Characterization of P-Property for some Z-Transformations on positive semidefinite cone

R. Balaji
CHARACTERIZATION OF $P$-PROPERTY FOR SOME $Z$-TRANSFORMATIONS ON POSITIVE SEMIDEFINITE CONE$^*$

R. BALAJI$^1$

Abstract. The $P$-property of the following two $Z$-transformations with respect to the positive semidefinite cone is characterized:

(i) $I - S$, where $S : S^{n\times n} \to S^{n\times n}$ is a nilpotent linear transformation,
(ii) $I - L_A^{-1}$, where $L_A$ is the Lyapunov transformation defined on $S^{n\times n}$ by $L_A(X) = AX + XA^T$.(Here $S^{n\times n}$ denotes the space of all symmetric $n \times n$ matrices and $I$ is the identity transformation.)

Key words. $P$-property, Stein-type transformations, Lyapunov transformations.

AMS subject classifications. 90C33, 17C55.

1. Introduction. An $n \times n$ matrix is said to be a $Z$-matrix if all the off-diagonal entries are non-positive. Several interesting properties on $Z$-matrices can be found in [1]. For a square matrix of order $n$, by an easy verification, we find that the following are equivalent:

1. $A$ is a $Z$-matrix.
2. If $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ then,
   $$x \geq 0, \quad y \geq 0 \quad (\text{entrywise non-negative}), \quad \text{and} \quad x^T y = 0 \implies y^T A x \leq 0.$$

Motivated by the above fact, we consider $Z$-transformations with respect to positive semidefinite cone.

Let $S^{n\times n}$ be the vector space of $n \times n$ symmetric matrices with real entries. A linear transformation $L : S^{n\times n} \to S^{n\times n}$ is called a $Z$-transformation with respect to the positive semidefinite cone if

$$X \succeq 0, \quad Y \succeq 0 \quad \text{and} \quad XY = 0 \implies \langle L(X), Y \rangle := \text{trace}(L(X)Y) \leq 0.$$(Here $X \succeq 0$ means $X$ is symmetric and positive semidefinite.) Significances of $Z$-transformations (especially in mathematical programming) can be found in [2]. An important result on $Z$-transformations is the following:

$^*$Received by the editors on April 29, 2011. Accepted for publication on October 4, 2011. Handling Editor: Michael Tsatsomeros.
$^1$Department of Mathematics, Indian Institute of Technology-Madras, Chennai-36, India (balaji5@iitm.ac.in).

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Theorem 1.1 (Theorem 6 [2]). Let $L : \mathbb{S}_n \times \mathbb{S}_n \to \mathbb{S}_n$ be a $Z$-transformation. Then the following are equivalent.

1. There exists a $X \succ 0$ such that $L(X) \succ 0$.
2. For every $Q \succeq 0$, there exists a unique $X \succeq 0$ such that $L(X) = Q$.
3. For every $Q \in \mathbb{S}_n$, there exists a $X \succeq 0$ such that $Y := L(X) + Q \succeq 0$ and $XY = 0$.

We will say that a transformation $S$ (defined on $\mathbb{S}_n$) has the property $(c)$ if:

$$X \succeq 0 \implies S(X) \succeq 0.$$ 

Transformations of the type $I - S$, where $I$ is the identity transformation on $\mathbb{S}_n$ and $S$ is a linear transformation with property $(c)$ are called Stein-type transformations. These transformations are important examples of $Z$-transformations. For a Stein-type transformation it is known that all the statements of Theorem 1.1 are equivalent to the condition $\rho(S) < 1$, where $\rho(S)$ is the spectral radius of $S$ (see [3]).

A transformation $L : \mathbb{S}_n \to \mathbb{S}_n$ is said to have the $P$-property if the following condition is satisfied:

$$XL(X) = L(X)X \quad \text{and} \quad XL(X) \preceq 0 \implies X = 0.$$ 

One of the unsolved problems on $Z$-transformations (see [2]) is to show that all the items in Theorem 1.1 are equivalent to the condition that $L$ has the $P$-property. Even for the Stein-type transformations, the problem remains unsolved. More precisely, if $I - S$ is a Stein-type transformation such that $\rho(S) < 1$, then the problem of determining whether $I - S$ has the $P$-property has no answer. It is natural to consider the simplest case, when $\rho(S) = 0$. In other words, assuming $S$ is nilpotent, we ask whether the Stein-type transformation $I - S$ has the $P$-property. First, we settle this question in this paper.

