 \} - \# \{ j = 1, 2, \ldots, q : j \neq j_0, \nu_j > v \text{ and } \kappa_j = 1 \} \}.
\]

Let \( \lambda_y \) be the point of (global) minimum of \( f \) having the largest index \( y \); then we let \( j_0 = j_{p_y} + 1 \). (Note that we cannot have \( y = s \) because \( \lambda_0 \) is not a point of minimum of \( f \).) Again, the minimal value of

\[
\# \{ j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1 \} - \# \{ j = 1, 2, \ldots, q : j \neq j_0, \nu_j > v \text{ and } \kappa_j = 1 \},
\]

where \( v \in \Omega \), is achieved at one of the points \( \lambda_{s-2}, \lambda_{s-4}, \ldots \). So, (5.5) becomes

\[
(5.6) \quad \min_{z = s-2, s-4, \ldots} \{ \# \{ j = 1, 2, \ldots, q : \nu_j > \lambda_z \text{ and } \kappa_j = -1 \} - \# \{ j = 1, 2, \ldots, q : j \neq j_0, \nu_j > \lambda_z \text{ and } \kappa_j = 1 \} \}.
\]

By the choice of \( j_0 = j_{p_y} + 1 \), we see that (5.6) is equal to

\[
(5.7) \quad 1 + \min_{z = s-2, s-4, \ldots} \{ \# \{ j = 1, 2, \ldots, q : \nu_j > \lambda_z \text{ and } \kappa_j = -1 \} - \# \{ j = 1, 2, \ldots, q : \nu_j > \lambda_z \text{ and } \kappa_j = 1 \} \},
\]
which is one more than the left hand side of (5.4), a contradiction again. □

The following result proved in [1] will be also needed for the proof of Theorem 5.1.

**Proposition 5.4.** Let $A, B \in \mathbb{R}^{n \times n}$, $A = A^T$, $B = -B^T$. Assume that there exists a $d$-dimensional subspace $M \subseteq \mathbb{R}^n$ which is simultaneously $A$-nonnegative, i.e., $(Ax, x) \geq 0$ for every $x \in M$, and $B$-neutral, i.e., $(Bx, y) = 0$ for all $x, y \in M$. Assume also that there exists a $d$-dimensional subspace $M' \subseteq \mathbb{R}^n$ which is simultaneously $A$-nonpositive and $B$-neutral. Then there exists a $d$-dimensional $(A, B)$-neutral subspace.

**Proof of Theorem 5.1.** By Lemma 5.2, there exists an $A$-nonnegative $B$-neutral subspace of dimension $\rho_+(A, B)$, and analogously there exists an $A$-nonpositive $B$-neutral subspace of dimension $\rho_+(-A, B)$. Since (cf. Remark 1.3 (1) and (2))

$$d := \min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A, B) = \min\{\rho_+(A, B), \rho_+(-A, B)\},$$

it follows that there exist an $A$-nonnegative $B$-neutral subspace and an $A$-nonpositive $B$-neutral subspace of the same dimension $d$. Now Proposition 5.4 implies that there exists a $d$-dimensional $(A, B)$-neutral subspace. □

6. **Proof of Theorem 1.2: general case.** Since by Proposition 2.1 an $(A, B)$-neutral subspace cannot have dimension greater than (1.2), we only have to prove existence of an $(A, B)$-neutral subspace $M$ having dimension $\min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A, B)$.

First, note that Lemma 3.1 leads to the following observation:

**Proposition 6.1.** Under the hypotheses of Lemma 3.1 part (b), if there is an $(A_j, B_j)$-neutral subspace $M_j$ of dimension $\min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A_j, B_j)$, $j = 1, 2$, then there is an $(A, B)$-neutral subspace $M$ of dimension $\min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A, B)$.

**Proof.** Let

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix},$$

and take advantage of (3.2). □

Without loss of generality we may (and do) assume that $A + \lambda B$ is in the canonical form as presented in Theorem 4.1.

Let $v_0 \times v_0$ be the size of the zero block (if present) in $A_0 + \lambda B_0$, let $v_1 \times v_1$ be the total size of blocks of types (sss3), (sss5), (sss6), (sss8) (if present) in $A_0 + \lambda B_0$, and let

$$(2\varepsilon_1 + 1) \times (2\varepsilon_1 + 1), \ldots, (2\varepsilon_s + 1) \times (2\varepsilon_s + 1).$$
be the sizes of blocks of type (sss1) (if present) in $A_0 + \lambda B_0$.

We shall calculate inertia of linear combinations of matrices in the blocks of types (sss0) - (sss8), and in each case show a neutral subspace of the requisite dimension. The calculations are straightforward.

