Positive and completely positive cones and Z-transformations

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ON COPOSITIVE AND COMPLETELY POSITIVE CONES, AND Z-TRANSFORMATIONS

M. SEETHARAMA GOWDA†

Abstract. A well-known result of Lyapunov on continuous linear systems asserts that a real square matrix $A$ is positive stable if and only if for some symmetric positive definite matrix $X$, $AX + XA^T$ is also positive definite. A recent result of Moldovan-Gowda says that a $Z$-matrix $A$ is positive stable if and only if for some symmetric strictly copositive matrix $X$, $AX + XA^T$ is also strictly copositive. In this paper, these results are unified/extended by replacing $\mathbb{R}^n$ and $\mathbb{R}_+^n$ by a closed convex cone $C$ satisfying $C - C = \mathbb{R}^n$. This is achieved by relating the $Z$-property of a matrix on this cone with the $Z$-property of the corresponding Lyapunov transformation $L_A(X) := AX + XA^T$ on the completely positive cone of $C$ and the $Z$-property of $L_A^T$ on the copositive cone of $C$ in $\mathbb{S}^n$ (the space of all real $n \times n$ symmetric matrices). A similar analysis is carried out for the Stein transformation $S_A(X) = X - AXA^T$.

Key words. Copositive matrix, Copositive and completely positive cones, $Z$-transformation, Lyapunov and Stein transformations.

AMS subject classifications. 15A48, 34A30, 34D23, 17B45.

1. Introduction. Given a closed convex cone $K$ in a real finite dimensional Hilbert space $(H, \langle \cdot, \cdot \rangle)$, and a linear transformation $L$ on $H$, we say that $L$ has the $Z$-property on $K$ (or that it is a $Z$-transformation on $K$) and write $L \in Z(K)$ if

$$[x \in K, y \in K^*, \text{ and } \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0, \tag{1.1}$$

where $K^*$ denotes the dual cone of $K$. As a generalization of a $Z$-matrix (which is a real square matrix with nonpositive off-diagonal entries), such transformations were introduced in [19] in the form of cross-positive matrices. $Z$-matrices/transformations have numerous properties and appear in many areas, e.g., see [3], [13]. Our motivation for this article comes from dynamical systems. Consider $\mathbb{S}^n$, the space of all $n \times n$ real symmetric matrices, with the inner product $\langle X, Y \rangle = \text{trace}(XY)$ and the cone $\mathbb{S}^n_+$ of all positive semidefinite matrices in $\mathbb{S}^n$. Then for any matrix $A \in \mathbb{R}^{n \times n}$, the Lyapunov transformation $L_A$ and Stein transformation $S_A$, respectively defined on $\mathbb{S}^n$ by

$$L_A(X) := AX + XA^T \quad \text{and} \quad S_A(X) := X - AXA^T,$$

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Copositive and Completely Positive Cones

are $Z$-transformations on $S^n_+$ [13]. These transformations have been well studied in dynamical systems theory, starting from Lyapunov’s paper [10] on continuous dynamical systems and Stein’s paper [22] on discrete dynamical systems. The celebrated result of Lyapunov deals with the stability of the linear system $\dot{x} + Ax = 0$, and, in particular with the equivalence of the following statements [7]:

(i) The system $\dot{x} + Ax = 0$ is asymptotically stable in $\mathbb{R}^n$ (which means that the trajectory of the system from any starting point in $\mathbb{R}^n$ converges to the origin as $t \to \infty$).

(ii) $A$ is positive stable (that is, all eigenvalues of $A$ lie in the open right-half plane).

(iii) There exists $X \in S^n$ such that $X$ and $L_A(X)$ are positive definite.

(iv) For every positive definite $Y \in S^n$, the equation $L_A(X) = Y$ has a (unique) positive definite solution $X$ in $S^n$.

For a discrete system of the form $x(k+1) = Ax(k)$, $k = 1, 2, \ldots$, similar equivalent statements can be made by replacing the positive stability of $A$ with Schur stability of $A$ (which means that all eigenvalues of $A$ lie in the open unit disk) and $L_A$ by $S_A$.

Now consider a linear system $\dot{x} + Ax = 0$ whose trajectories are constrained to lie in the nonnegative orthant $\mathbb{R}^n_+$. It is well known that this can happen if and only if $A$ is a $Z$-matrix. Analogous to Lyapunov’s result, we have the equivalence of the following when $A$ is a $Z$-matrix:

(i) The system $\dot{x} + Ax = 0$ is asymptotically stable in $\mathbb{R}^n_+$.

(ii) $A$ is positive stable.

(iii) There exists $X \in S^n$ such that $X$ and $L_A(X)$ are strictly copositive on $\mathbb{R}^n_+$.

(iv) For every $Y \in S^n$ that is strictly copositive on $\mathbb{R}^n_+$, the equation $L_A(X) = Y$ has a (unique) strictly copositive solution $X$ in $S^n$.

(v) There exists a vector $d > 0$ (i.e., $d$ belongs to the interior of $\mathbb{R}^n_+$) such that $Ad > 0$.

