The characteristic set with respect to the k-maximal vectors of a tree

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THE CHARACTERISTIC SET WITH RESPECT TO K-MAXIMAL VECTORS OF A TREE*

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Abstract. Let T be a tree on n vertices and L(T) be its Laplacian matrix. The eigenvalues and eigenvectors of T are respectively referred to those of L(T). With respect to a given eigenvector Y of T, a vertex u of T is called a characteristic vertex if Y[u] = 0 and there is a vertex w adjacent to u with Y[w] ≠ 0; an edge e = (u, w) of T is called a characteristic edge if Y[u]Y[w] < 0. C(T, Y) denotes the characteristic set of T with respect to the vector Y, which is defined as the collection of all characteristic vertices and characteristic edges of T corresponding to Y.

Let λ1(T) ≤ λ2(T) ≤ ⋯ ≤ λn(T) be the eigenvalues of a tree T on n vertices. An eigenvector is called a k-vector (k ≥ 2) of T if the eigenvalue λk(T) associated by this eigenvector satisfies λk(T) > λk−1(T). The k-vector Y of T is called k-maximal if C(T, Y) has maximum cardinality among all k-vectors of T. In this paper, the characteristic set with respect to any k-maximal vector of a tree is investigated by exploiting the relationship between the cardinality of the characteristic set and the structure of this tree. With respect to any k-maximal vector Y of a tree T, the structure of the trees T satisfying |C(T, Y)| = k − 1 − t for any t (0 ≤ t ≤ k − 2) are characterized.

Key words. Laplacian matrix, Characteristic set, k-Vector, k-Maximal vector.

AMS subject classifications. 05C50, 15A15.

1. Introduction. Let G = (V, E) be a simple graph with vertex set V = V(G) = {v1, v2, . . . , vn} and edge set E = E(G). The Laplacian matrix of G is defined as

\[ L = L(G) = D(G) - A(G), \]

where A(G) is the adjacency matrix of G and D(G) = diag{d(v1), d(v2), . . . , d(vn)}, the diagonal degree matrix of G. Since L(G) is positive semi-definite, its eigenvalues can be arranged as

\[ 0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G). \]

Henceforth λk(G) denotes the kth smallest eigenvalue of G. The kth smallest eigenvalue of G will be written as \( \lambda_k(G) \) if \( \lambda_k(G) > \lambda_{k-1}(G) \), and the corresponding eigenvectors will be called k-vectors of G.

For an eigenvector Y of a given graph G, a vertex v is called a characteristic vertex with respect to Y if Y[v] = 0 and there is a vertex w adjacent to v, such that Y[w] ≠ 0;

*Received by the editors on September 9, 2010. Accepted for publication on December 18, 2011. Handling Editor: Bryan L. Shader.
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an edge $e = (u, w)$ is called a characteristic edge of $G$ with respect to $Y$ if $Y[u]Y[w] < 0$. We denote by $C(G, Y)$ the characteristic set of $G$ with respect to the vector $Y$, which is defined as the collection of all characteristic vertices and characteristic edges of $G$ corresponding to $Y$. For convenience we relax the requirement that $Y$ be an eigenvector of $G$ in the definition of $C(G, Y)$, and allow $Y$ to be an arbitrary vector defined on the vertex set of $G$.

For a graph $G$, an eigenvector corresponding to the second smallest eigenvalue is called a Fiedler vector of $G$. It is known that $\lambda_2(G) > \lambda_1(G) = 0$ if and only if $G$ is connected [5]. Thus, each Fiedler vector of a connected graph is a 2-vector. Fiedler's remarkable result [5, Theorem 3.14] on the structure of Fiedler vectors (i.e., 2-vectors) of a connected graph motivated a lot of work on the structure of eigenvectors; see, e.g., [1, 2, 7, 8, 9, 10, 11, 12, 13, 14].

Merris introduced the notion of a characteristic set and showed that $|C(T, Y)| = 1$. In [11], Merris also showed that $C(T, Y)$ is fixed regardless of the choice of Fiedler vectors $Y$ of a given tree $T$; see [11] Theorem 2. With respect to any Fiedler vectors $Y$ of a given graph $G$, Bapat and Pati [11] investigated the cardinality of the characteristic set $C(G, Y)$. In [14], Pati extended the notation the characteristic set from Fiedler vectors to 3-vectors of trees and gave a complete description of 3-vectors of a given tree. Then Fan and Gong [2] further extended the concept of characteristic set to any $k$-vector of a tree.