If $S$ is a $Z$-transformation satisfying any of the items in Theorem 1.1, we find that $S^{-1}$ has property $(c)$. We now ask whether $I - S^{-1}$ has the $P$-property if $S$ is a $Z$-transformation with property $(c)$ and such that $\rho(S^{-1}) < 1$. One of the well-studied $Z$-transformations is the Lyapunov transformation for which we know that all the items of Theorem 1.1 are equivalent to the fact that $A$ is a positive stable matrix (See the definitions below for Lyapunov transformation and positive stable matrix). If $S = L_A^{-1}$, where $L_A$ is the Lyapunov transformation corresponding to a positive stable matrix $A$ with the property $\rho(L_A^{-1}) < 1$, then for the Stein-type transformation $I - L_A^{-1}$, we show that $I - L_A^{-1}$ has the $P$-property.

2. Preliminaries. All the matrices appearing here are assumed to be real. The following notations and definitions will be useful in the sequel.
Definition 2.1. Let $A$ be a square matrix. Then $A$ is said to be positive stable if every eigenvalue of $A$ has a positive real part.

Definition 2.2. For a square matrix $A$, the corresponding Lyapunov transformation $L_A : \mathbb{S}^{n \times n} \to \mathbb{S}^{n \times n}$ is defined by $L_A(X) := AX + XA^T$.

If $Q$ is an $n \times n$ matrix, and $\alpha = \{1, \ldots, k\}$ ($k < n$), $Q_{\alpha \alpha}$ will denote the $k \times k$ leading principal submatrix of $Q$.

Definition 2.3. Let $L : \mathbb{S}^{n \times n} \to \mathbb{S}^{n \times n}$ be a linear transformation. For any $\alpha = \{1, \ldots, k\}$, we define a linear transformation $L_{\alpha \alpha} : \mathbb{S}^{k \times k} \to \mathbb{S}^{k \times k}$ by

$$L_{\alpha \alpha}(Z) := [L(X)]_{\alpha \alpha} \quad (Z \in \mathbb{S}^{k \times k}),$$

where corresponding to $Z \in \mathbb{S}^{k \times k}$, $X \in \mathbb{S}^{n \times n}$ is the unique matrix such that

$$X_{ij} = \begin{cases} Z_{ij} & (i, j) \in \alpha \times \alpha \\ 0 & \text{else} \end{cases}.$$

We call $L_{\alpha \alpha}$ the principal subtransformation corresponding to $\alpha$.

If $\beta \in \mathbb{R}$, then we define $\beta^+ := \max(\beta, 0)$ and $\beta^- := \max(-\beta, 0)$. Suppose $D$ is a diagonal matrix with diagonal entries $d_i$. Then $D^+$ will denote the diagonal matrix whose diagonal entries are $d_i^+$. Similarly, $D^-$ will denote the diagonal matrix whose entries are $d_i^-$. If $X \in \mathbb{S}^{n \times n}$, then there exists an orthogonal matrix $U$ such that $UXU^T = D$, where $D$ is diagonal. Now we define $X^+ := UD^+U^T$ and $X^- := UD^-U^T$. It is easy to see that for every $X \in \mathbb{S}^{n \times n}$, $X = X^+ - X^-; X^+$ and $X^-$ are positive semidefinite.

We will use the fact that if $T$ is a linear transformation on $\mathbb{S}^{n \times n}$ with property (c), then its spectral radius is an eigenvalue of $T$ (see Theorem 0 in [4]).

Let $T : \mathbb{S}^{n \times n} \to \mathbb{S}^{n \times n}$ be a linear transformation. Then $T$ is a nilpotent transformation if there exists a positive integer $m$ such that $T^m = 0$.

3. Results. We prove our main results now.

3.1. Case 1. We intend to show that $I - S$ has the $P$-property if $S$ is nilpotent and has property (c). The result is trivial if $S = 0$ and so in the rest of the discussion, we assume $S$ is nonzero. Let $\nu$ be the least positive integer satisfying

$$(3.1) \quad S^\nu = 0, \quad \text{and} \quad S^{\nu-1} \neq 0.$$

First we prove the following basic lemma.

