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GENERALIZATIONS OF BRAUER’S EIGENVALUE LOCALIZATION THEOREM*

CHAOQIAN LI† AND YAOTANG LI†

Abstract. New eigenvalue inclusion regions are given by establishing the necessary and sufficient conditions for two classes of nonsingular matrices, named double $\alpha_1$-matrices and double $\alpha_2$-matrices. These results are generalizations of Brauer’s eigenvalue localization theorem and improvements over the results in [L. Cvetković, V. Kostić, R. Bru, and F. Pedroche. A simple generalization of Geršgorin’s theorem. Adv. Comput. Math., 35:271–280, 2011.].

Key words. Matrix eigenvalue, Brauer’s eigenvalue localization theorem, Double $\alpha_1$-matrices, Double $\alpha_2$-matrices.

AMS subject classifications. 15A18, 65F15.

1. Introduction. Let $\mathbb{C}^{n \times n}$ denote the collection of all $n \times n$ complex matrices and $N = \{1, 2, \ldots, n\}$. For a matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, we denote, for any $i, j, k \in N$,

$$r_i = \sum_{k \neq i} |a_{ik}|, \quad c_i = \sum_{k \neq i} |a_{ki}|,$$

$$\Gamma_i(A) = \{z \in \mathbb{C} : |z - a_{ii}| \leq r_i\},$$

$$\hat{\Gamma}_i(A) = \{z \in \mathbb{C} : |z - a_{ii}| \leq \min\{r_i, c_i\}\},$$

$$\mathcal{H} = \{i \in N : r_i > c_i\}, \quad \mathcal{L} = \{i \in N : r_i < c_i\},$$

$$\hat{\Gamma}_{i,j}(A) = \{z \in \mathbb{C} : |z - a_{ii}|(c_j - r_j) + |z - a_{jj}|(r_i - c_i) \leq c_j r_i - c_i r_j, i \in \mathcal{H}, j \in \mathcal{L}\},$$

$$\hat{\Gamma}_{i,j}(A) = \{z \in \mathbb{C} : \frac{|z - a_{ii}|}{c_i} \left(\frac{|z - a_{jj}|}{c_j}\right)^{\log \frac{c_j}{r_j}} \leq 1, i \in \mathcal{H}\{k : c_k = 0\}, j \in \mathcal{L}\{k : r_k = 0\}\},$$

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\[ K_{i,j}(A) = \{ z \in \mathbb{C} : |z - a_{ii}| |z - a_{jj}| \leq r_i r_j \} \]

and

\[ \bar{K}_{i,j}(A) = \{ z \in \mathbb{C} : |z - a_{ii}| |z - a_{jj}| \leq \min \{ r_i r_j, c_i c_j \} \}. \]

Eigenvalue localization has been a hot topic in matrix theory and its applications. Many researchers have obtained lots of eigenvalue inclusion regions; for details, see [1]–[7], [9]–[13]. We first recall the very well known eigenvalue localization theorem of Geršgorin [6].

**Theorem 1.1.** [6] Let \( A = [a_{ij}] \in \mathbb{C}^{n \times n} \) and \( \sigma(A) \) be the spectrum of \( A \). Then

\[ \sigma(A) \subseteq \Gamma(A) = \bigcup_{i \in \mathbb{N}} \Gamma_i(A). \]

Here, \( \Gamma(A) \) is called the Geršgorin set of \( A \). Recently, L. Cvetković et al. [4] gave the following two eigenvalue inclusion regions by the characterizations of two class of nonsingular \( H \)-matrices, and proved that these two regions stay within the set \( \Gamma(A) \cap \Gamma(A^T) \), where \( A^T \) is the transpose of \( A \).

**Theorem 1.2.** [4, Theorem 6] Let \( A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2 \). Then

\[ \sigma(A) \subseteq A_1(A) = \bar{\Gamma}(A) \bigcup \hat{\Gamma}(A), \]

where \( \bar{\Gamma}(A) = \bigcup_{i \in \mathbb{N}} \bar{\Gamma}_i(A) \) and \( \hat{\Gamma}(A) = \bigcup_{i \in \mathcal{H}, j \in \mathcal{L}} \hat{\Gamma}_{i,j}(A) \).

