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A NEW EIGENVALUE BOUND FOR THE HADAMARD PRODUCT OF AN M-MATRIX AND AN INVERSE M-MATRIX*

FUBIN CHEN†, YAOTANG LI‡, AND DEFENG WANG§

Abstract. If A and B are $n \times n$ nonsingular M-matrices, a new lower bound for the minimum eigenvalue $\tau(A \circ B^{-1})$ for the Hadamard product of A and B^{-1} is derived. This bound improves the result of [R. Huang. Some inequalities for the Hadamard product and the Fan product of matrices. *Linear Algebra Appl.*, 428:1551–1559, 2008.].

Key words. M-matrix, Hadamard product, Spectral radius, Lower bound.

AMS subject classifications. 15A06, 15A15, 15A48.

1. Introduction. For a positive integer n, N denotes the set $\{1, 2, \ldots, n\}$. The set of all $n \times n$ complex matrices is denoted by $\mathbb{C}^{n\times n}$ and $\mathbb{R}^{n\times n}$ denotes the set of all $n \times n$ real matrices.

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{n\times n}$. We write $A \geq B$ $(> B)$ if $a_{ij} \geq b_{ij}$ $(> b_{ij})$ for all $i, j \in \{1, 2, \ldots, n\}$. If 0 is the null matrix and $A \geq 0$ (> 0), we say that A is a nonnegative (positive) matrix. The spectral radius of A is denoted by $\rho(A)$. If A is a nonnegative matrix, the Perron-Frobenius theorem guarantees that $\rho(A)$ is an eigenvalue of A.

We let Z_n denote the class of all $n \times n$ real matrices all of whose off-diagonal entries are nonpositive. An $n \times n$ matrix A is called an M-matrix if there exists an $n \times n$ nonnegative matrix B and a nonnegative real number λ such that $A = \lambda I - B$ and $\lambda \geq \rho(B)$, I is the identity matrix; if $\lambda > \rho(B)$, we call A a nonsingular M-matrix; if $\lambda = \rho(B)$, we call A a singular M-matrix. Denote by M_n the set of nonsingular M-matrices.

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Let $A \in Z_n$ and let $\tau(A) = \min\{Re(\lambda) : \lambda \in \sigma(A)\}$. Basic for our purpose are the following simple facts (see Problems 16, 19 and 28 in Section 2.5 of [4]):

(1) $\tau(A) \in \sigma(A)$; $\tau(A)$ is called the minimum eigenvalue of A.

(2) If $A, B \in M_n$, and $A \geq B$, then $\tau(A) \geq \tau(B)$.

(3) If $A \in M_n$, then $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix A^{-1}, and $\tau(A) = \frac{1}{\rho(A^{-1})}$ is a positive real eigenvalue of A.

Let A be an irreducible nonsingular M-matrix. It is known that there exist positive vectors u and v such that $Au = \tau(A)u$ and $v^T A = \tau(A)v^T$, u and v being called right and left Perron eigenvectors of A, respectively.

For two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size, the Hadamard product of A and B is $A \circ B = (a_{ij}b_{ij})$. If A and B are two nonsingular M-matrices, then it is proved in [2] that $A \circ B^{-1}$ is a nonsingular M-matrix.

If $A = (a_{ij})$ is a nonsingular M-matrix, we write $N = D - A$, where $D =$diag(a_{ii}). Note that $a_{ii} > 0$ for all i if $A \in M_n$. Thus, we define $J_A = D^{-1}N$; J_A is nonnegative.

Let $A, B \in M_n$ and $B^{-1} = (\beta_{ij})$, in [4, Theorem 5.7.31] the following classical result is given:

$$\tau(A \circ B^{-1}) \geq \tau(A) \min_{1 \leq i \leq n} \beta_{ii}.$$

Recently, Huang [5, Theorem 9] improved this result and gave a new lower bound for $\tau(A \circ B^{-1})$, that is

$$\tau(A \circ B^{-1}) \geq \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}}.$$

In this paper, for two nonsingular M-matrices A and B, we give a new lower bound for $\tau(A \circ B^{-1})$; some examples are given to illustrate our result.

