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GENERALIZATIONS OF BRAUER’S EIGENVALUE LOCALIZATION THEOREM

CHAOQIAN LI† AND YAO TANG LI†

Abstract. New eigenvalue inclusion regions are given by establishing the necessary and sufficient conditions for two classes of nonsingular matrices, named double $\alpha_1$-matrices and double $\alpha_2$-matrices. These results are generalizations of Brauer’s eigenvalue localization theorem and improvements over the results in [L. Cvetković, V. Kostić, R. Bru, and F. Pedroche. A simple generalization of Geršgorin’s theorem. Adv. Comput. Math., 35:271–280, 2011].

Key words. Matrix eigenvalue, Brauer’s eigenvalue localization theorem, Double $\alpha_1$-matrices, Double $\alpha_2$-matrices.

AMS subject classifications. 15A18, 65F15.

1. Introduction. Let $\mathbb{C}^{n \times n}$ denote the collection of all $n \times n$ complex matrices and $N = \{1, 2, \ldots, n\}$. For a matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, we denote, for any $i, j, k \in N$,

$$r_i = \sum_{k \neq i} |a_{ik}|, \quad c_i = \sum_{k \neq i} |a_{ki}|,$$

$$\Gamma_i(A) = \{z \in \mathbb{C}: |z - a_{ii}| \leq r_i\},$$

$$\hat{\Gamma}_i(A) = \{z \in \mathbb{C}: |z - a_{ii}| \leq \min\{r_i, c_i\}\},$$

$$\mathcal{H} = \{i \in N : r_i > c_i\}, \quad \mathcal{L} = \{i \in N : r_i < c_i\},$$

$$\hat{\Gamma}_{i,j}(A) = \{z \in \mathbb{C} : |z - a_{ii}|(c_j - r_j) + |z - a_{jj}|(r_i - c_i) \leq c_j r_i - c_i r_j, i \in \mathcal{H}, j \in \mathcal{L}\},$$

$$\hat{\Gamma}_{i,j}(A) = \{z \in \mathbb{C} : \left|\frac{z - a_{ii}}{c_i}\right| \left|\frac{z - a_{jj}}{c_j}\right|^\frac{\text{ln} r_i}{\text{ln} r_j} \leq 1, i \in \mathcal{H}\backslash\{k : c_k = 0\},$$

\[j \in \mathcal{L}\backslash\{k : r_k = 0\}\}.$$
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\[ \mathcal{K}_{i,j}(A) = \{ z \in \mathbb{C} : |z - a_{ii}| |z - a_{jj}| \leq r_i r_j \} \]

and

\[ \mathcal{K}_{i,j}(A) = \{ z \in \mathbb{C} : |z - a_{ii}| |z - a_{jj}| \leq \min\{r_i r_j, c_i c_j\} \}. \]

Eigenvalue localization has been a hot topic in matrix theory and its applications. Many researchers have obtained lots of eigenvalue inclusion regions; for details, see [1]–[7], [9]–[13]. We first recall the very well known eigenvalue localization theorem of Gersgorin [6].

**Theorem 1.1.** [6] Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \) and \( \sigma(A) \) be the spectrum of \( A \). Then

\[ \sigma(A) \subseteq \Gamma(A) = \bigcup_{i \in \mathbb{N}} \Gamma_i(A). \]

Here, \( \Gamma(A) \) is called the Gersgorin set of \( A \). Recently, L. Cvetković et al. [4] gave the following two eigenvalue inclusion regions by the characterizations of two class of nonsingular \( H \)-matrices, and proved that these two regions stay within the set \( \Gamma(A) \cap \Gamma(A^T) \), where \( A^T \) is the transpose of \( A \).

**Theorem 1.2.** [4, Theorem 6] Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n}, n \geq 2 \). Then

\[ \sigma(A) \subseteq \mathcal{A}_1(A) = \tilde{\Gamma}(A) \bigcup \hat{\Gamma}(A), \]

where \( \tilde{\Gamma}(A) = \bigcup_{i \in \mathbb{N}} \tilde{\Gamma}_i(A) \) and \( \hat{\Gamma}(A) = \bigcup_{i \in \mathcal{H}, j \in \mathcal{L}} \hat{\Gamma}_{i,j}(A) \).