Lemma 3.1. Let $S$ be a nilpotent transformation. Assume that $S$ has property (c). Then the following are true:

(a) If $Q \succ 0$, then $Q \notin \text{Image}(S)$.
(b) If rank \( S(X) = m \), then there exists a \( P \succeq 0 \) such that rank \( S(P) \geq m \). In fact, if \( X \in \mathbb{S}^{n \times n} \), then

\[
\text{rank } S(X) \leq \text{rank } S(|X|),
\]

where \( |X| := X^+ + X^- \).

**Proof.** Let \( S \) satisfy (3.1). Suppose \( S(P) = Q \) for some \( Q \succ 0 \). If \( X \succeq 0 \), then there exists \( \epsilon > 0 \) such that \( Q - \epsilon X \succ 0 \). Since \( S \) has the property \((c)\) and satisfies (3.1), we have:

\[
(3.2) \quad S^{\nu-1}(Q - \epsilon X) + S^{\nu-1}(\epsilon X) = 0,
\]

\[
(3.3) \quad S^{\nu-1}(Q - \epsilon X) \succeq 0, \quad \text{and} \quad S^{\nu-1}(\epsilon X) \succeq 0.
\]

In view of (3.2) and (3.3), \( S^{\nu-1}(X) = 0 \). Therefore for any \( Y \in \mathbb{S}^{n \times n} \),

\[
S^{\nu-1}(Y) = S^{\nu-1}(Y^+) - S^{\nu-1}(Y^-) = 0
\]

and so \( S^{\nu-1} = 0 \) which is a contradiction to (3.1). This proves (a).

For any two positive semidefinite matrices \( U \) and \( V \) in \( \mathbb{S}^{n \times n} \),

\[
\text{rank}(U - V) \leq \text{rank}(U + V).
\]

The above inequality can be proved as follows. Let \( x \in \mathbb{R}^n \) be an element in the null space of \( U + V \). This gives \( Ux = -Vx \) and thus, \( x^TUx = -x^TVx \). Since \( U \) and \( V \) are symmetric and positive semidefinite, we get \( Ux = 0 = Vx \) and thus,

\[
\text{nullity } (U + V) \leq \text{nullity } (U - V).
\]

By rank nullity theorem, (3.4) follows.

By setting \( U = S(X^+) \) and \( V = S(X^-) \) in (3.4), we find from the property \((c)\) of \( S \) that the positive semidefinite matrix \( P := X^+ + X^- \) satisfies \( m \leq \text{rank } S(P) \). This proves (b). \( \square \)

We now prove the first main result.

**Theorem 3.2.** Suppose \( S : \mathbb{S}^{n \times n} \rightarrow \mathbb{S}^{n \times n} \) is a nilpotent transformation with property \((c)\). Then \( I - S \) has the \( P \)-property.

**Proof.** We prove the result by induction on \( n \). If \( n = 2 \), the result is true (see Theorem 13 in [2]). For \( k < n \), we will assume that the result holds and now we prove for \( k = n \). Let \( Q_0 \in \mathbb{S}^{n \times n} \) be such that