Recall that, for any 2-vector $Y$ of a tree $T$, $|C(T, Y)| = 1$ and $C(T, Y)$ is fixed regardless of the choice of 2-vectors $Y$, even though the eigenspace for $\lambda(G)$ (well known as the algebraic connectivity of $T$ [3]) is large (see [11], Theorem 2).

However, for $k \geq 3$, the characteristic set $C(T, Y)$ may depend upon the choice of the $k$-vectors. For example, consider the tree $T$ in Figure 1.1 (or see Figure 3.2 in [14]). One can find that $Y_1$, $Y_2$ and $Y_3$ are all 3-vectors of $T$, where

$Y_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$,

$Y_2 = \begin{bmatrix} .5929 & .4754 & .2638 & .4754 & .2638 & .4754 & .2638 & .4754 & .2638 & .4754 & .2638 & .4754 & .2638 & .4754 & .2638 & .4754 & .2638 & .4754 & .2638 & .4754 \end{bmatrix}^T$,

$Y_3 = \begin{bmatrix} .9098 & .7296 & .4049 & .0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$,

and

But one can verify that $C(T, Y_1) = \{11\}$, $C(T, Y_2) = \{4\}$, and $|C(T, Y_3)| = \{4, 11\}$.

For a given tree $T$ and a $k$-vector $\overrightarrow{Y}$ of $T$, $\overrightarrow{Y}$ is called $k$-maximal if $C(T, \overrightarrow{Y})$ has maximum cardinality over all $k$-vectors of $T$, i.e.,

$|C(T, \overrightarrow{Y})| = \max_{\overrightarrow{Y}} |C(T, \overrightarrow{Y})|$, where the maximum is taken over all $k$-vectors of $T$ (see [6]).
For any \( k \)-maximal vector of a tree, the following result is interesting.

**Proposition 1.1.** ([6], Theorem 3.2) Let \( T \) be a tree on \( n \) vertices. Suppose that both \( Y_1 \) and \( Y_2 \) are the \( k \)\((\geq 2)\)-maximal vectors of \( T \). Then

\[
C(T, Y_1) = C(T, Y_2).
\]

Proposition 1.1 implies that for any \( k \) with \( 2 \leq k \leq n \), the characteristic set \( C(T, Y) \) is fixed regardless the choice of the \( k \)-maximal vector \( Y \), i.e., the characteristic set is determined by the tree structure and independent of the \( k \)-maximal vectors, which is consistent with Merris’ result (see [11], Theorem 2). Henceforth, to exploit the relationship between the cardinality of the characteristic set and the tree structure, we focus on studying the \( k \)-maximal vectors of trees.

With respect to any \( k \)-(maximal) vector \( Y \) of a given tree \( T \), Fan et al. showed that [2, Corollary 2.5]

\[
1 \leq |C(T, Y)| \leq k - 1.
\]

In particular, they also gave a characterization for trees whose characteristic set \( C(T, Y) \) with respect to its any \( k \)-vector \( Y \) contains exactly one element, i.e., the \( k \)-simple trees; see [2 Theorem 2.11]. Naturally, the following problem is posed:

For a general tree \( T \) on \( n \) vertices and an arbitrary integer \( k \) \((\leq n)\), can we exploit the relationship between the cardinality of the characteristic set \( C(T, Y) \) with respect to its any \( k \)-maximal vector \( Y \) and the structure of such a tree \( T \)?

In this paper, we investigate the characteristic set with respect to any \( k \)-maximal vector of a given tree and consider the problem above. The rest paper is organized as follows. In Section 2, we first list several preliminary results. Then, for any \( k \)-(maximal) vector \( Y \) of a given tree \( T \), we establish some lemmas that relate characteristic vertex and the structure for the subvector of \( Y \). In Section 3, we study the cardinality of the characteristic set \( C(T, Y) \) with respect to any \( k \)-maximal vector \( Y \).
of a tree $T$, and determine the structure of the trees $T$ satisfying $|C(T, Y)| = k - 1 - t$, where $0 \leq t \leq k - 2$. In addition, examples that illustrate the occurrence of each of the case described in our theorems are given.