2. Some lemmas and the main result.

In order to prove our result, we first give some lemmas.

Lemma 2.1. [4, Lemma 5.1.2] *Let $A, B \in \mathbb{C}^{n\times n}$ and suppose that $D \in \mathbb{C}^{n\times n}$ and $E \in \mathbb{C}^{n\times n}$ are diagonal matrices, then*

$$D(A \circ B)E = (DAE) \circ B = (DA) \circ (BE) = (AE) \circ (DB) = A \circ (DBE).$$

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LEMMA 2.2. [5, Lemma 8] *Let $B = (b_{ij}) \in M_n$ be irreducible, and let $y = (y_i)$ be a positive vector such that $J_B y = \rho(J_B)y$. Then for $B^{-1} = (\beta_{ij})$, we have*

$$|\beta_{ji}| \leq \rho(J_B)\beta_{ii}\frac{y_j}{y_i}, \qquad i \neq j,$$

and

$$\beta_{ii} \geq \frac{1}{b_{ii}(1+\rho^2(J_B))}.$$

LEMMA 2.3. [3, Theorem 6.4.7] *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. Then all the eigenvalues of A lie in the region:*

$$\bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \left\{ z \in \mathbb{C} : |z-a_{ii}||z-a_{jj}| \leq \sum_{k\neq i}|a_{ki}| \sum_{k\neq j}|a_{kj}| \right\}.$$

By the definition of J_A, we have

$$\rho(J_{A^T}) = \rho(D^{-1}N^T) = \rho(ND^{-1}) = \rho(D^{-1}(ND^{-1})D) = \rho(D^{-1}N) = \rho(J_A).$$

THEOREM 2.4. *Let $A = (a_{ij}), B \in \mathbb{R}^{n\times n}$ be two nonsingular M-matrices and let $B^{-1} = (\beta_{ij})$. Then*

$$\begin{aligned}\tau(A \circ B^{-1}) \geq \min_{i\neq j} \frac{1}{2}\Big\{ a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \big[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 \\ + 4a_{ii}a_{jj}\beta_{ii}\beta_{jj}\rho^2(J_A)\rho^2(J_B)\big]^{\frac{1}{2}} \Big\}. \end{aligned} \tag{2.1}$$

Proof. It is evident that (2.1) is an equality for $n = 1$.

We next assume that $n \geq 2$.

If $A \circ B^{-1}$ is irreducible, then A and B are irreducible. Then J_A and J_B are also irreducible and nonnegative, so there exists a positive vector $u = (u_i)$ such that $J_{A^T}u = \rho(J_{A^T})u$. Note that $\rho(J_{A^T}) = \rho(J_A)$, so we have

$$\sum_{j\neq i} \frac{|a_{ji}|u_j}{u_i} = a_{ii}\rho(J_A).$$

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Let $\hat{A} = (\hat{a}_{ij}) = \hat{U}A\hat{U}^{-1}$ and $\hat{B}^{-1} = (\hat{\beta}_{ij}) = \hat{V}B^{-1}\hat{V}^{-1}$ in which $\hat{U}$ and $\hat{V}$ are the nonsingular diagonal matrices $\hat{U}$ =diag$(u_1, u_2, \ldots, u_n)$ and $\hat{V}$ =diag$\left(\frac{1}{v_1}, \frac{1}{v_2}, \ldots, \frac{1}{v_n}\right)$. Then, we have

$$\begin{aligned}
\hat{A} &= (\hat{a}_{ij}) = \hat{U}A\hat{U}^{-1} \\
&= \begin{bmatrix} u_1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \frac{1}{u_1} & & & \\ & \frac{1}{u_2} & & \\ & & \ddots & \\ & & & \frac{1}{u_n} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & \frac{a_{12}u_1}{u_2} & \cdots & \frac{a_{1n}u_1}{u_n} \\ \frac{a_{21}u_2}{u_1} & a_{22} & \cdots & \frac{a_{2n}u_2}{u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}u_n}{u_1} & \frac{a_{n2}u_n}{u_2} & \cdots & a_{nn} \end{bmatrix}.
\end{aligned}$$