Here, the strict copositivity (copositivity) of $X$ on $\mathbb{R}^n_+$ is defined by: $x^T X x > 0$ ($\geq 0$), for all $0 \neq x \in \mathbb{R}^n_+$. The equivalence of Items (i), (ii), and (v) is well known in the literature, e.g., see [17]. The new items (iii) and (iv) were proved by Moldovan and Gowda [18] by relying on the equivalence of the following statements for any $A \in \mathbb{R}^{n \times n}$:

(a) $A$ is a $Z$-matrix.

(b) $L_A$ has the $Z$-property on the cone of completely positive matrices in $S^n$.

(c) $L_{AT}$ has the $Z$-property on the cone of copositive matrices in $S^n$.

In a recent article [5], Bundfuss and Dür raise the question of studying the dynamics of $\dot{x} + Ax = 0$ which is constrained to a (polyhedral) cone $K$ by asking for the existence
of a symmetric matrix $X$ that is strictly copositive on $K$ for which $AX + XA^T$ is also strictly copositive on $K$. Motivated by the similarities in the above results of Lyapunov and Moldovan-Gowda, and the question of Bundfuss and Dür, in this paper, we present a unifying result (Theorem 3.8) by relating the Z-property of a matrix $A$ on a closed convex cone in $\mathbb{R}^n$ with the Z-property of $L_A$ ($L_A^T$) on the corresponding completely positive cone (respectively, copositive cone) in $S^n$.

Consider $\mathbb{R}^n$ with the usual inner product. Given a closed convex cone $C$ in $\mathbb{R}^n$ with dual $C^*$, we consider two related cones in $S^n$: The copositive cone of $C$ defined by

$$E = \text{copos}(C) := \{ A \in S^n : A \text{ copositive on } C \}$$

and the completely positive cone of $C$ defined by

$$K = \text{compos}(C) := \{ BB^T : \text{columns of } B \text{ belong to } C \}.$$ 

When $C = \mathbb{R}^n$, these two cones reduce to $S^n_+$ which is the underlying cone in semidefinite programming and semidefinite linear complementarity problems [1], [10], [11]. In the case of $C = \mathbb{R}^+_n$, these cones reduce, respectively, to the cones of copositive matrices and completely positive matrices which have appeared prominently in statistical and graph theoretic literature [4] and (recently) in the study of (combinatorial) optimization problems [6], [8].

With the notation $L \in Z(K)$ to mean that the transformation $L$ has the Z-property on $K$ and $L \in \Pi(K)$ to mean that $L(K) \subseteq K$, we show in this article (see Theorems 3.3 and 5.1) that

$$A \in Z(C) \Rightarrow L_A \in Z(K) \iff L_A^T \in Z(E) \quad \text{and}$$

$$A \in \Pi(C) \Rightarrow S_A \in Z(K) \iff S_A^T \in Z(E).$$

These results, along with the properties of Z-transformations, will allow us to extend the results of Lyapunov and Moldovan-Gowda, and (partially) answer the question of Bundfuss and Dür.

Here is an outline of the paper. Section 2 deals with the preliminaries. The Z-property of $L_A$ is covered in Section 3 and that of $S_A$ is covered in Section 5. In Section 4, we study Lyapunov-like transformations. Finally, Section 6 deals with some results relating Z-property, cone spectrum, and copositivity.

2. Preliminaries. Throughout this paper, $H$ denotes a finite dimensional real Hilbert space with inner product given by $\langle x, y \rangle$. For a set $K$ in $H$, $K^\circ$ and $K^\perp$ denote, respectively, the interior and orthogonal complement of $K$. For a closed convex cone $K$ in $H$ the dual is given by

$$K^* := \{ y \in H : \langle y, x \rangle \geq 0 \ \forall x \in K \}.$$
We use the notation \( K \ni x \perp y \in K^* \) to mean that \( x \in K \), \( y \in K^* \), and \( \langle x, y \rangle = 0 \).

Recall \[3\] that a closed convex cone \( K \) in \( H \) is a proper cone if \( K \) is reproducing (that is, \( K - K = H \)) and pointed (that is, \( K \cap -K = \{0\} \)) (or equivalently, \( K \) and \( K^* \) have nonempty interiors \[3\]).

For a linear transformation \( L \) on \( H \), \( L^* \) denotes its adjoint. It said to be

- **copositive on \( K \)** (strictly copositive on \( K \)) if \( \langle L(x), x \rangle \geq 0 \) (\( > 0 \)) for all \( 0 \neq x \in K \);
- **monotone** if \( \langle L(x), x \rangle \geq 0 \) for all \( x \in H \);
- **positive stable** (Schur stable) if all the eigenvalues of \( L \) lie in the open right-half plane (respectively, in the open unit disk).

In the space \( H = \mathbb{R}^n \), vectors are written as column vectors and the usual inner product is written as \( \langle x, y \rangle \) or as \( x^T y \). Following standard terminology,

- **Copositive matrices** (positive semidefinite matrices) are those which are copositive on \( \mathbb{R}^n_+ \) (respectively, on \( \mathbb{R}^n \));
- **Completely positive matrices** are of the form \( BB^T \) with columns of \( B \) coming from \( \mathbb{R}^n_+ \).