2. Preliminary results. Let $G$ be a connected graph on $n$ vertices, $L$, its Laplacian matrix, and $Y$, a vector defined on the vertex set of $G$. We will use following notation. For $U \subseteq V(G)$, $W \subseteq V(G)$, denote by $L[U, W]$ the submatrix of $L$ with rows corresponding to the vertices of $U$ and columns corresponding to the vertices of $W$, if $U = W$, $L[U, W]$ is simply written as $L[U]$; and similarly, denote by $Y[U]$ the subvector of $Y$ corresponding to the vertices of $U$. For convenience, we usually write $L[G_1, G_2]$ and $Y[G_1]$ instead of $L[V(G_1), V(G_2)]$ and $Y[V(G_1)]$ for subgraphs $G_1, G_2$ of $G$, respectively.

With respect to a vector $Y$ which gives a valuation of vertices of $G$, a vertex $v$ is called a zero (nonzero) vertex if $Y[v] = 0$ ($Y[v] \neq 0$), a component containing a nonzero vertex is called a nonzero component. Denote by $S(G)$ and $m_G(\lambda)$ the spectrum and the multiplicity of the eigenvalue $\lambda$ of the Laplacian matrix of a graph $G$, respectively.

Let $L$ be the Laplacian matrix of a graph $G = (V, E)$ and $Y$, an eigenvector of $L$ corresponding to the eigenvalue $\lambda$. Then the eigencondition at the vertex $v$ is the equation

$$\sum_{(i, v) \in E} L[i, v]Y[i] = (\lambda - L[v, v])Y[v].$$

An $n \times n$ matrix $A$ will be called acyclic if it is symmetric and if for any mutually distinct indices $k_1, k_2, \ldots, k_s$ ($s \geq 3$) in $\{1, 2, \ldots, n\}$, the equality

$$A[k_1, k_2]A[k_2, k_3] \cdots A[k_s, k_1] = 0$$

is fulfilled. Then the Laplacian matrix of a tree is acyclic. Denote by $m^+_A(\lambda)$ (respectively, $m^-_A(\lambda)$) the number of eigenvalues of the matrix $A$ greater than (respectively, less than) $\lambda$, and let $m_A(\lambda)$ the multiplicity of $\lambda$. The following results are known from the work of Fiedler.

**Lemma 2.1.** ([4], Lemma 1.12) Let

$$A = \begin{bmatrix} B & C \\ C^T & d \end{bmatrix}$$

be a partitioned symmetric real matrix, where $C$ is a vector. If there exists a vector $U$ such that $BU = 0$ and $C^TU \neq 0$. Then

$$m^-_A(0) = m^-_B(0) + 1 \quad \text{and} \quad m^+_A(0) = m^+_B(0) + 1.$$
Lemma 2.2. ([4], Theorem 2.3) Let $A$ be an $n \times n$ acyclic matrix. Let $Y$ be an eigenvector of $A$ corresponding to an eigenvalue $\lambda$.

Let there first be no “isolated” zero coordinate of $Y$, that is coordinate $Y[k] = 0$ such that $A[k,j]Y[j] = 0$ for all $j$. Then

$$m_A^+(\lambda) = a^+ + r, \quad m_A^-(\lambda) = a^- + r,$$

where $r$ is the number of zero coordinates of $Y$, $a^+$ is the number of those unordered pairs $(i,k)$ for which

$$A[i,k]Y[i]Y[k] < 0$$

and $a^-$ is the number of those unordered pairs $(i,k)$ $(i \neq k)$, for which

$$A[i,k]Y[i]Y[k] > 0.$$

If there are isolated zero coordinates of $Y$, $M$ is the set of indices corresponding to such coordinates and $A'$ the matrix obtained from $A$ by deleting all rows and columns with indices from $M$, then the numbers $m_A^+(\lambda)$, $m_A^-(\lambda)$ and $m_A(\lambda)$ satisfy

$$m_A^+(\lambda) = m_A'(\lambda) + c_1, \quad m_A^-(\lambda) = m_A'(\lambda) + c_2, \quad m_A(\lambda) = m_A'(\lambda) + c_0,$$

where $c_1$, $c_2$ and $c_0$ are nonnegative integers such that

$$c_1 + c_2 + c_0 = |M|,$$

the number of elements in $M$.