\[
\text{rank } S(Q_0) \geq \text{rank } S(Q) \quad \text{for all } Q \in \mathbb{S}^{n \times n}.
\]
In view of Item (b) in Lemma 3.1, without any loss of generality, we assume \( Q_0 \succeq 0 \).
If \( \hat{k} = \text{rank} S(Q_0) \), then Item (a) of Lemma 3.1 implies \( \hat{k} < n \).
There exists an orthogonal matrix \( U \) such that
\[
US(Q_0)U^T = \begin{bmatrix}
D & 0 \\
0 & 0
\end{bmatrix},
\]
\( D \in S^{\hat{k} \times \hat{k}} \) being diagonal and nonsingular. Define \( \tilde{S} : S^{n \times n} \to S^{n \times n} \) by
\[
\tilde{S}(X) := US(U^TXU)U^T.
\]
If \( \hat{Q}_0 = UQ_0U^T \), then
\[
\tilde{S}(\hat{Q}_0) = \begin{bmatrix}
D & 0 \\
0 & 0
\end{bmatrix}.
\]
By an easy verification, we find that \( \tilde{S} \) is nilpotent and has property (c). Further,
\[
(3.5) \quad \text{rank} \tilde{S}(\hat{Q}_0) \geq \text{rank} \tilde{S}(Q) \quad \text{for all } Q \in S^{n \times n}.
\]
We now claim that for any \( X \in S^{n \times n} \),
\[
(3.6) \quad \tilde{S}(X) = \begin{bmatrix}
E & 0 \\
0 & 0
\end{bmatrix}, \quad \text{for some } E \in S^{\hat{k} \times \hat{k}}.
\]
Let \( Q \succeq 0 \) and \( F := \tilde{S}(Q) \). As \( F = [f_{ij}] \succeq 0 \), \( f_{ii} = 0 \) if and only if the \( i \)th column of \( F \) is zero. Suppose \( f_{ii} > 0 \) for some \( i > \hat{k} \). Then
\[
\text{rank} \tilde{S}(\hat{Q}_0 + Q) = \text{rank}(\tilde{S}(\hat{Q}_0) + \tilde{S}(Q)) \geq \hat{k} + 1 > \hat{k}.
\]
Thus, we have \( \text{rank} \tilde{S}(\hat{Q}_0 + Q) > \text{rank} \tilde{S}(\hat{Q}_0) \) which is a contradiction to (3.5). So, for any \( Q \succeq 0 \),
\[
\tilde{S}(Q) = \begin{bmatrix}
E' & 0 \\
0 & 0
\end{bmatrix}, \quad E' \in S^{\hat{k} \times \hat{k}}.
\]
Since for any \( X \in S^{n \times n} \), \( \tilde{S}(X) = \tilde{S}(X^+) - \tilde{S}(X^-) \), using the \( e \)-property of \( \tilde{S} \), we see that (3.6) holds.
Let \( X = \begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix} \in S^{\hat{k} \times \hat{k}} \) be such that \( X(X - \tilde{S}(X)) \preceq 0 \). If
\[
\tilde{S}(X) = \begin{bmatrix}
F & 0 \\
0 & 0
\end{bmatrix}.
\]
Then from

\[(3.7) \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} - \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \preceq 0,\]

it follows that \(X_2^T X_2 + X_3^T \preceq 0\), and therefore, \(X_2\) and \(X_3\) are zero matrices. So, \(F = \tilde{S}_{\alpha\alpha}(X_1)\), where \(\alpha = \{1, \ldots, \hat{k}\}\). From (3.7) we now have

\[(3.8) X_1(X_1 - \tilde{S}_{\alpha\alpha}(X_1)) \preceq 0.\]

We next claim that \(\tilde{S}_{\alpha\alpha}\) has the property (c). Let \(X_0 \in \mathbb{S}^{\hat{k} \times \hat{k}}\) be positive semidefinite and

\[Y_0 = \tilde{S} \begin{bmatrix} X_0 & 0 \\ 0 & 0 \end{bmatrix}.\]

Since \(\tilde{S}\) has property (c), \(Y_0\) is a positive semidefinite matrix. Noticing that \(\tilde{S}_{\alpha\alpha}(X_0)\) is a leading principal submatrix of \(Y_0\), we conclude \(\tilde{S}_{\alpha\alpha}(X_0)\) is positive semidefinite. This proves our claim.

Now we assert that \(\tilde{S}_{\alpha\alpha}\) is nilpotent. Since \(\tilde{S}_{\alpha\alpha}\) has property (c), \(r := \rho(\tilde{S}_{\alpha\alpha})\) is an eigenvalue of \(\tilde{S}_{\alpha\alpha}\). Let \(X_0 \in \mathbb{S}^{\hat{k} \times \hat{k}}\) be a nonzero matrix in \(\mathbb{S}^{\hat{k} \times \hat{k}}\) such that

\[\tilde{S}_{\alpha\alpha}(X_0) = rX_0.\]

In view of (3.6) and the definition of \(\tilde{S}_{\alpha\alpha}\),

\[\tilde{S} \begin{bmatrix} X_0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} rX_0 & 0 \\ 0 & 0 \end{bmatrix}.\]

Hence, \(r\) is an eigenvalue of \(\tilde{S}\). Since \(\tilde{S}\) is nilpotent, \(r = 0\). Thus, \(\tilde{S}_{\alpha\alpha}\) is nilpotent.

By our induction assumption, \(I - \tilde{S}_{\alpha\alpha}\) must have \(P\)-property and hence from (3.8), \(X_1 = 0\); thus, \(X = 0\). This proves that \(I - \tilde{S}\) has the \(P\)-property. It is easy to see that \(I - S\) has the \(P\)-property if and only if \(I - \tilde{S}\) has the \(P\)-property. The proof is now complete.