Throughout this paper, \( K \) denotes a closed convex cone in \( H \) and \( C \) denotes a closed convex cone in \( \mathbb{R}^n \). Corresponding to \( C \), the copositive cone \( E \) and the completely positive cone \( K \) in \( S^n \) are defined, respectively, by (1.2) and (1.3).

**Proposition 2.1.** Let \( L \) be a self-adjoint linear transformation on \( H \) that is copositive on \( K \). Then

\[ x \in K, \langle L(x), x \rangle = 0 \Rightarrow L(x) \in K^*. \]

**Proof.** Suppose \( x \in K \) and \( \langle L(x), x \rangle = 0 \). Then for any \( y \in K \),

\[ 0 \leq \lim_{t \downarrow 0} \frac{1}{t} \langle L(x + ty), x + ty \rangle = 2\langle L(x), y \rangle. \]

This shows that \( L(x) \in K^* \). \( \square \)

**Proposition 2.2.** The following statements hold:

(i) \( E \) is a closed convex cone in \( S^n \) and \( K \subseteq S^n_+ \subseteq E \).

(ii) \( K \) is a closed convex cone; moreover, \( K \) is the dual of \( E \).

(iii) \( E \) (likewise, \( K \)) is proper if and only if \( C - C = \mathbb{R}^n \).
Proof. Item (i) is obvious and Item (ii) is well known, see [24]. We prove Item (iii). As $S^n_+ \subseteq \mathcal{E}$ and $S^n_+ - S^n_+ = S^n$, we have $\mathcal{E} - \mathcal{E} = S^n$. So, to see (iii), it is enough to show that $\mathcal{E} \cap -\mathcal{E} = \{0\}$ if and only if $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$.

Now, let $A \in S^n$. By an application of Proposition 2.1 with $L = A$ and $K = \mathcal{C}$, we have

$$A \in \mathcal{E} \cap -\mathcal{E} \iff x^T Ax = 0 \forall x \in \mathcal{C}$$

$$\iff -Ax, Ax \in \mathcal{C}^* \forall x \in \mathcal{C}$$

$$\iff A(\mathcal{C}) \subseteq \mathcal{C}^* \cap -\mathcal{C}^* = (\mathcal{C} - \mathcal{C})^\perp$$

$$\iff A(\mathcal{C} - \mathcal{C}) \subseteq (\mathcal{C} - \mathcal{C})^\perp.$$ 

Hence, when $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$, we have $A = 0$ for any $A \in \mathcal{E} \cap -\mathcal{E}$. On the other hand, when $\mathcal{C} - \mathcal{C} \neq \mathbb{R}^n$, (as $\mathcal{C} - \mathcal{C}$ is a subspace) there exists $0 \neq v \in (\mathcal{C} - \mathcal{C})^\perp$. Then $x^T (vv^T)x = 0$ for all $x \in \mathcal{C}$ and so $0 \neq A = vv^T \in \mathcal{E} \cap -\mathcal{E}$.

Finally, $\mathcal{E}$ is proper if and only if its dual $\mathcal{K}$ is proper. Thus, $\mathcal{K}$ is proper if and only if $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$.

3. The Z-property and Lyapunov transformations. The Z-property of a matrix or a linear transformation with respect to a cone is defined by [11]. The following result shows the importance of studying this property in dynamical systems.

Proposition 3.1. ([9], [19]) Suppose $L$ is a linear transformation on $H$ and $K$ be a proper cone in $H$. Then the following are equivalent:

(a) $L \in \mathcal{Z}(K)$.
(b) $e^{-tL}(K) \subseteq K$ for all $t \geq 0$ in $\mathbb{R}$.
(c) The trajectory of the dynamical system $\dot{x} + L(x) = 0$ with any initial point in $K$ stays in $K$.

As noted in the Introduction, when $\mathcal{C} = \mathbb{R}^n_+$, $A \in \mathcal{Z}(\mathcal{C})$ if and only if all the off-diagonal entries of $A$ are nonpositive. Here is a non-trivial example.

Example 3.2. Consider $\mathbb{R}^n$ with $n > 1$ and write any element in the form $x = [t, u^T]^T$, where $t \in \mathbb{R}$ and $u \in \mathbb{R}^{n-1}$. Let

$$\mathcal{C} = \mathcal{L}^n_+ := \left\{ x = \begin{bmatrix} t \\ u \end{bmatrix} : t \geq ||u|| \right\}.$$ 

This is a symmetric cone (that is, a self-dual, homogeneous, closed convex cone) in the Jordan spin algebra $\mathcal{L}^n$, called the Lorentz cone (or the second order cone or the ice-cream cone). For this (proper) cone, the copositive cone $\mathcal{E}$, the completely positive cone $\mathcal{K}$, and $\mathcal{Z}(\mathcal{L}^n_+)$ are described below.