Lemma 2.3. ([4], Corollary 2.5) Let $A$ be an $n \times n$ irreducible acyclic matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. If $\lambda_r$ corresponding to an eigenvector $Y$ with all coordinates different from zero, then $\lambda_r$ is simple and there are exactly $r - 1$ (unordered) pairs $(i,k), i \neq k$, for which

$$A[i,k]Y[i]Y[k] > 0.$$

Denote by $C_V(T,Y)$ the collection of all characteristic vertices in $C(T,Y)$ (or briefly $C_V$). From Lemma 2.3 for any $k$-vector $Y$ of a given tree $T$ on $n$ vertices, either $|C(T,Y)| = k - 1$ or $C_V(T,Y)$ contains characteristic vertices. Thus, as a consequence of Lemma 2.3 and (1.1), we have:

Corollary 2.4. Let $T$ be a tree on $n$ vertices and $Y$, a $k$-vector of $T$ with $2 \leq k \leq n$. If $|C(T,Y)| \leq k - 2$, then

$$|C_V(T,Y)| \geq 1.$$
In addition, the following two lemmas are needed for our discussion.

**Lemma 2.5.** (6, Lemma 2.2) Let $T$ be a tree on $n$ vertices with Laplacian matrix $L$. Suppose that $\lambda \in S(T)$ and $v \in V(T)$. Let also $Y$ be an eigenvector of $L$ corresponding to $\lambda$. If $v \in C(T, Y)$, then $m_{L[T-v]}(\lambda) = m_L(\lambda) + 1$.

**Lemma 2.6.** (6, Lemma 2.3) Let $T$ be a tree on $n$ vertices with Laplacian matrix $L$. Let also $\lambda \in S(T)$ and $v \in V(T)$. If $m_{L[T-v]}(\lambda) = m_L(\lambda) + 1$, then $Y[v] = 0$ for any eigenvector $Y$ of $L$ corresponding to $\lambda$.

Let $T = (V, E)$ be a tree on $n$ vertices with Laplacian matrix $L$ and let $\lambda$ be a nonzero eigenvalue of $L$. Suppose $W$ is a subset of $V$ and $T - W$ denotes the graph obtained from $T$ by deleting the vertices $W$ together with all edges incident to them. Suppose also that $M = \{T_i : i = 1, 2, \ldots, m\}$ is the collection of all components of $T - W$. According to whether or not the eigenvalue $\lambda$ is contained in $S(L[T_i])$, we partition $M$ as follows:

(a) $M_1(W; \lambda) = \{T_i : \lambda < \lambda_1(L[T_i]); T_i \in M\}$,
(b) $M_2(W; \lambda) = \{T_i : \lambda = \lambda_1(L[T_i]); T_i \in M\}$,
(c) $M_3(W; \lambda) = \{T_i : \lambda > \lambda_1(L[T_i]) \text{ and } \lambda \in S(L[T_i]); T_i \in M\}$, and
(d) $M_4(W; \lambda) = \{T_i : \lambda > \lambda_1(L[T_i]) \text{ and } \lambda \notin S(L[T_i]); T_i \in M\}$.

Let $Y$ be a $k$-vector of a tree $T$, $v \in C(T, Y)$ and $T'$ a component of $T - v$. According to whether the component $T'$ is of $M_1(v; k \lambda)$, $M_2(v; k \lambda)$, $M_3(v; k \lambda)$, or $M_4(v; k \lambda)$, we establish the following structural property for $Y[T']$.

**Lemma 2.7.** Let $T$ be a tree with Laplacian matrix $L$ and let $Y$ be a $k$-vector of $T$. Suppose $v \in C(T, Y)$ and $T'$ is a component of $T$ at $v$. Then

(a) $Y[T'] = 0$ if $T' \in M_1(v; k \lambda)$ or $T' \in M_4(v; k \lambda)$,
(b) $Y[T']$ is either zero, or positive, or negative if $T' \in M_2(v; k \lambda)$, and
(c) $Y[T']$ is either zero or non-zero containing both positive entries and negative entries if $T' \in M_3(v; k \lambda)$.