and

$$\begin{aligned}
\hat{B}^{-1} &= (\hat{\beta_{ij}}) = \hat{V}B^{-1}\hat{V}^{-1} \\
&= \begin{bmatrix} \frac{1}{v_1} & & & \\ & \frac{1}{v_2} & & \\ & & \ddots & \\ & & & \frac{1}{v_n} \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{bmatrix} \begin{bmatrix} v_1 & & & \\ & v_2 & & \\ & & \ddots & \\ & & & v_n \end{bmatrix} \\
&= \begin{bmatrix} \beta_{11} & \frac{\beta_{12}v_2}{v_1} & \cdots & \frac{\beta_{1n}v_n}{v_1} \\ \frac{\beta_{21}v_1}{v_2} & \beta_{22} & \cdots & \frac{\beta_{2n}v_n}{v_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_{n1}v_1}{v_n} & \frac{\beta_{n2}v_2}{v_n} & \cdots & \beta_{nn} \end{bmatrix}.
\end{aligned}$$

Also let $W = \hat{V}\hat{U}$. Then, W is nonsingular. From Lemma 2.1, we have

$$(VU)(A\circ B^{-1})(VU)^{-1} = VU(A\circ B^{-1})U^{-1}V^{-1} = (UAU^{-1})\circ(VB^{-1}V^{-1}) = \hat{A}\circ\hat{B}^{-1}.$$

Thus, we have $\tau(A \circ B^{-1}) = \tau(\hat{A} \circ \hat{B}^{-1})$ and

$$\hat{A} \circ \hat{B}^{-1} = (c_{ij}) = \begin{bmatrix} a_{11}\beta_{11} & \frac{a_{12}\beta_{12}u_1v_2}{u_2v_1} & \cdots & \frac{a_{1n}\beta_{1n}u_1v_n}{u_nv_1} \\ \frac{a_{21}\beta_{21}u_2v_1}{u_1v_2} & a_{22}\beta_{22} & \cdots & \frac{a_{2n}\beta_{2n}u_2v_n}{u_nv_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}\beta_{n1}u_nv_1}{u_1v_n} & \frac{a_{n2}\beta_{n2}u_nv_2}{u_2v_n} & \cdots & a_{nn}\beta_{nn} \end{bmatrix}.$$

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We next consider the minimum eigenvalue of $\hat{A} \circ \hat{B}^{-1}$. Let $\tau(\hat{A} \circ \hat{B}^{-1}) = \lambda$, so that $0 < \lambda < a_{ii}\beta_{ii}, \forall i \in N$. Thus, by Lemma 2.3, there is a pair (i,j) of positive integers with $i \neq j$ such that

$$|\lambda - a_{ii}\beta_{ii}|\,|\lambda - a_{jj}\beta_{jj}| \leq \sum_{k\neq i} |c_{ki}| \sum_{k \neq j} |c_{kj}|.$$

Observe that

$$\begin{aligned} \sum_{k\neq i} |c_{ki}| \sum_{k\neq j} |c_{kj}| &= \left(\sum_{k\neq i} \left| \frac{a_{ki}\beta_{ki}u_k v_i}{u_i v_k} \right| \right) \left(\sum_{k\neq j} \left| \frac{a_{kj}\beta_{kj}u_k v_j}{u_j v_k} \right| \right) \\ &\leq \left(\sum_{k\neq i} \left| \frac{a_{ki}u_k}{u_i} \right| \rho(J_B)\beta_{ii} \right) \left(\sum_{k\neq i} \left| \frac{a_{kj}u_k}{u_j} \right| \rho(J_B)\beta_{jj} \right) \\ &= a_{ii}a_{jj}\beta_{ii}\beta_{jj}\rho^2(J_A)\rho^2(J_B). \end{aligned}$$