Let $J := \text{diag}(1, -1, -1, \ldots, -1) \in \mathbb{R}^{n \times n}$ and $A \in \mathbb{R}^{n \times n}$. Then
(i) $A \in \mathcal{E}$ if and only if $A - \mu J$ is positive semidefinite for some $\mu \geq 0$, see \cite{15}, Lemma 2.2;

(ii) $A \in \mathcal{K}$ if and only if $A$ is a (finite) sum of matrices of the form $\begin{bmatrix} t^2 & tu^T \\ tu & uu^T \end{bmatrix}$, where $t \in \mathbb{R}$, $u \in \mathbb{R}^{n-1}$ with $t \geq ||u||$;

(iii) $A \in \mathcal{Z}(\mathcal{L}_+^n)$ if and only if $\alpha J - (JA + A^T J)$ is positive semidefinite for some $\alpha \in \mathbb{R}$, see Example 4 in \cite{13};

(iv) $A, -A \in \mathcal{Z}(\mathcal{L}^n_+)$ if and only if $JA + A^T J = \alpha J$ for some $\alpha \in \mathbb{R}$.

Note: Item (ii) follows from the definition and Item (iv) is a simple consequence of (iii).

We now come to one of the main results of the paper. Before stating this, we observe that for any $A \in \mathbb{R}^{n \times n}$,

$$L_A \in \mathcal{Z}(\mathcal{K}) \Leftrightarrow L_{A^T} \in \mathcal{Z}(\mathcal{E}).$$

This follows easily as $(L_A)^* = L_{A^T}$ and $\mathcal{E}^* = \mathcal{K}$ in $\mathcal{S}^n$.

**Theorem 3.3.** For any closed convex cone $\mathcal{C}$ in $\mathbb{R}^n$,

$$A \in \mathcal{Z}(\mathcal{C}) \Rightarrow L_A \in \mathcal{Z}(\mathcal{K}).$$

The reverse implication holds under the following condition on $\mathcal{C}$:

$$C \ni u \perp v \in C^*, \ u \neq 0 \Rightarrow \exists \ Y \in \mathcal{E} \text{ such that } Y u = v. \quad (3.1)$$

**Proof.** Let $A \in \mathcal{Z}(\mathcal{C})$ and $\mathcal{K} \ni X \perp Y \in K^* = \mathcal{E}$. Writing $X = \sum_1^N u_i u_i^T$, with $u_i \in \mathcal{C}$ for all $i$, we have

$$0 = \langle X, Y \rangle = \text{trace}(XY) = \sum_1^N u_i^T Y u_i.$$

This implies, as $Y$ is copositive on $\mathcal{C}$, $u_i^T Y u_i = 0$ for all $i$. From Proposition 2.1, $v_i := Y u_i \in \mathcal{C}^*$. So, for all $i$, $C \ni u_i \perp v_i \in C^*$. As $A \in \mathcal{Z}(\mathcal{C})$, $v_i^T A u_i = \langle Au_i, v_i \rangle \leq 0$ for all $i$. Now,

$$\langle L_A(X), Y \rangle = 2 \text{trace}(AXY) = 2 \sum_1^N \text{trace}(Au_i u_i^T Y) = 2 \sum_1^N v_i^T A u_i \leq 0.$$

Thus, $L_A \in \mathcal{Z}(\mathcal{K})$.

Now to see the reverse implication, assume that $\mathcal{C}$ satisfies (3.1), $L_A \in \mathcal{Z}(\mathcal{K})$, and let $u \in \mathcal{C}$, $v \in \mathcal{C}^*$ and $\langle u, v \rangle = 0$. We have to show that $\langle Au, v \rangle \leq 0$. We may assume
without loss of generality, that \( u \) is nonzero. Then there exists a \( Y \in \mathcal{E} \) such that
\[ Y u = v. \]
We have
\[ X = uu^T \in \mathcal{K}, \quad Y \in \mathcal{K}^* = \mathcal{E}, \quad \langle X, Y \rangle = u^T Y u = u^T v = 0. \]
Hence \( \text{trace}(L_A(X)Y) \leq 0 \). This leads to \( \text{trace}(AXY) \leq 0 \) and \( \langle Au, v \rangle = v^T Au = \text{trace}(AXY) \leq 0 \).

**Example 3.4.** When \( \mathcal{C} = \mathbb{R}^{n} \), we have \( \mathcal{C}^* = \{0\} \) and \( \mathcal{K} = S^n_+ \). In this case, every matrix \( A \in \mathbb{R}^{n \times n} \) belongs to \( \mathcal{Z}(\mathcal{C}) \) and consequently, for any \( A \in \mathbb{R}^{n \times n} \), both \( L_A \) and \( -L_A = L_{-A} \) belong to \( \mathcal{Z}(S^n_+) \).