**Proof.** From Lemma 2.5 we have $m_{L[T-v]}(k \lambda) = m_L(k \lambda) + 1$ as $v \in C(T, Y)$. Thus, $Y[v] = 0$ by Lemma 2.6. Combining with the equation $(L - k \lambda I)Y = 0$, we have

$$(L[T'] - k \lambda I)Y[T'] = 0.$$  

Then part (a) holds, since in that case $\det(L[T'] - k \lambda I) \neq 0$. Note that $L[T']$ is an M-matrix, then the eigenvector corresponding to its least eigenvalue is either positive or negative (see, for instance, [1, Lemma 1]). Consequently, part (b) follows. Part
(c) follows from the Perron-Frobenius theorem and the fact that the eigenvectors corresponding to the least eigenvalue are orthogonal to the eigenvectors corresponding to each other eigenvalue.

Furthermore, if the $k$-vector $Y$ is restricted to $k$-maximal, then Lemma 2.7(c) can be strengthened as follows:

**Lemma 2.8.** Let $T$ be a tree with Laplacian matrix $L$ and let $Y$ be a $k$-maximal vector of $T$. Suppose that $v \in \mathcal{C}(T, Y)$ and $T'$ is a component of $T$ at $v$. If $T' \in M_3(v; \lambda)$, then $Y[T']$ has at least one positive and at least one negative entry.

**Proof.** Assume to the contrary that $Y[T'] = 0$ by Lemma 2.7. By the definition of the $k$-maximal vector, it is sufficient to construct a $k$-vector $W$ of $L$ such that $|\mathcal{C}(T, W)| \geq |\mathcal{C}(T, Y)| + 1$.

Firstly, we have

$$C(T, Y) = C(T - T', Y[T - T']) + C(T', Y[T']) = C(T - T', Y[T - T'])$$

(2.1) the last equation holds from $C(T', Y[T']) = 0$ as $Y[T'] = 0$ by assumption. Note that if $\lambda$ is an eigenvalue of the principle submatrix $L[T']$, then there exists a nonzero vector, say $Y'$, such that $L[T']Y' = \lambda Y'$. Since $T' \in M_3(v; \lambda)$, $\lambda$ is not the least eigenvalue of $L[T']$. Applying the Perron-Frobenius theorem again, we see that the vector $Y'$ contains both positive entries and negative entries. Therefore, $|\mathcal{C}(T', Y')| \geq 1$.

Let $v' \in T'$ adjacent to $v$. Since $v \in \mathcal{C}(T, Y)$, there exists a vertex, say $v_1$, adjacent to $v$ such that $Y[v_1] \neq 0$. Without loss of generality, suppose $Y[v_1] = Y'[v']$. (otherwise, we can replace $Y'$ by $\alpha Y'$ for some nonzero scalar $\alpha (\neq 1)$.) Let $T_1$ be the component of $T$ at $v$ containing $v_1$ and $W$ be the vector obtained from $Y'$ by replacing $Y'[T']$ and $Y[T_1]$ by $Y'$ and $\ell Y[T_1]$, respectively, in which $t = (Y[v_1] - Y'[v'])/Y[v_1]$. We can readily verify that, corresponding to the vector $W$, the vertex $v$ satisfies the eigencondition. Then $W$ is also a $k$-vector of $T$ and $|\mathcal{C}(T_1, Y[T_1])| = |\mathcal{C}(T_1, tY[T_1])|$ as $t \neq 0$. Henceforth, with respect to the $k$-vector $W$, we have

$$|\mathcal{C}(T, W)| = |\mathcal{C}(T - T', W[T - T'])| + |\mathcal{C}(T', Y[T'])|$$

$$\geq |\mathcal{C}(T - T', W[T - T'])| + 1$$

$$= |\mathcal{C}(T - T', Y[T - T'])| + 1$$

$$= |\mathcal{C}(T, Y)| + 1.$$ 

The third equality follows from (2.1).

**3. The cardinality of the characteristic set with respect to $k$-maximal vectors of a tree.** In this section, we investigate $|\mathcal{C}(T, Y)|$ for a $k$-maximal vector $Y$ and characterize the structure of trees $T$ with $|\mathcal{C}(T, Y)| = k - 1 - t$ for some $t$ with $0 \leq t \leq k - 2$. 

Denote by $A \oplus B$ the direct sum of matrices $A$ and $B$. We begin our discussion with the result which reveals the secret why the upper bound in (1.1) for the cardinality of the characteristic set $|\mathcal{C}(T, Y)|$ with respect to $k$-maximal vector $Y$ of a given tree $T$ is sometimes not sharp.