(1) If $A' + \lambda B'$ is the block (sss0), then

$$
\text{In}_+((\cos \alpha)A' + i(\sin \alpha)B') = \text{In}_-((\cos \alpha)A' + i(\sin \alpha)B') = 0, \quad \forall \alpha \in [0, 2\pi).
$$

Clearly, there exists an $(A', B')$-neutral subspace of dimension $\min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A', B')$.

(2) If $A' + \lambda B'$ is the block (sss1), then

$$
\text{In}_+((\cos \alpha)A' + i(\sin \alpha)B') = \text{In}_-((\cos \alpha)A' + i(\sin \alpha)B') = \epsilon, \quad \forall \alpha \in [0, 2\pi),
$$

and $\text{span}(e_{2\ell+1}, \ldots, e_{2\ell+1})$ is an $(A', B')$-neutral subspace of dimension equal to $\min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A', B') = \epsilon + 1$.

(3) If $A' + \lambda B'$ is the block (sss3), then

$$
\text{In}_+((\cos \alpha)A' + i(\sin \alpha)B') = \text{In}_-((\cos \alpha)A' + i(\sin \alpha)B') = \begin{cases} 
    k/2 & \text{if } \cos \alpha \neq 0, \\
    k/2 - 1 & \text{if } \cos \alpha = 0,
\end{cases}
$$

and $\text{span}(e_1, \ldots, e_{k/2})$ is an $(A', B')$-neutral subspace of dimension equal to $\min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A', B') = k/2$.

(4) If $A' + \lambda B'$ is the block (sss5), then

$$
\text{In}_+((\cos \alpha)A' + i(\sin \alpha)B') = \text{In}_-((\cos \alpha)A' + i(\sin \alpha)B') = \begin{cases} 
    \ell/2 & \text{if } \sin \alpha \neq 0, \\
    \ell/2 - 1 & \text{if } \sin \alpha = 0,
\end{cases}
$$

and $\text{span}(e_1, \ldots, e_{\ell/2})$ is an $(A', B')$-neutral subspace of dimension equal to $\min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A', B') = \ell/2$.

(5) If $A' + \lambda B'$ is the block (sss6), then

$$
\text{In}_+((\cos \alpha)A' + i(\sin \alpha)B') = \text{In}_-((\cos \alpha)A' + i(\sin \alpha)B') = \ell/2, \quad \forall \alpha \in [0, 2\pi),
$$

and $\text{span}(e_1, \ldots, e_{\ell/2})$ is an $(A', B')$-neutral subspace of dimension equal to $\min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A', B') = \ell/2$.

(6) If $A' + \lambda B'$ is the block (sss8), then

$$
\text{In}_+((\cos \alpha)A' + i(\sin \alpha)B') = \text{In}_-((\cos \alpha)A' + i(\sin \alpha)B') = 2m, \quad \forall \alpha \in [0, 2\pi),
$$

and $\text{span}(e_1, \ldots, e_{2m})$ is an $(A', B')$-neutral subspace of dimension equal to $\min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A', B') = 2m$. 
If \( A' + \lambda B' = \begin{pmatrix} F_k + \lambda & 0 & 0 \\ 0 & 0 & F_{k+1} \\ 0 & -F_{k+1} & 0 \end{pmatrix} \oplus - \begin{pmatrix} F_{k'} + \lambda & 0 & 0 \\ 0 & 0 & F_{k'+1} \\ 0 & -F_{k'+1} & 0 \end{pmatrix}, \)

where \( k, k' \) are odd, then
\[
\begin{align*}
\text{In}_+ ((\cos \alpha)A' + i(\sin \alpha)B') &= \text{In}_- ((\cos \alpha)A' + i(\sin \alpha)B'), \\
&= \frac{k + k'}{2},
\end{align*}
\]

\[\forall \alpha \in [0, 2\pi) \text{ such that } \cos \alpha \neq 0.\]

Thus, \( \text{span} (e_1, \ldots, e_{(k-1)/2}, e_{(k+1)/2} + e_{k+(k'+1)/2}, e_{k+1}, \ldots, e_{k+(k'-1)/2}) \) is an \((A', B')\)-neutral subspace of dimension \( \min_{0 \leq \alpha < 2\pi} \Phi_\alpha (A', B') = \frac{k + k'}{2}. \)

