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Abraham Berman
Miriam Farber

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A LOWER BOUND FOR THE SECOND LARGEST LAPLACIAN EIGENVALUE OF WEIGHTED GRAPHS

ABRAHAM BERMAN† AND MIRIAM FARBER†

Abstract. Let $G$ be a weighted graph on $n$ vertices. Let $\lambda_{n-1}(G)$ be the second largest eigenvalue of the Laplacian of $G$. For $n \geq 3$, it is proved that $\lambda_{n-1}(G) \geq d_{n-2}(G)$, where $d_{n-2}(G)$ is the third largest degree of $G$. An upper bound for the second smallest eigenvalue of the signless Laplacian of $G$ is also obtained.

Key words. Weighted graph, Laplacian matrix, Second largest eigenvalue, Lower bound, Signless Laplacian, Merris graph.

AMS subject classifications. 15A42, 05C50, 05C69.

1. Introduction. Let $G = (E(G), V(G))$ be a simple graph (a graph without loops or multiple edges) with $|V(G)| = n$. We say that $G$ is a weighted graph if it has a weight (a positive number) associated with each edge. The weight of an edge $\{i, j\} \in E(G)$ will be denoted by $w_{ij}$. We define the adjacency matrix $A(G)$ of $G$ to be a symmetric matrix which satisfies

$$a_{ij} = \begin{cases} 0 & \text{if } \{i, j\} \notin E(G) \\ w_{ij} & \text{if } \{i, j\} \in E(G) \end{cases}.$$ 

The Laplacian matrix $L(G)$ is defined to be $D(G) - A(G)$ with $D(G) = \text{diag}(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$, where $\deg(v_i)$ is the sum of weights of all edges connected to $v_i$. The signless Laplacian matrix $Q(G)$ is defined by $D(G) + A(G)$. We denote by $0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ the eigenvalues of $L(G)$, and by $\mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$ the eigenvalues of $Q(G)$. We order the degrees of the vertices of $G$ as $d_1(G) \leq d_2(G) \leq \cdots \leq d_n(G)$. Various bounds for the Laplacian eigenvalues of unweighted graphs, in terms of their degrees, were studied in the past (e.g., [1]). Li and Pan [6] showed that for an unweighted connected graph $G$ with $n \geq 3$, $\lambda_{n-1}(G) \geq d_{n-1}(G)$. It is interesting to ask whether there exists a similar bound for weighted graphs. We will show it by using the following lemma ([5, p. 178]).

**Lemma 1.1.** Let $A$ be a symmetric matrix with eigenvalues $\theta_1(G) \leq \cdots \leq \theta_n(G)$. Then $\theta_k(A) = \max \left\{ \frac{(Af, f)}{(f, f)} \mid f \perp f_{k+1}, f_{k+2}, \ldots, f_n \right\} = \min \left\{ \frac{(Af, f)}{(f, f)} \mid f \perp f_1, f_2, \ldots, f_{k-1} \right\}$.
when \( f_1, f_2, \ldots, f_n \) are eigenvectors of the eigenvalues \( \theta_1, \theta_2, \ldots, \theta_n \), respectively.

2. The main result. We are ready now to present our main result.

**Theorem 2.1.** Let \( G \) be a simple weighted graph on \( n \) vertices with \( n \geq 3 \). Then

\[
\lambda_{n-1}(G) \geq d_{n-2}(G).
\]

**Proof.** First we check the case \( \lambda_{n-1}(G) = \lambda_n(G) \). Let \( u \) be the vertex with the largest degree in \( G \). From Lemma 1.1,

\[
\lambda_n(G) = \max \left\{ \frac{\langle L(G)f, f \rangle}{\langle f, f \rangle} \right\}.
\]

Define a vector \( v \) by

\[
v_i = \begin{cases} 0 & \text{if } i \neq u \\ 1 & \text{if } i = u \end{cases}.
\]

Then we have

\[
\lambda_n(G) \geq \frac{\langle L(G)v, v \rangle}{\langle v, v \rangle} = d_n(G).
\]

Hence, in this case, \( d_{n-2}(G) \leq d_n(G) \leq \lambda_n(G) = \lambda_{n-1}(G) \). Suppose then that \( \lambda_{n-1}(G) < \lambda_n(G) \). Let \( h \) be an eigenvector that corresponds to \( \lambda_n(G) \). Using Lemma 1.1 we have

\[
\lambda_{n-1}(G) = \max \left\{ \frac{\langle L(G)f, f \rangle}{\langle f, f \rangle} | f \perp h \right\}.
\]

(2.1)

Let \( s, t, q \) be the vertices with the largest degrees in the graph. Then there are two possibilities:

1) At least one of \( h_s, h_t, h_q \) is zero.