**Lemma 3.1.** Let $T$ be a tree with Laplacian matrix $L$ and let $Y$ be a $k$-maximal vector of $T$. Suppose that $v \in \mathcal{C}(T, Y)$ and $M_4(v; k \lambda) = \{ T_i : i = 1, 2, \ldots, p; p \geq 1 \}$. Let $t = \bigoplus_{i=1}^{p} m_{L[T_i]}^{(k \lambda)}$. Then

$$|\mathcal{C}(T, Y)| \leq k - 1 - t.$$

**Proof.** For convenience, let $T' = \bigcup_{i=1}^{p} T_i$. Firstly, we have

$$\mathcal{C}(T, Y) = \mathcal{C}(T - T', Y[T - T']) + \mathcal{C}(T', Y[T'])$$

as $Y[v] = 0$ by Lemma 2.6 and $Y[T'] = 0$ by Lemma 2.7. From Lemma 2.5 we have $m_{L[T - v]}^{(k \lambda)} = m_{L}^{(k \lambda)} + 1$. Therefore,

$$m_{L[T - v]}^{(k \lambda)} = m_{L}^{(k \lambda)} - 1,$$

which implies that

$$\lambda_{k-2}^{L[T - v]} < \lambda_{k-1}^{L[T - v]} = k \lambda.$$

Note that $S(L[T - v]) = S(L[T - T' - v]) \cup S(L[T'])$. Combining this with $t = \bigoplus_{i=1}^{p} m_{L[T_i]}^{(k \lambda)}$, we have

$$\lambda_{k-2-t}^{L[T - T' - v]} < \lambda_{k-1-t}^{L[T - T' - v]} = k \lambda = \lambda_{k-t}^{L[T - T' - v]}.$$

Applying Lemma 2.1 to the matrix $L[T - T']$, its principal submatrix $L[T - T' - v]$ and the vector $L[T - T', v]$, we have

$$\lambda_{k-1-t}^{L[T - T']} < k \lambda = \lambda_{k-t}^{L[T - T']}$$

i.e.,

$$m_{L[T - T']}^{(k \lambda)} = k - 1 - t.$$

Furthermore, applying Lemma 2.2 to $L[T - T']$ and $Y[T - T']$, we have $m_{L}^{(k \lambda)} = a^- + r$. One can find that $a^-$ and $r$ are exactly the number of characteristic edges and the characteristic vertices in $\mathcal{C}(T - T', Y[T - T'])$, respectively. Then

$$|\mathcal{C}(T - T', Y[T - T'])| = m_{L}^{(k \lambda)} = a^- + r \leq k - 1 - t,$$
where the last inequality follows from the fact that $L[T - T']$ has exactly

$$k - 1 - t = k - 1 - \bigoplus_{i=1}^{p} m_{L[T_i]}(k\lambda)$$

eigenvalues less than $k\lambda$. Thus, $|C(T, Y)| \leq k - 1 - t$ by (3.1), and the result follows.

Applying the method above repeatedly to every element of $C_{V}(T, Y)$, the following result can be obtained immediately.

**Theorem 3.2.** Let $T$ be a tree with its Laplacian matrix $L$ and let $Y$ be a $k$-maximal vector of $T$. Suppose that $M_{4}(C_{V}; k\lambda) = \{T_i : i = 1, 2, \ldots, p; p \geq 1\}$. Let $t = \bigoplus_{i=1}^{p} m_{L[T_i]}(k\lambda)$. Then

$$|C(T, Y)| \leq k - 1 - t.$$
Proof. Let $C_V(T, Y) = \{v_1, v_2, \ldots, v_m\}$ and $\{T_i : i = 1, 2, \ldots, p, p + 1, p + 2, \ldots, p + l\}$ be all components of $T - C_V$. Obviously $1 \leq m \leq |C(T, Y)|$. Note that $T$ contains no cycles. Thus each vertex $v_i \in C_V(T, Y)$ is adjacent to at least two nonzero components, and each pair of characteristic vertices is adjacent to at most one common nonzero component. Thus, $l \geq m + 1$. Hence, we can take $m$ mutually distinct nonzero components, say $T_1', T_2', \ldots, T_m'$, such that each component contains a vertex, labeling as $v_1', v_2', \ldots, v_m'$, respectively, such that, for each $i$, $Y[v_i'] \neq 0$ and $v_i' \in T_i'$ adjacent to $v_i$.