**Theorem 1.3.** [4, Theorem 7] Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n}, n \geq 2 \). Then

\[ \sigma(A) \subseteq \mathcal{A}_2(A) = \tilde{\Gamma}(A) \bigcup \hat{\Gamma}(A), \]

where \( \tilde{\Gamma}(A) = \bigcup_{i \in \mathbb{N}} \tilde{\Gamma}_i(A) \) and \( \hat{\Gamma}(A) = \bigcup_{i \in \mathcal{H}, j \in \mathcal{L}} \hat{\Gamma}_{i,j}(A) \).

In [1], Brauer obtained the following eigenvalue localization theorem.

**Theorem 1.4.** [1] Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n}, n \geq 2 \). Then

\[ \sigma(A) \subseteq \mathcal{K}(A) = \bigcup_{i,j \in \mathbb{N}, i \neq j} \mathcal{K}_{i,j}(A). \]

The set \( \mathcal{K}(A) \) is called the Brauer set of \( A \), and \( \mathcal{K}_{i,j}(A) \) is called the \((i,j)\)-th Brauer Cassini oval. It is well known that \( \mathcal{K}(A) \subseteq \Gamma(A) \) (see [12, 13]). Since \( A \) and
its transpose $A^T$ have the same spectrum, we have that $\sigma(A) = \sigma(A^T) \subseteq \Gamma(A^T)$, and thus, $\sigma(A) \subseteq (K(A) \cap K(A^T)) \subseteq (\Gamma(A) \cap \Gamma(A^T))$. We now let

$$K_i(A) = \bigcup_{i,j \in N \setminus i \neq j} K_{i,j}(A).$$

Note that $K_{i,j}(A) = K_{j,i}(A)$, $\bar{K}_{i,j}(A) = \bar{K}_{j,i}(A)$, $K_{i,j}(A) \subseteq K_{i,j}(A)$ and $\bar{K}_{i,j}(A) \subseteq K_{i,j}(A)$ for $i, j \in N$. These show that $K(A) \subseteq K(A)$ and $\bar{K}(A) \subseteq K(A)$, and thus,

$$\bar{K}(A) \subseteq \left( K(A) \cap K(A^T) \right).$$

An interesting problem arises: whether $\bar{K}(A)$ includes all eigenvalues of $A$ or not? The following example provides a negative answer.

**Example 1.5.** Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}.$$ 

By calculation, we get

$$\sigma(A) = \{-0.1149, 2.2541, 3.8608\},$$

$$\bar{K}_{1,2}(A) = \{z \in \mathbb{C} : |z - 1||z - 2| \leq 2\},$$

$$\bar{K}_{1,3}(A) = \{z \in \mathbb{C} : |z - 1||z - 3| \leq 3\}$$

and

$$\bar{K}_{2,3}(A) = \{z \in \mathbb{C} : |z - 2||z - 3| \leq 1\}.$$ 

Obviously, $-0.1149 \notin \bar{K}(A) = (\bar{K}_{1,2}(A) \cup \bar{K}_{1,3}(A) \cup \bar{K}_{2,3}(A))$.

In this paper, we also focus on the subject of eigenvalue localization. In Section 2, we establish necessary and sufficient conditions for two classes of nonsingular matrices, named double $\alpha_1$-matrices and double $\alpha_2$-matrices. In Section 3, new regions $K_1(A)$ and $K_2(A)$ including all the eigenvalues of $A$ are obtained, which include $K(A)$ and stay within the set $K(A) \cap K(A^T)$. Specially, we compare the new eigenvalue inclusion region $K_2(A)$ with $A_1(A)$ in Theorem 1.2 (Theorem 6 of [4]) and $A_2(A)$ in Theorem 1.3 (Theorem 7 of [4]), and prove $K_2(A) \subseteq A_1(A)$ and $K_2(A) \subseteq A_2(A)$.
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2. Necessary and sufficient conditions of double $\alpha_1$-matrices and double $\alpha_2$-matrices. In this section, double $\alpha_1$-matrices and double $\alpha_2$-matrices are presented. And their characterizations are given.

Definition 2.1. A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ is said to be a double $\alpha_1$-matrix, if there is $\alpha \in [0, 1]$ such that for all $i, j \in N, i \neq j$,

$$|a_{ii}| |a_{jj}| > \alpha r_i r_j + (1 - \alpha) c_i c_j.$$  

Definition 2.2. A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ is said to be a double $\alpha_2$-matrix, if there is $\alpha \in [0, 1]$ such that for all $i, j \in N, i \neq j$,

$$|a_{ii}| |a_{jj}| > (r_i r_j)\alpha (c_i c_j)^{1-\alpha}.$$  

(2.1)

As shown in [8], double $\alpha_2$-matrices are nonsingular. And moreover, from the generalized arithmetic-geometric mean inequality:

$$\alpha a + (1 - \alpha)b \geq a^\alpha b^{1-\alpha}$$

where $a, b \geq 0$ and $0 \leq \alpha \leq 1$, we easily get that double $\alpha_1$-matrices are also nonsingular.