**Theorem 1.3.** [4, Theorem 7] Let \( A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2 \). Then

\[ \sigma(A) \subseteq A_2(A) = \bar{\Gamma}(A) \bigcup \hat{\Gamma}(A), \]

where \( \bar{\Gamma}(A) = \bigcup_{i \in \mathbb{N}} \bar{\Gamma}_i(A) \) and \( \hat{\Gamma}(A) = \bigcup_{i \in \mathcal{H}, j \in \mathcal{L}} \hat{\Gamma}_{i,j}(A) \).

In [1], Brauer obtained the following eigenvalue localization theorem.

**Theorem 1.4.** [1] Let \( A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2 \). Then

\[ \sigma(A) \subseteq K(A) = \bigcup_{i,j \in \mathbb{N}, i \neq j} K_{i,j}(A). \]

The set \( K(A) \) is called the Brauer set of \( A \), and \( K_{i,j}(A) \) is called the \((i, j)\)-th Brauer Cassini oval. It is well known that \( K(A) \subseteq \Gamma(A) \) (see [12, 13]). Since \( A \) and
its transpose $A^T$ have the same spectrum, we have that $\sigma(A) = \sigma(A^T) \subseteq \Gamma(A^T)$, and thus, $\sigma(A) \subseteq (\mathcal{K}(A) \cap \mathcal{K}(A^T)) \subseteq (\Gamma(A) \cap \Gamma(A^T))$.

We now let

$$\bar{\mathcal{K}}(A) = \bigcup_{i,j \in \mathbb{N}, i \neq j} \bar{\mathcal{K}}_{i,j}(A). \tag{1.1}$$

Note that $\mathcal{K}_{i,j}(A) = \mathcal{K}_{j,i}(A)$, $\bar{\mathcal{K}}_{i,j}(A) = \bar{\mathcal{K}}_{j,i}(A)$, $\bar{\mathcal{K}}_{i,j}(A) \subseteq \mathcal{K}_{i,j}(A)$ and $\bar{\mathcal{K}}_{i,j}(A) \subseteq \mathcal{K}_{i,j}(A^T)$ for $i, j \in \mathbb{N}$, $i \neq j$. These show that $\mathcal{K}(A) \subseteq \mathcal{K}(A)$ and $\mathcal{K}(A) \subseteq \mathcal{K}(A^T)$, and thus,

$$\bar{\mathcal{K}}(A) \subseteq \left(\mathcal{K}(A) \cap \mathcal{K}(A^T)\right).$$

An interesting problem arises: whether $\bar{\mathcal{K}}(A)$ includes all eigenvalues of $A$ or not? The following example provides a negative answer.

**Example 1.5.** Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}. $$

By calculation, we get

$$\sigma(A) = \{-0.1149, 2.2541, 3.8608\},$$

$$\bar{\mathcal{K}}_{1,2}(A) = \{z \in \mathbb{C} : |z - 1||z - 2| \leq 2\},$$

$$\bar{\mathcal{K}}_{1,3}(A) = \{z \in \mathbb{C} : |z - 1||z - 3| \leq 3\}$$

and

$$\bar{\mathcal{K}}_{2,3}(A) = \{z \in \mathbb{C} : |z - 2||z - 3| \leq 1\}.$$

Obviously, $-0.1149 \notin \bar{\mathcal{K}}(A) = (\bar{\mathcal{K}}_{1,2}(A) \cup \bar{\mathcal{K}}_{1,3}(A) \cup \bar{\mathcal{K}}_{2,3}(A))$.

In this paper, we also focus on the subject of eigenvalue localization. In Section 2, we establish necessary and sufficient conditions for two classes of nonsingular matrices, named double $\alpha_1$-matrices and double $\alpha_2$-matrices. In Section 3, new regions $\mathcal{K}_1(A)$ and $\mathcal{K}_2(A)$ including all the eigenvalues of $A$ are obtained, which include $\bar{\mathcal{K}}(A)$ and stay within the set $\mathcal{K}(A) \cap \mathcal{K}(A^T)$. Specially, we compare the new eigenvalue inclusion region $\mathcal{K}_2(A)$ with $\mathcal{A}_1(A)$ in Theorem 1.2 (Theorem 6 of [4]) and $\mathcal{A}_2(A)$ in Theorem 1.3 (Theorem 7 of [4]), and prove $\mathcal{K}_2(A) \subseteq \mathcal{A}_1(A)$ and $\mathcal{K}_2(A) \subseteq \mathcal{A}_2(A)$.
2. Necessary and sufficient conditions of double $\alpha_1$-matrices and double $\alpha_2$-matrices. In this section, double $\alpha_1$-matrices and double $\alpha_2$-matrices are presented. And their characterizations are given.