**Corollary 3.3.** Let \(\{A_1, \ldots, A_\nu\}\) be a finite set of \(n \times n\) nilpotent matrices. Assume that \(A_i A_j = A_j A_i\) for all \(i\) and each \(A_i\) is nilpotent. Then the transformation \(X - \sum_{i=1}^\nu A_iX A_i^T\) has the \(P\)-property.

**Proof.** Let \(M_{A_i}(X) = A_iX A_i^T\). Then, using \(A_i A_j = A_j A_i\), we verify that \(M_{A_i} M_{A_j} = M_{A_j} M_{A_i}\). Now it is easy to see that \(\sum_{i=1}^\nu M_{A_i}\) is nilpotent, and hence, \(X - \sum_{i=1}^\nu A_iX A_i^T\) has the \(P\)-property. \(\square\)
3.2. Case 2. Now we shall show that if a matrix $A$ is positive stable and $\rho(L_A^{-1}) < 1$, then $I - L_A^{-1}$ has the $P$-property. Note that by Lyapunov theorem (cf. Theorem 6 [3]), $L_A^{-1}$ will have the property (c). Hence, $I - L_A^{-1}$ is a Stein-type transformation and satisfy all the items in Theorem 1.1. Before proving the main result, we will prove some intermediate lemmas.

**Lemma 3.4.** Let $A$ be a positive stable matrix of order $n$ and $\rho(L_A^{-1}) < 1$. Then

1. $\text{trace } A > \frac{n}{2}$.
2. If there exist a nonsingular $X$ and $Y := X - L_A^{-1}(X)$ such that $XY = YX$ and $XY \preceq 0$, then $X$ must be indefinite.

**Proof.** If $\lambda$ is an eigenvalue of $A$, then it is straightforward to verify that $\lambda + \lambda^*$ is an eigenvalue of $L_A$. In other words, $2 \text{Re}(\lambda)$ is an eigenvalue of the linear transformation $L_A$. Our assumptions on $A$ now imply that $0 < \frac{1}{2} \text{Re}(\lambda) < 1$, and hence, $\text{Re}(\lambda) > \frac{1}{2}$. As $A$ is a real matrix, we now deduce that the sum of all the eigenvalues of $A$ is greater than $\frac{n}{2}$. This proves 1.

Suppose $X \succeq 0$ is a nonsingular matrix such that $XY \preceq 0$. Because $XY = YX$, there exists an orthogonal matrix $U$ such that $X = UD U^T$ and $Y = UE U^T$, where $D$ and $E$ are diagonal matrices and now $XY \preceq 0$ implies that

$$ (3.9) \quad DE \preceq 0. $$

The matrix $D$ must be positive definite as $X$ is a nonsingular positive semidefinite matrix and by (3.9), we conclude $E \preceq 0$; hence,

$$ Y \preceq 0. $$

This means that $X - L_A^{-1}(X) \preceq 0$. The matrix $A$ is positive stable, and hence by Lyapunov theorem $I - L_A^{-1}$, is a $Z$-transformation. From the assumption $\rho(L_A^{-1}) < 1$, it follows from Item 2 of Theorem 1.1 that

$$ (I - L_A^{-1})(X) \preceq 0 \quad \implies \quad X \preceq 0. $$

Therefore, $X$ cannot be positive semidefinite. This is a contradiction.

In a similar manner, it follows that $X$ cannot be negative semidefinite. This proves 2. \[ \Box \]

**Lemma 3.5.** If $A$ is positive stable and $\rho(L_A^{-1}) < 1$, then

1. There does not exist a nonsingular matrix $X$ commuting with $Y := X - L_A^{-1}(X)$, such that $XY \preceq 0$. 

2. If $X$ is either positive semidefinite or negative semidefinite and if $Y := X - L_A^{-1}(X)$ is such that $XY = YX$, then

$$XY \preceq 0 \Rightarrow X = 0.$$ 

Proof. Let $X$ be a nonsingular matrix such that $XY = YX$ and $XY \preceq 0$, where $Y := X - L_A^{-1}(X)$. In view of previous lemma, $X$ must be indefinite.