(8) If \( A' + \lambda B' \) is the block (sss4), then
\[
\begin{align*}
\text{In}_+ ((\cos \alpha)A' + i(\sin \alpha)B') &= \text{In}_- ((\cos \alpha)A' + i(\sin \alpha)B'), \\
&= \ell/2,
\end{align*}
\]

\[\forall \alpha \in [0, 2\pi) \text{ such that } \sin \alpha \neq 0, \]

and \( \text{span} (e_{\ell/2+1}, \ldots, e_{\ell}) \) is an \((A', B')\)-neutral subspace of dimension equal to \( \min_{0 \leq \alpha < 2\pi} \Phi_\alpha (A', B') = \ell/2. \)

(9) If \( A' + \lambda B' \) is the block (sss7) (of size \( 2m \times 2m \)), with \( m \) even, then
\[
\begin{align*}
\text{In}_+ ((\cos \alpha)A' + i(\sin \alpha)B') &= \text{In}_- ((\cos \alpha)A' + i(\sin \alpha)B'), \\
&= m,
\end{align*}
\]

\[\forall \alpha \in [0, 2\pi) \text{ such that } \tan \alpha \neq \pm \nu, \]

and \( \text{span} (e_1, \ldots, e_m) \) is an \((A', B')\)-neutral subspace of dimension equal to \( \min_{0 \leq \alpha < 2\pi} \Phi_\alpha (A', B') = m. \)

(10) Assume
\[
A' + \lambda B' = \xi_1 (\Omega_{2m_1} (\nu) + \lambda \tilde{\Omega}_{2m_1}) \oplus \xi_2 (\Omega_{2m_2} (\nu) + \lambda \tilde{\Omega}_{2m_2}),
\]

where \( \nu > 0, m_1, m_2 \) are odd, and
\[
(6.1) \quad \xi_1 (-1)^{\frac{m_1-1}{2}} = -\xi_2 (-1)^{\frac{m_2+1}{2}}.
\]

Then
\[
(6.2) \quad \text{In}_+ ((\cos \alpha)A' + i(\sin \alpha)B') = \text{In}_- ((\cos \alpha)A' + i(\sin \alpha)B') = m_1 + m_2
\]
for all $\alpha \in [0, 2\pi)$ except those values for which $\tan \alpha = \pm \nu$. Indeed, a calculation shows that the direct sum of the middle $2 \times 2$ block in

$$\xi_1((\cos \alpha)\Omega_{2m_1}(\nu) + i(\sin \alpha)\Omega_{2m_2})$$

and of the middle $2 \times 2$ block in

$$\xi_2((\cos \alpha)\Omega_{2m_1}(\nu) + i(\sin \alpha)\Omega_{2m_2})$$

is

$$(6.3) \quad \xi_1((\cos \alpha)\nu\Xi_{2}^{m_1+1} + i(\sin \alpha)\Xi_{2}) \oplus \xi_2((\cos \alpha)\nu\Xi_{2}^{m_2+1} + i(\sin \alpha)\Xi_{2}).$$

Now (6.2) follows easily from (6.3). Also, the $4 \times 4$ matrix (6.3) has the following $2$-dimensional neutral subspace $\mathcal{M}_0$ independent of $\alpha$ (the hypothesis (6.1) is essential here):

$$(6.4) \quad \mathcal{M}_0 = \begin{cases} \text{span}(e_1 + e_3, e_2 + e_4) & \text{if } m_1 = 4k + 3, \ m_2 = 4\ell + 3, \\ \text{span}(e_1 + e_3, e_2 + e_4) & \text{if } m_1 = 4k + 1, \ m_2 = 4\ell + 1, \\ \text{span}(e_1 + e_4, e_2 + e_3) & \text{if } m_1 = 4k + 3, \ m_2 = 4\ell + 1, \\ \text{span}(e_1 + e_4, e_2 + e_3) & \text{if } m_1 = 4k + 1, \ m_2 = 4\ell + 3, \end{cases}$$

where $k$ and $\ell$ are nonnegative integers. Let

$$\mathcal{M} = \text{span}(e_1, \ldots, e_{m_1}, e_{m_1} + e_1, \ldots, e_{m_1} + e_{m_1} + e_1, e_{m_1} + e_{m_1} + e_1, e_{m_1} + e_{m_1} + e_1)$$

if $(m_1 - 1)/2$ and $(m_2 - 1)/2$ have the same parity, and

$$\mathcal{M} = \text{span}(e_1, \ldots, e_{m_1}, e_{m_1} + e_1, \ldots, e_{m_1} + e_{m_1} + e_1, e_{m_1} + e_{m_1} + e_1 + e_{m_1} + e_1)$$

if $(m_1 - 1)/2$ and $(m_2 - 1)/2$ have different parity. It follows from (6.4) that $\mathcal{M}$ is an $(A', B')$-neutral subspace of dimension $\min_{0 \leq \alpha < 2\pi} \Phi_{\alpha}(A', B') = m_1 + m_2$.