Thus, we have

$$|\lambda - a_{ii}\beta_{ii}|\,|\lambda - a_{jj}\beta_{jj}| \leq a_{ii}a_{jj}\beta_{ii}\beta_{jj}\rho^2(J_A)\rho^2(J_B).$$

Then, we have

$$\lambda \geq \frac{1}{2}\left\{ a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \left[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 + 4a_{ii}a_{jj}\beta_{ii}\beta_{jj}\rho^2(J_A)\rho^2(J_B) \right]^{\frac{1}{2}} \right\}.$$

That is,

$$\begin{aligned} \tau(A \circ B^{-1}) &\geq \frac{1}{2}\Big\{ a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \big[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 \\ &\qquad + 4a_{ii}a_{jj}\beta_{ii}\beta_{jj}\rho^2(J_A)\rho^2(J_B) \big]^{\frac{1}{2}} \Big\} \\ &\geq \min_{i\neq j} \frac{1}{2}\Big\{ a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \big[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 \\ &\qquad + 4a_{ii}a_{jj}\beta_{ii}\beta_{jj}\rho^2(J_A)\rho^2(J_B) \big]^{\frac{1}{2}} \Big\}. \end{aligned}$$

Now, assume that $A \circ B^{-1}$ is reducible. It is known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [1]). If we denote by $D = (d_{ij})$ the $n \times n$ permutation matrix with $d_{12} = d_{23} = \cdots = d_{n-1,n} = d_{n1} = 1$, then both $A - tD$ and $B - tD$ are irreducible nonsingular M-matrices for any chosen positive real number t, sufficiently small such that all the leading principal minors of both $A - tD$ and $B - tD$ are positive. Now we substitute $A - tD$ and $B - tD$ for A and B, respectively in the previous case, and then letting $t \longrightarrow 0$, the result follows by continuity. $\square$

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THEOREM 2.5. *Let $A = (a_{ij}), B \in \mathbb{R}^{n\times n}$ be two nonsingular M-matrices and let $B^{-1} = (\beta_{ij})$. Then*

$$\min_{i\neq j} \frac{1}{2}\left\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \left[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 + 4a_{ii}a_{jj}\beta_{ii}\beta_{jj}\rho^2(J_A)\rho^2(J_B)\right]^{\frac{1}{2}}\right\}$$
$$\geq \frac{1-\rho(J_A)\rho(J_B)}{1+\rho^2(J_B)} \min_{1\leq i\leq n} \frac{a_{ii}}{b_{ii}}.$$

Proof. Without loss of generality, for $i \neq j$, assume that

$$a_{ii}\beta_{ii} - a_{ii}\beta_{ii}\rho(J_A)\rho(J_B) \leq a_{jj}\beta_{jj} - a_{jj}\beta_{jj}\rho(J_A)\rho(J_B). \tag{2.2}$$

Thus, (2.2) is equivalent to

$$a_{jj}\beta_{jj}\rho(J_A)\rho(J_B) \leq a_{ii}\beta_{ii}\rho(J_A)\rho(J_B) + a_{jj}\beta_{jj} - a_{ii}\beta_{ii} \tag{2.3}$$