For each $i$ $(1 \leq i \leq m)$, from Lemmas 2.3 and 2.4 we have $Y[v_i] = 0$. Thus,

$$L[T_i']Y[T_i'] = \lambda Y[T_i'],$$

for each $i$.

Write $T - C_V(T, Y)$ as $T'$ for simplicity. Let $Y_1 = [Y[T_1']^T 0 \cdots 0]^T$, where the zeros are appended so that $(L[T'] - \lambda I)Y_1 = 0$. One can readily verify that $L[v_1, T']Y_1 = L[v_1, v_1'][Y_1[v_1']] \neq 0$, since the vector $L[v_1, T_1']$ has exactly one nonzero coordinate $L[v_1, v_1']$ and $Y_1[v_1'] \neq 0$. Thus, applying Lemma 2.1,

$$m_{L(T')}^{(k\lambda)} = m_{L(T')}(k\lambda) + 1.$$

Further, let $Y_2 = [Y[T_2']^T 0 \cdots 0]^T$, where the zeros are appended so that $(L[T' \cup \{v_1\}] - \lambda I)Y_2 = 0$. Thus, by a similar discussion, $L[v_2, T' \cup \{v_1\}]Y_2 = L[v_1, T_2']Y[T_2'] = L[v_2, v_2'][Y_2[v_2']] \neq 0$. Applying Lemma 2.1 again, we have

$$m_{L(T' \cup \{v_1, v_2\})}^{(k\lambda)} = m_{L(T' \cup \{v_1\})}^{(k\lambda)} + 1 = m_{L(T')}(k\lambda) + 2.$$

Using the above operation repeatedly, we have

$$m_L^{(k\lambda)} = m_{L(T' \cup C_V)}^{(k\lambda)} = m_{L[T' \cup \{v_1, \ldots, v_{m-1}\}]}^{(k\lambda)} + 1 \ldots = m_{L[T']}^{(k\lambda)} + m - 1 = m_{L[T']}^{(k\lambda)} + m.$$

Thus,

$$m_{L[T']}^{(k\lambda)} = m_L^{(k\lambda)} - m.$$

Consequently,

$$\lambda_{k-1-m}(L[T']) < k \lambda \text{ and } \lambda_{k-m}(L[T']) = k \lambda,$$
since $k\lambda$ is an eigenvalue of $L[T']$ by Lemma 2.4.

Without loss of generality, suppose $M_2(C_V;k\lambda) = \{T_1^*, T_2^*, \ldots, T_q^*\}$. For each $i = 1, 2, \ldots, q$, let $t_i = m_{L[T_i^*]}(k\lambda)$. From Lemma 3.3, $|C(T_i^*, Y[T_i^*])| = m_{L[T_i^*]}(k\lambda) = t_i$ holds for each $i$. Then
\[
\sum_{i=1}^{q} |C(T_i^*, Y[T_i^*])| = \sum_{i=1}^{q} t_i =: t^*.
\]

Hence,
\[
|C(T, Y)| = |C(T - C_V, Y[T - C_V])| + \sum_{i=1}^{q} |C(T_i, Y[T_i])| = \sum_{i=1}^{q} t_i = t^* + m.
\]

On the other hand, note that $S(T') = \bigcup_{i=1}^{p+q} S(T_i)$ and each eigenvalue corresponding to the component being of $M_1(C_V;k\lambda)$ or $M_2(C_V;k\lambda)$ is no less than $k\lambda$, then $t + t^* = k - 1 - m$. Hence,
\[
|C(T, Y)| = t^* + m = k - 1 - t. \quad \Box
\]

Putting Theorem 3.4 together with Lemma 2.3, we can give the characterization for the structure of the trees with any possible cardinality of the characteristic set with respect to its $k$-maximal vector.

**Theorem 3.5.** Let $T$ be a tree with Laplacian matrix $L$, and let $Y$ be an arbitrary $k$-maximal vector of $T$. Then $|C(T, Y)| = k - 1$ if and only if every coordinate of $Y$ different from zero, or $M_2(C_V;k\lambda) = \emptyset$ holds for each $v \in C(T, Y)$.