Repeatedly using Proposition 6.1, items (1) - (10) above, and Theorem 4.1, and replacing if necessary $A$ and $B$ by $-A$ and $-B$, respectively, we see that the proof of Theorem 1.2 is reduced to the consideration of the following case:

$$(6.5) \quad A + \lambda B = \oplus_{j=1}^{q} \xi_{j}(\Omega_{2m_j}(\nu_j) + \widetilde{\Omega}_{2m_j})$$

$$\oplus \oplus_{i=1}^{s} \left( F_{k_i} + \lambda \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & F_{k_{i-1}} \\ 0 & -F_{k_{i-1}} & 0 \end{bmatrix} \right),$$

where $m_1, \ldots, m_q$ are odd and $k_1, \ldots, k_s$ are odd, and $\xi_j$ are signs $\pm 1$ (the cases when $q = 0$, i.e., the first part of (6.5) is missing, or $s = 0$, i.e., the second part of (6.5)
is missing, are not excluded); also, if \( \nu_{j_1} = \nu_{j_2} \) then the signs of the corresponding blocks in (6.5) are the same.

Applying a suitable simultaneous permutation of rows and columns to \( A + \lambda B \) in (6.5), we obtain \( A' + \lambda B' \) in the following block form:

\[
A' + \lambda B' = \begin{bmatrix}
0_k & 0 & A_1 + \lambda B_1 \\
0 & A_0 + \lambda B_0 & * \\
A_1^T - \lambda B_1^T & * & * \\
\end{bmatrix},
\]

where

\[
k = \left( \sum_{j=1}^{q} (m_j - 1) \right) + \left( \sum_{i=1}^{s} \frac{k_i - 1}{2} \right).
\]

In (6.6), \( A_1 + \lambda B_1 \) is a \( k \times k \) block diagonal matrix pencil with the diagonal blocks of the forms

\[
\begin{bmatrix}
\ldots & 0 & 0 & \nu_j \Xi_{m_j+1}^2 \\
\ldots & 0 & -\nu_j \Xi_{m_j+1}^2 & -I_2 \\
\ldots & \nu_j \Xi_{m_j+1}^2 & -I_2 & 0 \\
\ldots & \vdots & \vdots & \vdots \\
\end{bmatrix} + \lambda
\begin{bmatrix}
\ldots & 0 & 0 & \Xi_{m_j}^2 \\
\ldots & 0 & -\Xi_{m_j}^2 & 0 \\
\ldots & \Xi_{m_j}^2 & 0 & 0 \\
\ldots & \vdots & \vdots & \vdots \\
\end{bmatrix},
\]

where \( j = 1, 2, \ldots, q \), and of the forms

\[
F_{m_i-1} + \lambda G_{m_i-1}' \quad i = 1, 2, \ldots, s,
\]

where

\[
G_m' = \begin{bmatrix}
0 & 0 \\
0 & F_{m-1} \\
\end{bmatrix} \in \mathbb{R}^{m \times m},
\]

and

\[
A_0 + \lambda B_0 := \left( \oplus_{j=1}^{q} \xi_j (-1)^{m_j-1} (\nu_j \Xi_{m_j+1}^2 + \lambda \Xi_{m_j}^2) \right) \oplus I_s
\]

\[
= \left( \oplus_{j=1}^{q} \xi_j (-\nu_j I_2 + \lambda \Xi_2) \right) \oplus I_s.
\]

Note that \((\cos \alpha)A_1 + i(\sin \alpha)B_1\) is invertible for all but finitely many values of \( \alpha \in [0, 2\pi) \). By Lemma 3.3, we have

\[
\min_{0 \leq \alpha < 2\pi} \Phi_{\alpha}(A', B') = k + \min_{0 \leq \alpha < 2\pi} \Phi_{\alpha}(A_0, B_0).
\]

On the other hand, by Theorem 5.1, there exists an \((A_0, B_0)\)-neutral subspace \( \mathcal{M}_0 \) of dimension \( \min_{0 \leq \alpha < 2\pi} \Phi_{\alpha}(A_0, B_0) \). Then clearly

\[
\mathcal{M} := \begin{bmatrix} R^k \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{M}_0 \\ 0 \end{bmatrix}
\]
Neutral Subspaces of Pairs of Symmetric/Skewsymmetric Real Matrices

is an \((A', B')\)-neutral subspace of dimension \(k + \min_{0 \leq \alpha < 2\pi} \Phi_{\alpha}(A_0, B_0)\). In view of (6.7), we have proved Theorem 1.2 for the pair \((A, B)\).

REFERENCES