**Corollary 3.6.** Suppose \( \mathcal{C} \) is a closed convex pointed cone in \( \mathbb{R}^{n} \). Then
\[ A \in \mathcal{Z}(\mathcal{C}) \Leftrightarrow L_A \in \mathcal{Z}(\mathcal{K}). \]

**Proof.** We show that the given \( \mathcal{C} \) satisfies condition \( [3, 1] \) and quote the previous theorem. To this end, let \( \mathcal{C} \ni u \perp v \in \mathcal{C}^* \), \( u \neq 0 \). Since \( \mathcal{C} \) is pointed, \( \mathcal{C}^* \) has nonempty interior. Let \( w \in (\mathcal{C}^*)^o \) such that \( w^T u = 1 \). Define \( Y := vw^T + wv^T \). Clearly \( Y \in S^n \) and for all \( x \in \mathcal{C} \),
\[ x^T Y x = x^T v \ w^T x + x^T w \ v^T x \geq 0; \]
thus, \( Y \in \mathcal{E} \). Also, \( Y u = v w^T u + w v^T u = v \).}

**Example 3.5.** In \( \mathbb{R}^{2} \), let \( \mathcal{C} \) be the closed upper half-plane. In this case, \( \mathcal{C}^* \) is the nonnegative \( y \)-axis and \( \mathcal{E} = S^2_+ \); hence \( \mathcal{K} = S^2_+ \). Now consider a matrix \( A \in \mathbb{R}^{2 \times 2} \) whose \( (2, 1) \) entry is one. Then for the standard coordinate vectors \( e_1 \) and \( e_2 \), we have \( \mathcal{C} \ni e_1 \perp e_2 \in \mathcal{C}^* \). However, \( \langle Ae_1, e_2 \rangle = 1 \). Therefore, \( A \notin \mathcal{Z}(\mathcal{C}) \) while \( L_A \in \mathcal{Z}(\mathcal{K}) \).

**Proposition 3.7.** ([3], [13]) Suppose \( L \) is a linear transformation on \( H, \mathcal{K} \) a proper cone in \( H \), and \( L \in \mathcal{Z}(\mathcal{K}) \). Then the following are equivalent:

1. There exists \( d \in \mathcal{K}^o \) such that \( L(d) \in \mathcal{K}^o \).
2. \( L \) is invertible with \( L^{-1}(\mathcal{K}^o) \subseteq \mathcal{K}^o \).
(3) $L$ is positive stable.
(4) $L + tI$ is invertible for all $t \in [0, \infty)$.
(5) All real eigenvalues of $L$ are positive.
(6) There exists $e \in (K^\circ)^*$ such that $L^*(e) \in (K^\circ)^\circ$.

Moreover, when $H = \mathbb{R}^n$, $K = \mathbb{R}_+^n$, and $L = A$, the above properties (for a $Z$-matrix $A$) are further equivalent to

(7) $A$ is a $P$-matrix, that is, all principal minors of $A$ are positive.
(8) There exists a positive definite diagonal matrix $D$ in $S^n$ such that $AD+DA^T$ is positive definite.

As a consequence, we have the following.

**Theorem 3.8.** Suppose $C$ is a closed convex cone in $\mathbb{R}^n$ such that $C - C = \mathbb{R}^n$. Then the following are equivalent:

(a) $A$ is positive stable.
(b) The system $\dot{x} + Ax = 0$ is asymptotically stable in $C$ (that is, from any starting point in $C$, its trajectory converges to the origin).

When $A \in Z(C)$, these are further equivalent to:

(c) There exists $D \in K^\circ$ such that $AD+DA^T \in K^\circ$.
(d) There exists $D \in E^\circ$ such that $A^T D + DA \in E^\circ$.

If, in addition, $C$ is also proper, then the above conditions are equivalent to

(e) There exists $d \in C^\circ$ such that $Ad \in C^\circ$.

**Proof.** The proof of (a) $\Rightarrow$ (b) is standard, see the proof of Theorem 3.1 in [7]. The proof of (b) $\Rightarrow$ (a) is as in [7], except that the starting point should be allowed to vary in the interior of $C$ (which is nonempty because $C - C = \mathbb{R}^n$).

Now assume that $A \in Z(C)$. Since $C - C = \mathbb{R}^n$, by Proposition 2.2 both $E$ and $K$ are proper; hence they have nonempty interiors. Also, since $A \in Z(C)$, by Theorem 3.3 $L_A \in Z(K)$ and $L_A^T \in Z(E)$. Since the eigenvalues of $L_A$ on $\mathcal{L}$ are of the form $\lambda + \mu$, where $\lambda$ and $\mu$ are eigenvalues of $A$, see [26], it follows that $A$ is positive stable if and only if $L_A$ is positive stable. Now the equivalence of Items (a), (c), and (d) follows from the previous result applied to $L_A$ on $K$.

When $C$ is proper, the previous result can be applied to $A$ and $C$ (note that $Z(C)$) to get the equivalence of (a) and (e).

**Remark 3.9.** (i) In Item (d) of the above theorem, the matrix $D \in E^\circ$ is necessarily strictly copositive on $C$: For any nonzero $u \in C$, $u^T(D-\varepsilon I)u \geq 0$ for small $\varepsilon > 0$. 