2) All the numbers \( h_s, h_t, h_q \) are different from zero.

In case 1), we assume without loss of generality that \( h_t = 0 \). Define a vector \( g \) by

\[
g_i = \begin{cases} 0 & \text{if } i \neq t \\ 1 & \text{if } i = t \end{cases}.
\]

Since \( g \) is orthogonal to \( h \), we get from (2.1) that

\[
\lambda_{n-1}(G) \geq \frac{\langle L(G)g, g \rangle}{\langle g, g \rangle},
\]

and hence,

\[
\lambda_{n-1}(G) \geq \sum_{uv \in E(G)} w_{uv}(y_u - y_v)^2 + \sum_{t \in V(G)} w_{tt}(g_t - g_t)^2.
\]
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\[ \sum_{tv \in E(G)} w_{tv} = \deg(t) \geq \min \{ \deg(s), \deg(t), \deg(q) \} = d_{n-2}(G), \]
and we are done.

In case 2), at least two of \( h_s, h_t, h_q \) have the same sign. Suppose without loss of
generality that \( h_s, h_t \) have the same sign. Define a vector \( g \) by

\[ g_i = \begin{cases} 
    0 & \text{if } i \neq t, s \\
    1 & \text{if } i = t \\
    -\delta & \text{if } i = s 
\end{cases} \]

with \( \delta > 0 \) such that \( g \) is orthogonal to \( h \) (such a positive \( \delta \) exists since \( h_s \) and \( h_t \) are
with the same sign). Therefore,

\[ \lambda_{n-1}(G) \geq \frac{\langle L(G)g, g \rangle}{\langle g, g \rangle} = \sum_{uv \in E(G)} w_{uv}(g_u - g_v)^2 \]
\[ = \sum_{tv \in E(G), v \neq s} w_{tv}(g_t - g_v)^2 + \sum_{us \in E(G), u \neq t} w_{us}(g_u - g_s)^2 + w_{ts}(g_t - g_s)^2 \]
\[ \leq \sum_{tv \in E(G), v \neq s} w_{tv} + \delta^2 \left( \sum_{us \in E(G), u \neq t} w_{us} \right) + w_{ts}(1 + 2\delta + \delta^2) \]
\[ = \frac{\deg(t) - w_{ts} + \delta^2(\deg(s) - w_{ts}) + w_{ts}(1 + 2\delta + \delta^2)}{1 + \delta^2} \]
\[ = \frac{\deg(t) + \deg(s)\delta^2 + 2w_{ts}\delta}{1 + \delta^2}, \]
and since \( \delta > 0 \) we have:

\[ \lambda_{n-1}(G) \geq \frac{\deg(t) + \deg(s)\delta^2 + 2w_{ts}\delta}{1 + \delta^2} \geq \frac{\deg(t) + \deg(s)\delta^2}{1 + \delta^2} \]
\[ \geq \min \{ \deg(s), \deg(t) \} \geq \min \{ \deg(s), \deg(t), \deg(q) \} = d_{n-2}(G) \]
and we are done. \( \square \)

**Remark 2.2.** As we mentioned before, for connected unweighted graphs with
\( n \geq 3 \), \( \lambda_{n-1}(G) \geq d_{n-1}(G) \) ([6]). This is not true for weighted graphs as is shown by
Figure 2.1:
Note that the eigenvalues of $L(G)$ are 0, 9, 23, so $9 = \lambda_{n-1}(G) < d_{n-1}(G) = 10$.

3. Application. For a weighted graph $G$, we define $m_{L(G)}(I)$ to be the number of the eigenvalues of $L(G)$ that fall inside an interval $I$ (counting multiplicities). The independence number of $G$ is denoted by $\alpha(G)$. Merris [7] showed that if $G$ is a simple unweighted graph on $n$ vertices, then $m_{L(G)}([d_1(G), n]) \geq \alpha(G)$. Graphs which attain equality in the expression above were studied by Goldberg and Shapiro [4]. By similar technique to the one used by Merris in [7], we can show the following version for weighted graphs.

**Theorem 3.1.** Let $G$ be a simple weighted graph on $n$ vertices. Then we have $m_{L(G)}([d_1(G), \infty]) \geq \alpha(G)$.

Various examples of weighted graphs that attain equality can be found, and some of them are mentioned in [4] (for the special case of unweighted graph). This suggests the following question: Does there exist a graph for which there is no way to assign weights to the edges so that $m_{L(G)}([d_1(G), \infty]) = \alpha(G)$?