**Definition 2.1.** A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ is said to be a double $\alpha_1$-matrix, if there is $\alpha \in [0, 1]$ such that for all $i, j \in N, i \neq j$,

$$|a_{ii}| |a_{jj}| > \alpha r_i r_j + (1 - \alpha) c_i c_j.$$  

**Definition 2.2.** A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ is said to be a double $\alpha_2$-matrix, if there is $\alpha \in [0, 1]$ such that for all $i, j \in N, i \neq j$,

$$|a_{ii}| |a_{jj}| > (r_i r_j)^\alpha (c_i c_j)^{1 - \alpha}. \quad (2.1)$$

As shown in [8], double $\alpha_2$-matrices are nonsingular. And moreover, from the generalized arithmetic-geometric mean inequality:

$$\alpha a + (1 - \alpha)b \geq a^\alpha b^{1-\alpha}$$

where $a, b \geq 0$ and $0 \leq \alpha \leq 1$, we easily get that double $\alpha_1$-matrices are also nonsingular.

Now we establish necessary and sufficient conditions for double $\alpha_1$-matrices and double $\alpha_2$-matrices, respectively. First, some notations are given. For a matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2$, we denote

$$\mathcal{R} = \{(i, j) : r_i r_j > c_i c_j, \ i \neq j, \ i, j \in N\},$$

$$\mathcal{C} = \{(i, j) : c_i c_j > r_i r_j, \ i \neq j, \ i, j \in N\},$$

$$\mathcal{E} = \{(i, j) : r_i r_j = c_i c_j, \ i \neq j, \ i, j \in N\}.$$  

Note here that $(i, j) \in \mathcal{R}$ ($\mathcal{C}$ or $\mathcal{E}$) implies $(j, i) \in \mathcal{R}$ ($\mathcal{C}$ or $\mathcal{E}$, respectively).

**Theorem 2.3.** A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2$, is a double $\alpha_2$-matrix if and only if the following two conditions hold:

(i) $|a_{ii}| |a_{jj}| > \min\{r_i r_j, c_i c_j\}$ for all $i, j \in N, i \neq j$.

(ii) $\log_{r_i r_j} \frac{|a_{ii}| |a_{jj}|}{c_i c_j} > \log_{r_{m n}} \frac{c_m c_n}{a_m a_n}$ for $(i, j) \in \mathcal{R}\{((l, k) : c_l c_k = 0)\}$, and $(m, n) \in \mathcal{C}\{((l, k) : r_l r_k = 0)\}$.  

Proof. Firstly, suppose that $A$ is a double $\alpha_2$-matrix. Then there is $\alpha \in [0, 1]$ such that
\[ |a_{ii}| |a_{jj}| > (r_ir_j)^\alpha (c_ic_j)^{1-\alpha} \]
for all $i, j \in N, i \neq j$. Condition (i) follows from the fact
\[ (r_ir_j)^\alpha (c_ic_j)^{1-\alpha} \geq \min\{r_ir_j, c_ic_j\} \]
Now, for $(i, j) \in R \setminus \{(l, k) : c_lc_k = 0\}$, we have
\[ |a_{ii}| |a_{jj}| > c_ic_j \]
Note that $r_ir_j > c_ic_j$, taking the logarithm of the above inequality for the base $r_ir_j > c_ic_j$, and using the monotonicity, we obtain that
\[ \log \frac{r_ir_j}{c_ic_j} |a_{ii}| |a_{jj}| > \alpha. \]
Thus, condition (ii) holds.

Conversely, suppose that the conditions (i) and (ii) hold. For each $(i, j) \in E$, condition (i) directly implies inequality (2.1). And for $(i, j) \in R$ such that $c_ic_j = 0$, or $(m, n) \in C$ such that $r_mr_n = 0$, inequality (2.1) follows immediately. Thus, it remains to prove that inequality (2.1) holds for all $(i, j) \in (R \setminus \{(l, k) : c_lc_k = 0\}) \cup (C \setminus \{(l, k) : r_lr_k = 0\})$.