Now we establish necessary and sufficient conditions for double $\alpha_1$-matrices and double $\alpha_2$-matrices, respectively. First, some notations are given. For a matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2$, we denote

$$\mathcal{R} = \{(i, j) : r_i r_j > c_i c_j, \ i \neq j, \ i, j \in N\},$$

$$\mathcal{C} = \{(i, j) : c_i c_j > r_i r_j, \ i \neq j, \ i, j \in N\},$$

$$\mathcal{E} = \{(i, j) : r_i r_j = c_i c_j, \ i \neq j, \ i, j \in N\}.$$  

Note here that $(i, j) \in \mathcal{R}$ (or $\mathcal{C}$) implies $(j, i) \in \mathcal{R}$ (or $\mathcal{C}$, respectively).

Theorem 2.3. A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2$, is a double $\alpha_2$-matrix if and only if the following two conditions hold:

(i) $|a_{ii}| |a_{jj}| > \min\{r_i r_j, \ c_i c_j\}$ for all $i, j \in N, i \neq j$.

(ii) $\log \frac{c_i c_j}{r_i r_j} \frac{|a_{ii}| |a_{jj}|}{c_i c_j} > \log \frac{c_m c_n}{r_m r_n} \frac{c_m c_n}{a_{mm} a_{nn}}$ for $(i, j) \in \mathcal{R}\{(l, k) : c_l c_k = 0\}$, and $(m, n) \in \mathcal{C}\{(l, k) : r_l r_k = 0\}$. 
Proof. Firstly, suppose that \( A \) is a double \( \alpha \)-matrix. Then there is \( \alpha \in [0, 1] \) such that

\[
|a_{ij}| > \alpha (c_i c_j)^{1-\alpha}
\]

for all \( i, j \in N, i \neq j \). Condition (i) follows from the fact

\[(r_i r_j)^\alpha (c_i c_j)^{1-\alpha} \geq \min\{r_i r_j, c_i c_j\}.
\]

Now, for \( (i, j) \in \mathcal{R}\setminus\{(l, k) : c_l c_k = 0\} \), we have

\[
\frac{|a_{ii}| |a_{jj}|}{c_i c_j} > \left(\frac{r_i r_j}{c_i c_j}\right)^\alpha.
\]

Note that \( r_i r_j > c_i c_j \), taking the logarithm of the above inequality for the base \( r_i r_j / c_i c_j \) > 1, and using the monotonicity, we obtain that

\[
\log_{r_i r_j / c_i c_j} \frac{|a_{ii}| |a_{jj}|}{c_i c_j} > \alpha.
\]

Similarly, for \( (m, n) \in \mathcal{C}\setminus\{(l, k) : r_l r_k = 0\} \), we obtain that

\[
\log_{\frac{c_m c_n}{|a_{mm}||a_{nn}|}} \frac{c_m c_n}{|a_{mm}||a_{nn}|} < \alpha.
\]

Thus, condition (ii) holds.

Conversely, suppose that the conditions (i) and (ii) hold. For each \( (i, j) \in \mathcal{E} \), condition (i) directly implies inequality (2.1). And for \( (i, j) \in \mathcal{R} \) such that \( c_i c_j = 0 \), or \( (m, n) \in \mathcal{C} \) such that \( r_m r_n = 0 \), inequality (2.1) follows immediately. Thus, it remains to prove that inequality (2.1) holds for all \( (i, j) \in (\mathcal{R}\setminus\{(l, k) : c_l c_k = 0\})\cup(\mathcal{C}\setminus\{(l, k) : r_l r_k = 0\}) \).