As $XY = YX$ and $XY \preceq 0$, there is an orthogonal matrix $U$ such that

$$UXU^T = \begin{bmatrix} D & 0 \\ 0 & -E \end{bmatrix}, \quad UYU^T = \begin{bmatrix} -F & 0 \\ 0 & G \end{bmatrix},$$

where the matrices $D$ and $E$ are positive definite; $F$ and $G$ are positive semidefinite. Further $D$, $E$, $F$, and $G$ are diagonal. Note that $X - Y = L_A^{-1}(X)$, and thus, $X = L_A(X - Y)$. We now have

$$UXU^T = UL_A(X-Y)U^T$$

$$= UL_A(U^T(U(X-Y)U^T)U)^T$$

$$= UL_A(U^T\begin{bmatrix} D + F & 0 \\ 0 & -E - G \end{bmatrix}U)U^T.$$  

(3.10)

Let $d_i$, $e_i$, $f_i$ and $g_i$ be the diagonal entries of $D$, $E$, $F$ and $G$, respectively. Assume that the order of $D$ and $F$ is $\nu$. If $a_{11}, a_{22}, \ldots, a_{nn}$ are the diagonal entries of $UAU^T$, then we find from the above equations that

$$a_{kk} = \begin{cases} \frac{d_k}{2(d_k + f_k)} & \text{if } k = 1, \ldots, \nu \\ \frac{e_k}{2(e_k + g_k)} & \text{if } k = \nu + 1, \ldots, n. \end{cases}$$

Thus, $\text{trace } A = \text{trace } (UAU^T) \leq \frac{n}{2}$. This contradicts Lemma 3.4. Therefore item 1 is proved.

The proof of item 2 follows easily by replacing $E = 0$ in the above.

Theorem 3.6. Let $A$ be an $n \times n$ positive stable matrix with real entries. If $L_A$ is the corresponding Lyapunov transformation then the following are equivalent:

(i) $\rho(L_A^{-1}) < 1$.

(ii) $I - L_A^{-1}$ has the $P$-property.
Proof. Since $I - L_A^{-1}$ is a Stein-type-transformation, (ii) ⇒ (i) follows immediately from the fact that $\rho(L_A^{-1})$ is an eigenvalue of $L_A^{-1}$. We now prove (i) ⇒ (ii).

Let $X$ be such that

$$X(X - L_A^{-1}(X)) \preceq 0.$$ 

Put $Y := X - L_A^{-1}(X)$. In view of Lemma 3.4 and Lemma 3.5, we see that $X$ must be indefinite and $X$ is singular. Since $X$ and $Y$ commute and $XY \preceq 0$, there is an orthogonal matrix $U$ such that

$$UXU^T = \begin{bmatrix} D & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad UYU^T = \begin{bmatrix} -F & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & L \end{bmatrix},$$

where the matrices $D$ and $E$ are positive definite; $F$ and $G$ are positive semidefinite. Further, $D$, $E$, $F$, $G$ and $L$ are diagonal. Assume that $D$ and $E$ are of order $\nu_1$ and $\nu_2$, respectively.

Now working similarly as in (3.10) of previous lemma, it is easy to show that

$$(3.11) \quad \begin{bmatrix} D & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = L_{UAU^T} \begin{bmatrix} D + F & 0 & 0 \\ 0 & -E - G & 0 \\ 0 & 0 & -L \end{bmatrix}.$$ 

Put $\tilde{A} = UAU^T$. It is straightforward to verify that $\rho(L_{\tilde{A}}) = \rho(L_A)$. First we consider the case $L = 0$. We now define two diagonal matrices of order $\nu_1 + \nu_2$ viz.

$$\tilde{D} := \begin{bmatrix} D & 0 \\ 0 & -E \end{bmatrix}, \quad \tilde{E} := \begin{bmatrix} D + F & 0 \\ 0 & -E - G \end{bmatrix}.$$ 

It is easy to note that $\tilde{D}$ and $\tilde{E}$ are nonsingular.

Let $\tilde{A} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, where $A_1$ is of order $\nu_1 + \nu_2$. Since $L = 0$, from (3.11), we have

$$\begin{bmatrix} \tilde{D} & 0 \\ 0 & 0 \end{bmatrix} = L_{\tilde{A}} \begin{bmatrix} \tilde{E} & 0 \\ 0 & 0 \end{bmatrix}.$$ 