From (2.1) and (2.3), we have

$$\begin{aligned}
&\frac{1}{2}\left\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - [(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 + 4a_{ii}a_{jj}\beta_{ii}\beta_{jj}\rho^2(J_A)\rho^2(J_B)]^{\frac{1}{2}}\right\}\\
&\geq \frac{1}{2}\Big\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \big[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2\\
&\quad + 4a_{ii}\beta_{ii}\rho(J_A)\rho(J_B)\left[a_{ii}\beta_{ii}\rho(J_A)\rho(J_B) + a_{jj}\beta_{jj} - a_{ii}\beta_{ii}\right]\big]^{\frac{1}{2}}\Big\}\\
&= \frac{1}{2}\Big\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \big[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2\\
&\quad + 4a_{ii}^2\beta_{ii}^2\rho^2(J_A)\rho^2(J_B) + 4a_{ii}\beta_{ii}\rho(J_A)\rho(J_B)(a_{jj}\beta_{jj} - a_{ii}\beta_{ii})\big]^{\frac{1}{2}}\Big\}\\
&= \frac{1}{2}\left\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \left[(a_{jj}\beta_{jj} - a_{ii}\beta_{ii} + 2a_{ii}\beta_{ii}\rho(J_A)\rho(J_B))^2\right]^{\frac{1}{2}}\right\}\\
&= \frac{1}{2}\left\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - (a_{jj}\beta_{jj} - a_{ii}\beta_{ii} + 2a_{ii}\beta_{ii}\rho(J_A)\rho(J_B))\right\}\\
&= a_{ii}\beta_{ii} - a_{ii}\beta_{ii}\rho(J_A)\rho(J_B)\\
&= a_{ii}\beta_{ii}(1-\rho(J_A)\rho(J_B))\\
&\geq \frac{1-\rho(J_A)\rho(J_B)}{1+\rho^2(J_B)}\frac{a_{ii}}{b_{ii}}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\tau(A\circ B^{-1}) &\geq \min_{i\neq j}\frac{1}{2}\Big\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \big[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2\\
&\qquad + 4a_{ii}a_{jj}\beta_{ii}\beta_{jj}\rho^2(J_A)\rho^2(J_B)\big]^{\frac{1}{2}}\Big\}\\
&\geq \frac{1-\rho(J_A)\rho(J_B)}{1+\rho^2(J_B)} \min_{1\leq i\leq n}\frac{a_{ii}}{b_{ii}}. \qquad \square
\end{aligned}$$

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REMARK 2.6. Theorem 2.5 shows that the result of Theorem 2.4 is better than the result of Theorem 9 in [5].

3. Examples.

EXAMPLE 3.1. Let

$$A = \begin{bmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ 0 & 0 & -0.5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}.$$

Then

$$A \circ B^{-1} = \begin{bmatrix} 0.4 & -0.1 & 0 & 0 \\ -0.1167 & 0.3667 & -0.1 & 0 \\ 0 & -0.1167 & 0.4 & -0.1 \\ 0 & 0 & -0.1 & 0.4 \end{bmatrix}.$$

By calculating with Matlab 7.0, we have $\rho(J_A) = 0.809, \rho(J_B) = 0.7652$, and $\tau(A \circ B^{-1}) = 0.2148$. By Theorem 9 in [5], we have

$$\tau(A \circ B^{-1}) \geq \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}} = 0.048.$$

By our Theorem 2.4, we have

$$\tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2} \Big\{ a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \big[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 + 4a_{ii}a_{jj}\beta_{ii}\beta_{jj}\rho^2(J_A)\rho^2(J_B)\big]^{\frac{1}{2}} \Big\} = 0.1524.$$

which approaches the real value 0.2148. This numerical example shows that the result in Theorem 2.4 is better than that in Theorem 9 in [5] in some cases.

EXAMPLE 3.2. Let

$$A = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -0.5 \\ -0.5 & 1 \end{bmatrix}.$$

Then

$$A \circ B^{-1} = \begin{bmatrix} 1.7142 & -0.5714 \\ -0.2857 & 2.2858 \end{bmatrix}.$$

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By calculating with Matlab 7.0, we have $\rho(J_A) = 0.7071, \rho(J_B) = 0.3536$, and $\tau(A \circ B^{-1}) = 1.0144$. By Theorem 9 in [5], we have

$$\tau(A \circ B^{-1}) \geq \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}} = 0.6666.$$

By our Theorem 2.4, we have

$$\begin{aligned}\tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2}\Big\{ & a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \big[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 \\ & + 4a_{ii}a_{jj}\beta_{ii}\beta_{jj}\rho^2(J_A)\rho^2(J_B)\big]^{\frac{1}{2}}\Big\} = 1.0144.\end{aligned}$$

It is a surprise to see that our bound is the minimum eigenvalue of $A \circ B^{-1}$. This numerical example shows that the bound of Theorem 2.4 is sharp.

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