For each \( (i, j) \in \mathcal{R}\setminus\{(l, k) : c_l c_k = 0\} \), we have \( r_i r_j > c_i c_j \), which, from condition (i), leads to \( |a_{ii}| |a_{jj}| > c_i c_j \). Using the properties of the log function for the base greater than one, we obtain

\[
(2.2) \quad \log_{r_i r_j / c_i c_j} \frac{|a_{ii}| |a_{jj}|}{c_i c_j} > 0.
\]

Similarly, for each \( (m, n) \in \mathcal{C}\setminus\{(l, k) : r_l r_k = 0\} \), we have

\[
(2.3) \quad \log_{\frac{c_m c_n}{|a_{mm}||a_{nn}|}} \frac{c_m c_n}{|a_{mm}||a_{nn}|} < 1.
\]

From inequalities (2.2), (2.3) and condition (ii), we have that there is \( \alpha \) such that, for each \( (i, j) \in \mathcal{R}\setminus\{(l, k) : c_l c_k = 0\} \) and each \( (m, n) \in \mathcal{C}\setminus\{(l, k) : r_l r_k = 0\} \),

\[
(2.4) \quad \max \left\{ 0, \log_{\frac{c_m c_n}{|a_{mm}||a_{nn}|}} \frac{c_m c_n}{|a_{mm}||a_{nn}|} \right\} < \alpha < \min \left\{ \log_{\frac{c_i c_j}{c_l c_k}} \frac{|a_{ii} a_{jj}|}{c_i c_j}, 1 \right\}.
\]
From the left inequality and right inequality of inequality (2.4), we get, respectively, that for each $(i, j) \in \mathbb{R}\{(l, k) : c_ic_k = 0\}$,
\[
\frac{|a_{ii}a_{jj}|}{c_ic_j} > \left(\frac{r_ir_j}{c_ic_j}\right)^\alpha
\]
and for each $(m, n) \in \mathbb{C}\{(l, k) : r_lr_k = 0\}$,
\[
\frac{c_mc_n}{|a_{mm}a_{nn}|} > \left(\frac{c_mc_n}{r_mr_n}\right)^\alpha.
\]
Thus, the proof is completed.

Similar to the proof of Theorem 2.3, we can obtain the following necessary and sufficient conditions for double $\alpha_1$-matrices, and its proof is omitted.

**Theorem 2.4.** A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, is a double $\alpha_1$-matrix if and only if the following two conditions hold:

(i) $|a_{ii}|a_{jj}| > \min\{r_ir_j, c_ic_j\}$ for all $i, j \in \mathbb{N}, i \neq j$.

(ii) $\frac{|a_{ii}|a_{jj}| - c_ic_j}{r_ir_j - c_ic_j} > \frac{c_mc_n - |a_{mm}|a_{nn}|}{c_mc_n - r_mr_n}$ for all $(i, j) \in \mathbb{R}$, and all $(m, n) \in \mathbb{C}$.

3. Eigenvalue localizations. By the necessary and sufficient conditions of double $\alpha_1$-matrices and double $\alpha_2$-matrices in Section 2, we give two new eigenvalue inclusion regions.

**Theorem 3.1.** Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, and $\sigma(A)$ be the spectrum of $A$. Then
\[
\sigma(A) \subseteq K_2(A) = \tilde{K}(A) \cup \hat{K}(A),
\]
where $\tilde{K}(A)$ is given by (1.1), $\hat{K}(A) = \bigcup_{(i,j) \in \mathbb{R}, (m,n) \in \mathbb{C}} \hat{K}_{i,j,m,n}(A)$ and
\[
\hat{K}_{i,j,m,n}(A) = \{z \in \mathbb{C} : \frac{\lambda - a_{ii}}{c_ic_j} \left(\frac{\lambda - a_{jj}}{c_ic_j}\right) \log \frac{c_mc_n}{r_mr_n} \leq 1, (i, j) \in \mathbb{R}\{(l, k) : c_ic_k = 0\}, (m, n) \in \mathbb{C}\{(l, k) : r_lr_k = 0\}\}.
\]

**Proof.** For any $\lambda \in \sigma(A)$, $\lambda I - A$ is singular. Note that the moduli of every off-diagonal entry of $\lambda I - A$ is the same as $A$. Hence, the sets $\mathbb{R}$ and $\mathbb{C}$ for the matrix $\lambda I - A$ remain the same. If $\lambda \notin K_2(A)$, then $\lambda I - A$ satisfies conditions (i) and (ii) of Theorem 2.3, hence $\lambda I - A$ is a double $\alpha_2$-matrix, which implies that $\lambda I - A$ is nonsingular. This is a contradiction. Hence, $\lambda \in K_2(A)$, that is, $\sigma(A) \subseteq K_2(A)$. QED
Remark 3.2. (i) From the original definition of double $\alpha_2$-matrices, we can derive directly the following eigenvalue inclusion region (see [9]):

$$K_2(A) = \bigcap_{0 \leq \alpha \leq 1} \bigcup_{i,j \in N, i \neq j} \left\{ \mathbb{C} : |z - a_{ii}| \leq (r_ir_j)^\alpha (c_ic_j) \right\}.$$  

Obviously, the form of $K_2(A)$ obtained in (3.1) is much more convenient than that in (3.2).