When \( C - C = \mathbb{R}^n \) and \( A \in Z(C) \), the equation
\[
A^T X + X A = Y, \quad Y \text{ strictly copositive on } C
\]
has a unique solution \( X \) which is also strictly copositive on \( C \) for some \( Y \) (equivalently for all \( Y \)) if and only if \( A \) is positive stable. This follows from Items (1) and (2) in Proposition \ref{prop:copositive} with \( K = \mathcal{E} \) and \( L = L_A^T \). When \( C \) is proper and \( A \) is positive stable, this unique solution is given by
\[
X = \int_0^\infty e^{-tA^T} Y e^{-tA} dt.
\]

(iii) The results of Lyapunov and Moldovan-Gowda (stated in the Introduction) follow by taking \( C = \mathbb{R}^n \) and \( C = \mathbb{R}^n_+ \) respectively.

(iv) Suppose \( C \) is proper. When the conditions of the above result are in place, (any) trajectory of the system \( \dot{x} + Ax = 0 \) from any starting point in \( C \) stays in \( C \) and converges to the origin as \( t \to \infty \). In this setting, \( f(x) := d^T x \) (with \( d \) as in Item (e)) acts as a linear Lyapunov function and \( g(x) := x^T D x \) (with \( D \) as in Item (d)) acts as a quadratic Lyapunov function.

(v) Instead of our condition \( C - C = \mathbb{R}^n \) in Theorem \ref{thm:copositive}, Stern \cite{Stern} assumes that \( C \) in \( \mathbb{R}^n \) satisfies \( C \cap -C = \{0\} \). He proves that when \( A \in Z(C) \) and \( C \cap -C = \{0\} \), the system \( \dot{x} + Ax = 0 \) is asymptotically stable if and only if the following implication holds:
\[
[x \in C, \quad -Ax \in C] \Rightarrow x = 0.
\]
It may be noted that if, in addition, \( C - C = \mathbb{R}^n \), that is, if \( C \) is proper, then the above condition is equivalent to Item (e) in Theorem \ref{thm:copositive}.

The following result (partially) answers a question of Bundfuss and Dür \cite{BundfussDuer}:

**Corollary 3.10.** Suppose \( C = M(\mathbb{R}^n_m) \) is a polyhedral cone in \( \mathbb{R}^n \), where \( M \) is an \( n \times m \)-matrix. Assume that \( M \) has rank \( n \) and \( A \in Z(C) \). Then there exists a symmetric matrix \( D \) such that \( D \) and \( A^T D + DA \) are strictly copositive on \( C \) if and only if \( A \) is positive stable.

4. **Lyapunov-like transformations.** Motivated by Example \ref{ex:lyapunov-like}, a linear transformation \( L \) on \( H \) is said to be *Lyapunov-like* with respect to a closed convex cone \( K \) in \( H \) if both \( L \) and \( -L \) have the \( Z \)-property on \( K \). This simply means that
\[
K \ni x \perp y \in K^* \Rightarrow \langle L(x), y \rangle = 0.
\]

For any matrix \( A \in \mathbb{R}^{n \times n} \), the Lyapunov transformation \( L_A \) is Lyapunov-like with respect to \( S_+^n \) in \( S^n \) (see Example \ref{ex:lyapunov-like}). In the setting of the cone \( \mathbb{R}_+^n \) in \( \mathbb{R}^n \), Lyapunov-
like matrices are just diagonal matrices. Because of Proposition 3.1, Lyapunov-like transformations are intimately connected to automorphism groups and Lie algebras.

In the rest of this section, we assume that \( \mathcal{C} \) is a proper cone in \( \mathbb{R}^n \) and use the notation \( \mathcal{B}(\mathcal{S}^n, \mathcal{S}^n) \) to denote the set of all (bounded) linear transformations on \( \mathcal{S}^n \).

We consider two automorphism groups:

- \( \text{Aut}(\mathcal{C}) := \{ A \in \mathbb{R}^{n \times n} : A(\mathcal{C}) = \mathcal{C} \} \).
- \( \text{Aut}(\mathcal{K}) := \{ L \in \mathcal{B}(\mathcal{S}^n, \mathcal{S}^n) : L(\mathcal{K}) = \mathcal{K} \} \).

(Note that elements of these groups are necessarily invertible, as \( \mathcal{C} \) and \( \mathcal{K} \) have nonempty interiors.) Since these groups can be regarded as matrix groups, the corresponding Lie algebras are given, see [2], by:

- \( \text{Lie}(\text{Aut}(\mathcal{C})) := \{ A \in \mathbb{R}^{n \times n} : e^{tA} \in \text{Aut}(\mathcal{C}) \forall t \in \mathbb{R} \} \).
- \( \text{Lie}(\text{Aut}(\mathcal{K})) := \{ L \in \mathcal{B}(\mathcal{S}^n, \mathcal{S}^n) : e^{tL} \in \text{Aut}(\mathcal{K}) \forall t \in \mathbb{R} \} \).

Note that in these Lie algebras, the Lie bracket is the one induced by the (associative) product of matrices/transformations: \( [A, B] = AB - BA \), etc.

In view of Proposition 3.1 we have

\[ A, -A \in \mathcal{Z}(\mathcal{C}) \iff A \in \text{Lie}(\text{Aut}(\mathcal{C})) \quad \text{and} \quad L, -L \in \mathcal{Z}(\mathcal{K}) \iff L \in \text{Lie}(\text{Aut}(\mathcal{K})). \]

**THEOREM 4.1.** For any proper cone \( \mathcal{C} \) in \( \mathbb{R}^n \), the mapping \( A \mapsto L_A \) is an injective Lie algebra homomorphism from \( \text{Lie}(\text{Aut}(\mathcal{C})) \) to \( \text{Lie}(\text{Aut}(\mathcal{K})) \).