A first simple example is $K_n$ ($n \geq 3$). There is no way to assign weights to the edges of the complete graph so that $m_{L(K_n)}([d_1(K_n), \infty]) = 1$. This follows from Theorem 2.1, since

$$\lambda_n(K_n) \geq \lambda_{n-1}(K_n) \geq d_{n-1}(K_n) \geq d_1(K_n).$$

Hence, for any weighting of $K_n$, $m_{L(K_n)}([d_1(K_n), \infty]) \geq 2$. Are there other examples? The answer is still yes. Using Theorem 2.1, we can construct a family of such graphs in the following way: First, we take two graphs $G$ and $H$, each one of them is on at least four vertices, such that $\alpha(G), \alpha(H) \leq 2$. We obtain a new graph $K$ by adding an edge between one vertex of $G$ and one vertex of $H$. If $\alpha(K) \leq 3$, then there is no way to put weights on its edges such that $m_{L(K)}([d_1(K), \infty]) = \alpha(K)$. To show it, suppose in contradiction that there is such way. We look at the graph $G \cup H$ with weights induced by $K$ (i.e., all the edges in $G \cup H$ have the same weight as they have in $K$). Recall that
$n \geq 4$, hence from Theorem 2.1 we have $\lambda_{n-1}(G) \geq d_2(G)$, $\lambda_{n-1}(H) \geq d_2(H)$, and hence $G \cup H$ has at least four eigenvalues greater than or equal to $\min\{d_2(G), d_2(H)\}$. Since $d_1(K) \leq \min\{d_2(G), d_2(H)\}$, using the interlacing theorem for adding an edge (which could be found in [3, p. 291] for unweighted graphs, but it is also true in the weighted case), we get that there are at least four eigenvalues of $L(K)$ which are above $d_1(K)$, so $\alpha(K) \geq 4$, contradicting the assumption that $\alpha(K) \leq 3$. To construct such graphs $K$, we can take $G$ and $H$ to be complete graphs (see Figure 3.1).

![Fig. 3.1.](image1)

$G$ and $H$ can be chosen also to be noncomplete, but here one has to be careful in choosing the vertices. Since $\alpha(G \cup H)=4$, we must add an edge that will reduce the independence number of $K$ to 3 (see Figure 3.2).

![Fig. 3.2.](image2)

4. The signless Laplacian. It was proven in [2] that for a simple unweighted noncomplete graph $G$ with $n$ vertices ($n \geq 2$), $\mu_{n-1}(G) \geq \lambda_2(G)$. In this section, we deal with the relations between $\mu_2(G)$ and $\lambda_{n-1}(G)$. First, using techniques similar to those of the proof of Theorem 2.1, we prove the following:

Theorem 4.1. Let $G$ be a simple weighted graph on $n$ vertices. Then $\mu_2(G) \leq d_3(G)$. 

Proof. For the signless Laplacian, we have
\[
\langle Q(G)g, g \rangle = \sum_{uv \in E(G)} w_{uv}(g_u + g_v)^2 \sum_{z \in V(G)} g_z^2 \langle g, g \rangle.
\]
Here we denote by \( h \) an eigenvector that corresponds to \( \mu_1(G) \), and hence from Lemma 1.1,
\[
\mu_2(G) = \min \left\{ \frac{\langle Q(G)f, f \rangle}{\langle f, f \rangle} \mid f \perp h \right\}.
\]
We denote by \( s, t, q \) be the three vertices with the smallest degrees in \( G \), and again, at least two of \( h_s, h_t, h_q \) have the same sign. We construct the vector \( g \) in the same way as in Theorem 2.1, and conclude with
\[
\mu_2(G) \leq \frac{\langle Q(G)g, g \rangle}{\langle g, g \rangle} = \sum_{tv \in E(G), v \neq s} w_{ts}(1 + 0)^2 + \sum_{us \in E(G), u \neq t} w_{us}(0 + (-\delta))^2 + w_{ts}(1 + (-\delta))^2
\]
\[
= \frac{\deg(t) + \delta^2 \deg(s) - 2w_{ts}\delta}{1 + \delta^2} \leq \frac{\deg(t) + \delta^2 \deg(s)}{1 + \delta^2} \leq d_3(G).
\]
We conclude the paper with the following corollary, which follows directly from Theorems 2.1 and 4.1.

Corollary 4.2. Let \( G \) be a simple weighted graph on \( n \) vertices, \( n \geq 5 \). Then \( \mu_2(G) \leq \lambda_{n-1}(G) \).

REFERENCES