From the above equation, we have

$$(3.12) \quad \begin{bmatrix} \tilde{D} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 \tilde{E} + \tilde{E} A_1^T & \tilde{E} A_3^T \\ A_3 \tilde{E} & 0 \end{bmatrix};$$

hence, $A_3 \tilde{E} = 0$. The matrix $\tilde{E}$ must be nonsingular and therefore $A_3 = 0$. Thus, every eigenvalue of $A_1$ must be an eigenvalue of $A$ and so $A_1$ is positive stable. We
claim that $r := \rho(L^{-1}_{A_1}) < 1$. Since $A_1$ is positive stable, $L^{-1}_{A_1}$ will have the property (c) (by Lyapunov theorem) and so $r$ is an eigenvalue of $L^{-1}_{A_1}$. Let $V$ be such that $L^{-1}_{A_1}(V) = rV$. Let $\tilde{V}$ be the $n \times n$ matrix defined by

$$
\tilde{V} = \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix}.
$$

It is easy to see that $L^{-1}_{A_1}(\tilde{V}) = r\tilde{V}$ and since $L^{-1}_{A_1}(V) = rV$, we deduce $r < 1$.

From (3.12), we have $\tilde{D} = A_1\tilde{E} + \tilde{E}A_1^T$, and thus, $L^{-1}_{A_1}(\tilde{D}) = \tilde{E}$. Now we have

$$
\begin{align*}
\tilde{D}(\tilde{D} - L^{-1}_{A_1}(\tilde{D})) &= \tilde{D}(\tilde{D} - \tilde{E}) \\
&= \begin{bmatrix} D & 0 \\ 0 & -E \end{bmatrix} \begin{bmatrix} -F & 0 \\ 0 & G \end{bmatrix} \\
&\leq 0.
\end{align*}
$$

Thus, $\tilde{D}$ is a nonsingular matrix such that $\tilde{D}$ and $\tilde{D} - L^{-1}_{A_1}(\tilde{D})$ commute and $\tilde{D}(\tilde{D} - L^{-1}_{A_1}(\tilde{D})) \preceq 0$. This contradicts the previous lemma.

We now consider the case where $L$ is nonzero. First assume $L$ is nonsingular. Since $L$ is a diagonal matrix, the diagonal entries of $L$ must be nonzero now. In this case using (3.11), we compute the diagonal entries $\alpha_{kk}$ of $\tilde{A}$:

$$
\alpha_{kk} = \begin{cases} 
\frac{d_k}{2(d_k + f_k)} & \text{if } k = 1, \ldots, \nu_1 \\
\frac{e_k}{2(e_k + g_k)} & \text{if } k = \nu_1 + 1, \ldots, \nu_1 + \nu_2 \\
0 & \text{if } k > \nu_1 + \nu_2.
\end{cases}
$$

Now it is easy to see that $\text{trace}\tilde{A} \leq \frac{1}{2}(\nu_1 + \nu_2) < \frac{n}{2}$ which contradicts Lemma 3.4.

Finally, we consider the case $L$ is singular but nonzero. In this case, we can write $UXU^T$ and $UYU^T$ as follows:

$$
UXU^T = \begin{bmatrix} D & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad UYU^T = \begin{bmatrix} -F & 0 & 0 & 0 \\ 0 & G & 0 & 0 \\ 0 & 0 & L_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

where the matrix $L_1$ is nonsingular. Suppose the order of $L_1$ is $\nu_3$. Let the matrix $\tilde{A}$ be partitioned conformally (as above in $UXU^T$ and $UYU^T$) into

$$
\tilde{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.
$$
Following the same arguments as above, we see that $A_{41}$, $A_{42}$, and $A_{43}$ are zero matrices. Further if $\hat{A}$ is the $(\nu_1 + \nu_2 + \nu_3) \times (\nu_1 + \nu_2 + \nu_3)$ leading principal submatrix of $A$, then we see that

\[
\begin{bmatrix}
D & 0 & 0 \\
0 & -E & 0 \\
0 & 0 & 0
\end{bmatrix}
= L_{\hat{A}} \begin{bmatrix}
-F & 0 & 0 \\
0 & G & 0 \\
0 & 0 & L_1
\end{bmatrix},
\]

$\hat{A}$ is positive stable and $\rho(L_{\hat{A}}^{-1}) < 1$. Invoking Lemma 3.4, we find that

\[
\text{trace } \hat{A} > \frac{1}{2}(\nu_1 + \nu_2 + \nu_3).
\]

However, calculating the trace of $\hat{A}$ by finding the sum of all the diagonal entries of $\hat{A}$ from (3.13), we see that

\[
\text{trace } \hat{A} \leq \frac{1}{2}(\nu_1 + \nu_2).
\]

This is a contradiction. The proof is now complete. \qed

REFERENCES