**Proof.** For \( A \in \text{Lie}(\text{Aut}(\mathcal{C})) \), we have \( A, -A \in \mathcal{Z}(\mathcal{C}) \). By Theorem 3.3, \( L_A, -L_A \in \mathcal{Z}(\mathcal{K}) \), that is, \( L_A \in \text{Lie}(\text{Aut}(\mathcal{K})) \). Clearly, the mapping \( A \mapsto L_A \) is linear. That it is a Lie algebra homomorphism follows from the identity \( L_{[A,B]} = [L_A,L_B] \). To show that this is injective, suppose \( L_A = 0 \), that is, \( AX + XA^T = 0 \) for all \( X \in \mathcal{S}^n \). By taking \( X = I \) (Identity), we see that \( A + A^T = 0 \), that is, \( A \) is skew-symmetric. By taking \( X \) to be a diagonal matrix with distinct elements, we see that \( A = 0 \). \( \square \)

5. **The \( \mathcal{Z} \)-property of Stein transformations.** Recall that for a matrix \( A \in \mathbb{R}^{n \times n} \), the corresponding Stein transformation \( S_A \) is defined on \( \mathcal{S}^n \) by \( S_A(X) := X - AXA^T \). We also recall that \( \Pi(\mathcal{C}) := \{ A \in \mathbb{R}^{n \times n} : A(\mathcal{C}) \subseteq \mathcal{C} \} \). As in the case of Lyapunov transformations, we have \( S_A \in \mathcal{Z}(\mathcal{K}) \iff S_A^T \in \mathcal{Z}(\mathcal{E}) \).

**THEOREM 5.1.** Let \( \mathcal{C} \) be any closed convex cone in \( \mathbb{R}^n \). Then

\[ \pm A \in \Pi(\mathcal{C}) \Rightarrow S_A \in \mathcal{Z}(\mathcal{K}). \]

**Proof.** Without loss of generality, let \( A \in \Pi(\mathcal{C}) \). Let \( X = \sum_{i=1}^{N} u_i u_i^T \in \mathcal{K}, Y \in \mathcal{K}^* = \mathcal{E}, \) and \( \langle X, Y \rangle = 0 \), where \( u_i \in \mathcal{C} \) for all \( i \). Then \( w_i := Au_i \in \mathcal{C} \) for all \( i \).
Now, as $Y$ is copositive on $C$, 

$$\text{trace}(AXA^TY) = \sum_{i=1}^{N} w_i^T Y w_i \geq 0.$$ 

Hence 

$$\langle S_A(X), Y \rangle = \langle X, Y \rangle - \langle AXA^T, Y \rangle = -\text{trace}(AXA^TY) \leq 0.$$ 

This proves that $S_A \in Z(K)$. 

**Example 5.2.** By taking $C = \mathbb{R}^n$ in the above theorem, we see that for any matrix $A \in \mathbb{R}^{n \times n}$, $S_A \in Z_S(n)$. Now, let $C$ be the closed upper half-plane in $\mathbb{R}^2$ so that $K = E = S_+^2$. Then for any $2 \times 2$ real matrix $A$, $S_A \in Z(S_+^2)$, while it is easy to construct a $2 \times 2$ real matrix which is not in $\Pi(C)$. Thus, the converse in the above theorem does not hold.

Analogous to Theorem 3.8, we have

**Theorem 5.3.** Suppose $C$ is a closed convex cone in $\mathbb{R}^n$ such that $C - C = \mathbb{R}^n$. Then the following are equivalent:

(a) $A$ is Schur stable.

(b) The system $x(k+1) = Ax(k)$, $k = 0, 1, 2, \ldots$ is asymptotically stable in $C$ (that is, from any starting point in $C$, its trajectory converges to the origin).

When $\pm A \in \Pi(C)$, these are further equivalent to:

(c) There exists $D \in K^\circ$ such that $S_A(D) \in K^\circ$.

(d) There exists $D \in E^\circ$ such that $S_A(D) \in E^\circ$.

Note: $S_A$ is positive stable if and only if $A$ is Schur stable, see [10].

6. Cone spectrum, copositivity, and Z-transformations. In this section, we relate the $Z$-property, copositivity, and cone spectrum. Let $L$ be a linear transformation on $H$ and $K$ be a nonzero closed convex cone in $H$. Then the cone spectrum of $L$ with respect to $K$ is the set of all real $\lambda$ for which there is an $x$ such that 

$$0 \neq x \in K, L(x) - \lambda x \in K^* \quad \text{and} \quad \langle x, L(x) - \lambda x \rangle = 0.$$ 

We denote this set by $\sigma(L, K)$.

The following result gives the nonemptiness of the cone spectrum.