(ii) Since $K_2(A) = K_2(A^T)$, we have that $K_2(A) \subseteq (K(A) \cap K(A^T)).$

Similar to the proof of Theorem 3.1, we can obtain easily the following eigenvalue localization theorem.

**Theorem 3.3.** Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, and $\sigma(A)$ be the spectrum of $A$. Then

$$\sigma(A) \subseteq K_1(A) = \bar{K}(A) \bigcup \tilde{K}(A),$$

where $\bar{K}(A)$ is given by (1.1), $\tilde{K}(A) = \bigcup_{(i,j) \in \mathcal{R}, (m,n) \in \mathcal{C}} \tilde{K}_{i,j,m,n}(A)$ and

$$\tilde{K}_{i,j,m,n}(A) = \{z \in \mathbb{C} : |\lambda - a_{ii}| |\lambda - a_{jj}| (c_mc_n - r_mr_n) + |\lambda - a_{mm}| |\lambda - a_{nn}| (r_ir_j - c_ic_j) \leq c_mc_n r_mr_n - c_ic_j r_mr_n, (i,j) \in \mathcal{R}, (m,n) \in \mathcal{C} \}.$$

Similar to Remark 3.2, we also obtain that $K_1(A) \subseteq (K(A) \cap K(A^T))$. Next, we compare $K_2(A)$ in Theorem 3.1 with $A_1(A)$ in Theorem 1.2 (Theorem 6 of [4]) and $A_2(A)$ in Theorem 1.3 (Theorem 7 of [4]).

**Theorem 3.4.** Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$. And $A_2(A)$ and $K_2(A)$ are defined in Theorems 1.3, and 3.1, respectively. Then

$$K_2(A) \subseteq A_2(A).$$

**Proof.** We prove $K_2(A) \subseteq A_2(A)$. Equivalently, we prove that if $z \notin A_2(A)$, then $z \notin K_2(A)$. In fact, if $z \notin A_2(A)$, from Theorem 1.3, we have that for any $i \in N$,

$$|z - a_{ii}| > \min \{r_i, c_i\},$$  

and for $i \in \mathcal{H} \setminus \{k : c_k = 0\}$ and $j \in \mathcal{L} \setminus \{k : r_k = 0\}$,

$$\frac{|z - a_{ii}|}{c_i} \left( \frac{|z - a_{jj}|}{c_j} \right) \log \frac{c_i}{c_j} > 1.$$
From Theorems 5 and 7 of [4], inequalities (3.3) and (3.4) imply that for any \( i \in N \),
\[
|z - a_{ii}| > (r_i)^{\alpha} (c_i)^{1-\alpha}
\]
for some \( \alpha \in [0, 1] \). Hence, for any \( i, j \in N \) and \( i \neq j \), we have
\[
|z - a_{ii}| |z - a_{jj}| > (r_i)^{\alpha} (c_i)^{1-\alpha} (r_j)^{\alpha} (c_j)^{1-\alpha} = (r_i r_j)^{\alpha} (c_i c_j)^{1-\alpha}
\]
for some \( \alpha \in [0, 1] \). This implies that \( zI - A \) is a double \( \alpha_2 \)-matrix. From Theorem 2.3, the following two inequalities hold:

\[
|z - a_{ii}| |z - a_{jj}| > \min\{r_i r_j, c_i c_j\}
\]
for all \( i, j \in N, i \neq j \), and

\[
\frac{\log |z - a_{ii}| |z - a_{jj}|}{c_i c_j} \frac{c_m c_n}{c_i c_j} > \frac{c_m c_n}{c_i c_j} |z - a_{mm}| |z - a_{nn}|
\]
for \( (i, j) \in \mathcal{R} \setminus \{(l, k) : c_l c_k = 0\} \), and \( (m, n) \in \mathcal{C} \setminus \{(l, k) : r_l r_k = 0\} \). Moreover, inequality (3.6) is written equivalently as

\[
\frac{|z - a_{ii}| |z - a_{jj}|}{c_i c_j} \left( \frac{|z - a_{mm}| |z - a_{nn}|}{c_m c_n} \right) \log \frac{\c_i \c_j}{\c_m \c_n} > 1.
\]
Hence, from inequalities (3.5) and (3.7), \( z \notin \mathcal{K}(A) \) and \( z \notin \hat{\mathcal{K}}(A) \), that is, \( z \notin \mathcal{A}_2(A) \). The proof is completed. \( \square \)