**Theorem 3.6.** Let $T$ be a tree with Laplacian matrix $L$, and let $Y$ be an arbitrary $k$-maximal vector of $T$. Suppose $t$ is an arbitrary integer with $1 \leq t \leq k - 2$. Then $|C(T, Y)| = k - 1 - t$ if and only if $C_V(T, Y) \neq \emptyset$ and $M_4(C_V;k\lambda) = \{T_i : i = 1, 2, \ldots, p(p \geq 1)\}$, where $t = \sum_{i=1}^{p} m_{L[T_i]}(k\lambda)$.

Below we give an example to show the occurrence of each of the case described in the above theorems.

**Example 3.7.** Let $T$ be a tree on $n = 2m + 4p + q + 1$ ($p + q \geq 2$) vertices obtained from a star on $m + p + q + 1$ vertices by appending $m$ pendent edges to $m$ pendent vertices and $p$ paths with length 3 to other $p$ pendent vertices, respectively, see Figure 3.1.

By a little calculation, we have $k\lambda(T) = 1$ with multiplicity $p+q-1$, $M_1(u; 1) = \emptyset$, $M_2(u; 1) = \{T[v_{4p+2m+1}] : i = 1, 2, \ldots, q\}$, $M_4(u; 1) = \{T[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}] : i = \}$
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1, 2, \ldots, p$, and $M_4(u; 1) = \{ T[v_{4p+i}, v_{4p+m+i}] : i = 1, 2, \ldots, m \}$, where $k = m + p + 2$ and $T[S]$ is the subgraph of $T$ induced by its vertex subset $S$. Then the $k$-vector $Y$ has the following partitioned form:

$$Y = [t_1 X_1^T \ t_2 X_2^T \ \cdots \ \ t_p X_p^T \ W_1^T \ \cdots \ W_m^T \ s_1 \ \cdots \ s_q \ 0],$$

where $X_i = [-1 \ -1 \ 0 \ 1]^T$ for $i = 1, 2, \ldots, p$, $W_l = [0 \ 0]^T$ for $l = 1, 2, \ldots, m$, and each $t_i$ (or $s_j$) is real such that the vertex $u$ satisfies eigencondition.

One can see that, for each $k$-vector $Y$, $t_i X_i$ is either zero or non-zero containing both positive entries and negative entries for each $i$, $s_j$ is either zero, or positive, or negative for each $j$, and $W_l = 0$ for each $l$, which is consistent with Lemma 2.7. On the other hand, from the partitioned form of $Y$, $u \in \mathcal{C}(T, Y)$ if some $t_i$ (or $s_j$) is nonzero. Moreover, we have $|\mathcal{C}(T[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}], Y[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}])| = |\mathcal{C}(T[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}], t_i X_i)|$ is either 1 or 0 according to the real $t_i$ is nonzero or not. Thus,

$$|\mathcal{C}(T, Y)| = 1 + \sum_{i=1}^{p} |\mathcal{C}(L[T[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}], Y[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}])| \leq 1 + p,$$

from which we have that $Y$ is $k$-maximal if $t_i \neq 0$ for each $i$, which is consistent with Lemma 2.8. One can also find that, with respect to any $k$-maximal vector $Y$, $|\mathcal{C}(T, Y)| = 1 + p$ regardless of the choice of the integer $m$. Hence, $|\mathcal{C}(T, Y)| = 1 + p = k - 1$ if $m = 0$ (in such a case $M_4(u; 1) = \emptyset$), and $|\mathcal{C}(T, Y)| = 1 + p \leq k - 2$ otherwise, which is consistent with Lemma 3.1

Moreover, if $Y$ is $k$-maximal, then $C_Y(T, Y) = \{u\} \cup \{v_{2p+i} : i = 1, 2, \ldots, p\}$, and $M_1(C_Y; 1) = M_3(C_Y; 1) = \emptyset$, $M_2(C_Y; 1) = \{ T[v_{4p+2m+i}] : i = 1, 2, \ldots, q \} \cup \{ T[v_{3p+j}] : j = 1, 2, \ldots, p \} \cup \{ T[u, v_{p+l}] : l = 1, 2, \ldots, p \}$, $M_4(C_Y; 1) = \{ T[v_{4p+i}, v_{4p+m+i}] : i = 1, 2, \ldots, m \}$, which is consistent with Theorems 3.4, 3.5 and 3.6.
REFERENCES