For each $(i, j) \in R \setminus \{(l, k) : c_lc_k = 0\}$, we have $r_ir_j > c_ic_j$, which, from condition (i), leads to $|a_{ii}| |a_{jj}| > c_ic_j$. Using the properties of the log function for the base greater than one, we obtain
\[ \log \frac{r_ir_j}{c_ic_j} |a_{ii}| |a_{jj}| > 0. \]
Similarly, for each $(m, n) \in C \setminus \{(l, k) : r_lr_k = 0\}$, we have
\[ \log \frac{r_mr_n}{a_{mm}a_{nn}} < 1. \]
From inequalities (2.2), (2.3) and condition (ii), we have that there is $\alpha$ such that, for each $(i, j) \in R \setminus \{(l, k) : c_lc_k = 0\}$ and each $(m, n) \in C \setminus \{(l, k) : r_lr_k = 0\}$,
\[ \max \left\{0, \log \frac{c_mc_n}{a_{mm}a_{nn}} \right\} < \alpha < \min \left\{ \log \frac{r_ir_j}{c_ic_j}, 1 \right\}. \]
From the left inequality and right inequality of inequality (2.4), we get, respectively, that for each $(i, j) \in \mathbb{R}\{ (l, k) : c_l c_k = 0 \}$,

$$\frac{|a_{ii} a_{jj}|}{c_i c_j} > \left( \frac{r_i r_j}{c_i c_j} \right)^\alpha$$

and for each $(m, n) \in \mathbb{C}\{ (l, k) : r_l r_k = 0 \}$,

$$\frac{c_m c_n}{|a_{mm} a_{nn}|} > \left( \frac{r_m r_n}{c_m c_n} \right)^\alpha.$$  

Thus, the proof is completed. $\blacksquare$

Similar to the proof of Theorem 2.3, we can obtain the following necessary and sufficient conditions for double $\alpha_1$-matrices, and its proof is omitted.

**Theorem 2.4.** A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, is a double $\alpha_1$-matrix if and only if the following two conditions hold:

1. $|a_{ii}| |a_{jj}| > \min\{r_i r_j, c_i c_j\}$ for all $i, j \in \mathbb{N}$, $i \neq j$.
2. $\frac{|a_{ii} a_{jj}|}{r_i r_j c_i c_j} > \frac{c_m c_n |a_{mm} a_{nn}|}{c_m c_n r_m r_n}$ for all $(i, j) \in \mathbb{R}$, and all $(m, n) \in \mathbb{C}$.

**3. Eigenvalue localizations.** By the necessary and sufficient conditions of double $\alpha_1$-matrices and double $\alpha_2$-matrices in Section 2, we give two new eigenvalue inclusion regions.

**Theorem 3.1.** Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, and $\sigma(A)$ be the spectrum of $A$. Then

$$\sigma(A) \subseteq \mathcal{K}_2(A) = \hat{\mathcal{K}}(A) \bigcup \hat{\mathcal{K}}(A),$$

where $\hat{\mathcal{K}}(A)$ is given by (1.1), $\hat{\mathcal{K}}(A) = \bigcup_{(i, j) \in \mathbb{R}, (m, n) \in \mathbb{C}} \hat{\mathcal{K}}_{i, j, m, n}(A)$ and

$$\hat{\mathcal{K}}_{i, j, m, n}(A) = \{ z \in \mathbb{C} : \frac{|\lambda - a_{ii}| |\lambda - a_{jj}|}{c_i c_j} \left( \frac{\left| \lambda - a_{mm} \right| \left| \lambda - a_{nn} \right|}{c_m c_n} \right)^{\log \frac{c_m c_n}{|a_{mm} a_{nn}|}} \frac{r_i r_j}{c_i c_j} \leq 1, \ (i, j) \in \mathbb{R}\{ (l, k) : c_l c_k = 0 \}, \ (m, n) \in \mathbb{C}\{ (l, k) : r_l r_k = 0 \} \}.$$

**Proof.** For any $\lambda \in \sigma(A)$, $\lambda I - A$ is singular. Note that the moduli of every off-diagonal entry of $\lambda I - A$ is the same as $A$. Hence, the sets $\mathcal{R}$ and $\mathcal{C}$ for the matrix $\lambda I - A$ remain the same. If $\lambda \notin \mathcal{K}_2(A)$, then $\lambda I - A$ satisfies conditions (i) and (ii) of Theorem 2.3, hence $\lambda I - A$ is a double $\alpha_2$-matrix, which implies that $\lambda I - A$ is nonsingular. This is a contradiction. Hence, $\lambda \in \mathcal{K}_2(A)$, that is, $\sigma(A) \subseteq \mathcal{K}_2(A)$. $\blacksquare$
Remark 3.2. (i) From the original definition of double $\alpha_2$-matrices, we can derive directly the following eigenvalue inclusion region (see [9]):

$\mathcal{K}_2(A) = \bigcap_{0 \leq \alpha \leq 1} \bigcup_{i,j \in \mathbb{N}, i \neq j} \left\{ z \in \mathbb{C} : |z - a_{ii}| |z - a_{jj}| \leq (r_i r_j)^{\alpha} (c_i c_j)^{1-\alpha} \right\}$.