**Lemma 3.5.** Let \( A = [a_{ij}] \in \mathbb{C}^{n \times n}, \ n \geq 2 \). And \( \mathcal{A}_1(A) \) and \( \mathcal{A}_2(A) \) are defined in Theorems 1.2, and 1.3, respectively. Then

\[
\mathcal{A}_2(A) \subseteq \mathcal{A}_1(A).
\]

**Proof.** Similar to the proof of Theorem 3.4 and from the fact that if
\[
|z - a_{ii}| > \alpha r_i + (1 - \alpha)c_i, \ i \in N
\]
for some \( \alpha \in [0, 1] \), then
\[
|z - a_{ii}| > r_i^\alpha c_i^{1-\alpha},
\]
we can easily get that if \( z \notin \mathcal{A}_1(A) \), then \( z \notin \mathcal{A}_2(A) \), that is, \( \mathcal{A}_2(A) \subseteq \mathcal{A}_1(A) \). \( \square \)

From Theorem 3.4 and Lemma 3.5, we have easily the following result.

**Corollary 3.6.** Let \( A = [a_{ij}] \in \mathbb{C}^{n \times n}, \ n \geq 2 \). And \( \mathcal{A}_1(A) \) and \( \mathcal{K}_2(A) \) are defined in Theorems 1.2 and 3.1, respectively. Then

\[
\mathcal{K}_2(A) \subseteq \mathcal{A}_1(A).
\]
Similar to the proof of Lemma 3.5, we can establish easily the following comparison result.

**Theorem 3.7.** Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$. And $\mathcal{K}_1(A)$ and $\mathcal{K}_2(A)$ are defined in Theorems 3.3, and 3.1, respectively. Then

$$\mathcal{K}_2(A) \subseteq \mathcal{K}_1(A).$$

**Remark 3.8.** In Theorem 3.7, it is proved that $\mathcal{K}_2(A) \subseteq \mathcal{K}_1(A)$. However, $\mathcal{K}_2(A)$ is determined with more difficulty than $\mathcal{K}_1(A)$ because it is difficult to compute exactly $\log \left( \frac{r_{ij}}{c_{ij}} \right)$ in some cases.

**Example 3.9.** Let

$$A = \begin{bmatrix}
1 & 0.5 & 0.5 & 0 \\
1 & -1 & 0.5 & 0 \\
0.5 & 0 & i & 0.05 \\
0.1 & 0 & 0.1i & i
\end{bmatrix}.$$

The eigenvalue inclusion regions of Theorems 1.2, 1.3, 1.4, 3.3 and 3.1 are given, respectively, by Figs. 3.1, 3.2, 3.3, 3.8 and 3.9. And $\tilde{\mathcal{K}}(A)$, $\hat{\mathcal{K}}(A)$ and $\check{\mathcal{K}}(A)$ are shown in Figs. 3.5, 3.6 and 3.7, respectively. Note that the exact eigenvalues are plotted with asterisks. As we can see, $\tilde{\mathcal{K}}(A)$ fails to capture all the eigenvalues of $A$, so, the necessity of $\hat{\mathcal{K}}(A)$ or $\check{\mathcal{K}}(A)$ is evident. Also, it is easy to see that $\mathcal{K}_1(A) \subset A_1(A)$, $\mathcal{K}_2(A) \subset A_2(A) \subset A_1(A)$ and $\mathcal{K}_2(A) \subset \mathcal{K}_1(A) \subset \left( \mathcal{K}(A) \cap \mathcal{K}(A^T) \right)$. This example shows that the two new eigenvalue inclusion regions are smaller than the intersection of the Brauer sets of a matrix and its transpose, and the region of Theorem 3.1 is smaller than those of Theorem 6 and Theorem 7 in [4].

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**Fig. 3.1.** $A_1(A)$  
**Fig. 3.2.** $A_2(A)$
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