**Proposition 6.1.** Let $K$ be a nonzero closed convex cone in $H$ and $L$ be linear on $H$.

(i) If $K$ is proper, then $\sigma(L, K) \neq \emptyset$.
(ii) If $L$ is self-adjoint, then $\sigma(L, K) \neq \emptyset$. In fact,

$$\lambda^* := \min \{ \langle (L(x), x) : x \in K, ||x|| = 1 \} \in \sigma(L, K).$$

Proof. The proof of (i) is given in [20], Corollary 2.1. While a proof of (ii) is given in [14], Corollary 2.4 and [21], Example 1, we offer a direct and simple proof. Let $\lambda^* = \langle (L(x^*), x^*) \rangle$, where $x^* \in K$ with $||x^*|| = 1$. Define $S := L - \lambda^* I$. Then for all $0 \neq x \in K$, we have

$$\langle S(x), x \rangle = ||x||^2 \left\{ \langle (L(x), x) \rangle, \frac{x}{||x||} - \lambda^* \right\} \geq 0.$$  

This means that the self-adjoint transformation $S$ is copositive on $K$. Since $\langle S(x^*), x^* \rangle = 0$, we have, from Proposition 2.1, $y^* = S(x^*) \in K^*$. Thus, we have

$$x^* \in K, \quad y^* := L(x^*) - \lambda^* x^* \in K^* \quad \text{and} \quad \langle x^*, y^* \rangle = 0.$$  

Hence, $\lambda^* \in \sigma(L, K)$. \qed

The above result, together with the observation that every $\lambda \in \sigma(L, K)$ is of the form $\lambda = \frac{\langle (L(x), x) \rangle}{||x||}$ for some nonzero $x \in K$, gives the following:

Corollary 6.2. Suppose $\sigma(L, K)$ is nonempty. If $L$ is copositive on $K$, then $\lambda \geq 0$ for all $\lambda \in \sigma(L, K)$. The converse holds when $L$ is self-adjoint.

In what follows, we write $\sigma(L)$ for the spectrum of $L$.

Theorem 6.3. Suppose $K$ is proper and $L \in \mathbb{Z}(K)$. Then

$$\min Re \sigma(L) \in \sigma(L, K) \subseteq \sigma(L).$$

Proof. Let $\mu^* := \min \{ Re \lambda : \lambda \in \sigma(L) \}$. Since $K$ is proper and $L \in \mathbb{Z}(K)$, by Theorem 6 in [19], there exists a nonzero $u \in K$ such that $L(u) = \mu^* u$. Clearly, $\mu^* \in \sigma(L, K)$. This proves the first part of the inclusion. The second part is proved in Theorem 9, [27]; here is its short proof: Let $\mu \in \sigma(L, K)$ so that for some nonzero $x \in K$, $y = L(x) - \mu x \in K^*$ and $\langle x, y \rangle = 0$. As $L - \mu I \in \mathbb{Z}(K)$, $\langle (L - \mu I)x, y \rangle \leq 0$. This leads to $y = 0$, that is, $L(x) = \mu x$ proving $\mu \in \sigma(L)$. \qed

Remark 6.4. In Theorem 6.3, the equivalence of (a) and (d) was proved under the assumptions that $C - \mathcal{C} = \mathbb{R}^n$ and $A \in \mathbb{Z}(\mathcal{C})$. When $\mathcal{C}$ is proper and $A \in \mathbb{Z}(\mathcal{C})$, the following simple proof (an adaptation of the standard argument) can be given. We only prove the implication (d) $\Rightarrow$ (a). Assume that for some (symmetric) $D$ that is strictly copositive on $\mathcal{C}$, $A^T D + DA = Y$ is also strictly copositive on $\mathcal{C}$. Let $\mu^* = \min Re \sigma(A)$ so that by Theorem 6 in [19], there is a nonzero $u \in \mathcal{C}$ such that $Au = \mu^* u$. Then $0 < u^T Y u = u^T (A^T D + DA) u = 2 \mu^* u^T D u$. Since $u^T D u$ is also positive, we see that $\mu^* > 0$. This means that $A$ is positive stable.
The following result extends a result of J. Tao [25] proved in the setting of symmetric cones.

**Corollary 6.5.** Let $K$ be proper, $L \in Z(K)$ and copositive on $K$. Then $L$ is semi-positive stable (that is, all eigenvalues of $L$ lie in the closed right-half plane). If, in addition, $L$ is self-adjoint or $K$ is self-dual, then $L$ is monotone.

**Proof.** That $L$ is semi-positive stable follows from Corollary 6.2 and Theorem 6.3. If $L$ is self-adjoint, then all eigenvalues of $L$ are nonnegative, and hence $L$ is monotone. When $K$ is self dual, $L \in Z(K) \Leftrightarrow L^* \in Z(K^*) \Leftrightarrow L^* \in Z(K)$. In this case, $L + L^*$ is self-adjoint, copositive on $K$, and belongs to $Z(K)$. By the previous case, $L + L^*$ (and hence $L$) is monotone. □

**A concluding remark.** In a follow up paper [12], it is shown that the mapping $A \mapsto L_A$ in Theorem 4.1 is actually a bijection.

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**REFERENCES**


Coprime and Completely Positive Cones