(3.2)

Obviously, the form of $\mathcal{K}_2(A)$ obtained in (3.1) is much more convenient than that in (3.2).

(ii) Since $\mathcal{K}_2(A) = \mathcal{K}_2(A^T)$, we have that $\mathcal{K}_2(A) \subseteq (\mathcal{K}(A) \bigcap \mathcal{K}(A^T))$.

Similar to the proof of Theorem 3.1, we can obtain easily the following eigenvalue localization theorem.

**Theorem 3.3.** Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, and $\sigma(A)$ be the spectrum of $A$. Then

$\sigma(A) \subseteq \mathcal{K}_1(A) = \bar{\mathcal{K}}(A) \bigcup \check{\mathcal{K}}(A)$,

where $\bar{\mathcal{K}}(A)$ is given by (1.1), $\check{\mathcal{K}}(A) = \bigcup_{(i,j) \in \mathcal{R}, (m,n) \in \mathcal{C}} \check{\mathcal{K}}_{i,j,m,n}(A)$ and

$\check{\mathcal{K}}_{i,j,m,n}(A) = \left\{ z \in \mathbb{C} : |\lambda - a_{ii}| |\lambda - a_{jj}| (c_m c_n - r_m r_n) + |\lambda - a_{mm}| |\lambda - a_{nn}| (r_i r_j - c_i c_j) \leq c_m c_n r_i r_j - c_i c_j r_m r_n, (i,j) \in \mathcal{R}, (m,n) \in \mathcal{C} \right\}$.

Similar to Remark 3.2, we also obtain that $\mathcal{K}_1(A) \subseteq (\mathcal{K}(A) \bigcap \mathcal{K}(A^T))$. Next, we compare $\mathcal{K}_2(A)$ in Theorem 3.1 with $\mathcal{A}_1(A)$ in Theorem 1.2 (Theorem 6 of [4]) and $\mathcal{A}_2(A)$ in Theorem 1.3 (Theorem 7 of [4]).

**Theorem 3.4.** Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$. And $\mathcal{A}_2(A)$ and $\mathcal{K}_2(A)$ are defined in Theorems 1.3, and 3.1, respectively. Then

$\mathcal{K}_2(A) \subseteq \mathcal{A}_2(A)$.

**Proof.** We prove $\mathcal{K}_2(A) \subseteq \mathcal{A}_2(A)$. Equivalently, we prove that if $z \notin \mathcal{A}_2(A)$, then $z \notin \mathcal{K}_2(A)$. In fact, if $z \notin \mathcal{A}_2(A)$, from Theorem 1.3, we have that for any $i \in \mathbb{N}$,

(3.3) $|z - a_{ii}| > \min\{r_i, c_i\}$,

and for $i \in \mathcal{H}\{k : c_k = 0\}$ and $j \in \mathcal{L}\{k : r_k = 0\}$,

(3.4) $\frac{|z - a_{ii}|}{c_i} \left( \frac{|z - a_{jj}|}{c_j} \right)^{\log r_i c_i r_j} > 1$. 

From Theorems 5 and 7 of [4], inequalities (3.3) and (3.4) imply that for any \( i \in N \),

\[
|z - a_{ii}| > (r_i)^\alpha(c_i)^{1-\alpha}
\]

for some \( \alpha \in [0, 1] \). Hence, for any \( i, j \in N \) and \( i \neq j \), we have

\[
|z - a_{ii}| |z - a_{jj}| > (r_i)^\alpha(c_i)^{1-\alpha}(r_j)^\alpha(c_j)^{1-\alpha} = (r_i r_j)^\alpha(c_i c_j)^{1-\alpha}
\]

for some \( \alpha \in [0, 1] \). This implies that \( zI - A \) is a double \( \alpha_2 \)-matrix. From Theorem 2.3, the following two inequalities hold:

\[
|z - a_{ii}| |z - a_{jj}| > \min\{r_i r_j, c_i c_j\}
\]

for all \( i, j \in N, i \neq j \), and

\[
\log \frac{c_i c_j}{r_i r_j} |z - a_{ii}| |z - a_{jj}| > \log \frac{c_m c_n}{r_m r_n} |z - a_{mm}| |z - a_{nn}|
\]

for \( (i, j) \in \mathcal{R}\{\{l, k\} : c_l c_k = 0\} \), and \( (m, n) \in \mathcal{C}\{\{l, k\} : r_l r_k = 0\} \). Moreover, inequality (3.6) is written equivalently as

\[
\left|\frac{z - a_{ii}}{c_i c_j} |z - a_{jj}| \right| \left|z - a_{mm}||z - a_{nn}| \right| = (r_i r_j)^\alpha(c_i c_j)^{1-\alpha} > 1.
\]

Hence, from inequalities (3.5) and (3.7), \( z \notin \mathcal{K}(A) \) and \( z \notin \mathcal{K}(A) \), that is, \( z \notin \mathcal{K}_2(A) \). The proof is completed.

**Lemma 3.5.** Let \( A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2 \). And \( \mathcal{A}_1(A) \) and \( \mathcal{A}_2(A) \) are defined in Theorems 1.2 and 1.3, respectively. Then

\[
\mathcal{A}_2(A) \subseteq \mathcal{A}_1(A).
\]

**Proof.** Similar to the proof of Theorem 3.4 and from the fact that if

\[
|z - a_{ii}| > \alpha r_i + (1 - \alpha)c_i, \ i \in N
\]

for some \( \alpha \in [0, 1] \), then

\[
|z - a_{ii}| > r_i^\alpha c_i^{1-\alpha},
\]

we can easily get that if \( z \notin \mathcal{A}_1(A) \), then \( z \notin \mathcal{A}_2(A) \), that is, \( \mathcal{A}_2(A) \subseteq \mathcal{A}_1(A) \). \( \Box \)

From Theorem 3.4 and Lemma 3.5, we have easily the following result.

**Corollary 3.6.** Let \( A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2 \). And \( \mathcal{A}_1(A) \) and \( \mathcal{K}_2(A) \) are defined in Theorems 1.2 and 3.1, respectively. Then

\[
\mathcal{K}_2(A) \subseteq \mathcal{A}_1(A).
\]
Similar to the proof of Lemma 3.5, we can establish easily the following comparison result.

**Theorem 3.7.** Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$. And $\mathcal{K}_1(A)$ and $\mathcal{K}_2(A)$ are defined in Theorems 3.3, and 3.1, respectively. Then

$$\mathcal{K}_2(A) \subseteq \mathcal{K}_1(A).$$

**Remark 3.8.** In Theorem 3.7, it is proved that $\mathcal{K}_2(A) \subseteq \mathcal{K}_1(A)$. However, $\mathcal{K}_2(A)$ is determined with more difficulty than $\mathcal{K}_1(A)$ because it is difficult to compute exactly $\log \left( \frac{\max c_m c_n}{\min r_i r_j} \right)$ in some cases.

**Example 3.9.** Let

$$A = \begin{bmatrix}
1 & 0.5 & 0.5 & 0 \\
1 & -1 & 0.5 & 0 \\
0.5 & 0 & i & 0.05 \\
0.1 & 0 & 0.1i & i
\end{bmatrix}.$$  

The eigenvalue inclusion regions of Theorems 1.2, 1.3, 1.4, 3.3 and 3.1 are given, respectively, by Figs. 3.1, 3.2, 3.3, 3.8 and 3.9. And $\tilde{K}(A)$, $\bar{K}(A)$ and $\hat{K}(A)$ are shown in Figs. 3.5, 3.6 and 3.7, respectively. Note that the exact eigenvalues are plotted with asterisks. As we can see, $\tilde{K}(A)$ fails to capture all the eigenvalues of $A$, so, the necessity of $\bar{K}(A)$ or $\hat{K}(A)$ is evident. Also, it is easy to see that $\mathcal{K}_1(A) \subset \mathcal{A}_1(A)$, $\mathcal{K}_2(A) \subset \mathcal{A}_2(A) \subset \mathcal{A}_1(A)$ and $\mathcal{K}_2(A) \subset \mathcal{K}_1(A) \subset (\mathcal{K}(A) \cap \mathcal{K}(A^T))$. This example shows that the two new eigenvalue inclusion regions are smaller than the intersection of the Brauer sets of a matrix and its transpose, and the region of Theorem 3.1 is smaller than those of Theorem 6 and Theorem 7 in [4].

**Fig. 3.1.** $\mathcal{A}_1(A)$ **Fig. 3.2.** $\mathcal{A}_2(A)$
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