A new eigenvalue bound for the Hadamard product of an M-matrix and an inverse M-matrix

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A NEW EIGENVALUE BOUND FOR THE HADAMARD PRODUCT OF AN $M$-MATRIX AND AN INVERSE $M$-MATRIX\footnote{Received by the editors on November 20, 2011. Accepted for publication on March 24, 2012. Handling Editor: Roger A. Horn.}

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Abstract. If $A$ and $B$ are $n \times n$ nonsingular $M$-matrices, a new lower bound for the minimum eigenvalue $\tau(A \circ B^{-1})$ for the Hadamard product of $A$ and $B^{-1}$ is derived. This bound improves the result of \cite{Huang}.\footnote{Department of Architecture and Engineering, Oxbridge College, Kunming University of Science and Technology, Kunming, Yunnan, 650106, P.R. China (chenfubinyn@163.com). Supported by Scientific Research Fund of Yunnan Provincial Education Department (No. 2010Y073) and Scientific Research Fund of Oxbridge College (No. JQ10003).}

Key words. $M$-matrix, Hadamard product, Spectral radius, Lower bound.

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1. Introduction. For a positive integer $n$, $N$ denotes the set \{1, 2, \ldots, $n$\}. The set of all $n \times n$ complex matrices is denoted by $\mathbb{C}^{n \times n}$ and $\mathbb{R}^{n \times n}$ denotes the set of all $n \times n$ real matrices.

Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and $B = (b_{ij}) \in \mathbb{R}^{n \times n}$. We write $A \geq B$ ($> B$) if $a_{ij} \geq b_{ij}$ ($> b_{ij}$) for all $i, j \in \{1, 2, \ldots, n\}$. If $0$ is the null matrix and $A \geq 0$ ($> 0$), we say that $A$ is a nonnegative (positive) matrix. The spectral radius of $A$ is denoted by $\rho(A)$. If $A$ is a nonnegative matrix, the Perron-Frobenius theorem guarantees that $\rho(A)$ is an eigenvalue of $A$.

We let $Z_n$ denote the class of all $n \times n$ real matrices all of whose off-diagonal entries are nonpositive. An $n \times n$ matrix $A$ is called an $M$-matrix if there exists an $n \times n$ nonnegative matrix $B$ and a nonnegative real number $\lambda$ such that $A = \lambda I - B$ and $\lambda \geq \rho(B)$, $I$ is the identity matrix; if $\lambda > \rho(B)$, we call $A$ a nonsingular $M$-matrix; if $\lambda = \rho(B)$, we call $A$ a singular $M$-matrix. Denote by $M_n$ the set of nonsingular $M$-matrices.

\cite{Huang}. Some inequalities for the Hadamard product and the Fan product of matrices. Linear Algebra Appl., 428:1551–1559, 2008.\footnote{School of Mathematics and Statistics, Yunnan University, Kunming, Yunnan, 650091, P.R. China (liyaotang@ynu.edu.cn). Supported by National Natural Science Foundations of China (No. 10961027, No. 71161020) and IRTSTYN, and the Natural Science Foundation of Yunnan Province (No. 2009CD011).}

\cite{Huang}. Some inequalities for the Hadamard product and the Fan product of matrices. Linear Algebra Appl., 428:1551–1559, 2008.\footnote{Department of Forest Product Industry, Yunnan Forestry Technological College, Kunming, Yunnan, 650224, P.R. China (wangdefengyn@126.com).}
Let $A \in \mathbb{Z}_n$ and let $\tau(A) = \min\{\Re(\lambda) : \lambda \in \sigma(A)\}$. Basic for our purpose are the following simple facts (see Problems 16, 19 and 28 in Section 2.5 of [4]):

1. $\tau(A) \in \sigma(A)$; $\tau(A)$ is called the minimum eigenvalue of $A$.

2. If $A, B \in M_n$, and $A \geq B$, then $\tau(A) \geq \tau(B)$.

3. If $A \in M_n$, then $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix $A^{-1}$, and $\tau(A) = \frac{1}{\rho(A^{-1})}$ is a positive real eigenvalue of $A$.

Let $A$ be an irreducible nonsingular $M$-matrix. It is known that there exist positive vectors $u$ and $v$ such that $Au = \tau(A)u$ and $v^T A = \tau(A)v^T$, $u$ and $v$ being called right and left Perron eigenvectors of $A$, respectively.

For two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size, the Hadamard product of $A$ and $B$ is $A \circ B = (a_{ij}b_{ij})$. If $A$ and $B$ are two nonsingular $M$-matrices, then it is proved in [2] that $A \circ B^{-1}$ is a nonsingular $M$-matrix.

If $A = (a_{ij})$ is a nonsingular $M$-matrix, we write $N = D - A$, where $D = \text{diag}(a_{ii})$. Note that $a_{ii} > 0$ for all $i$ if $A \in M_n$. Thus, we define $J_A = D^{-1}N$; $J_A$ is nonnegative.

Let $A, B \in M_n$ and $B^{-1} = (\beta_{ij})$, in [4, Theorem 5.7.31] the following classical result is given:

$$\tau(A \circ B^{-1}) \geq \tau(A) \min_{1 \leq i \leq n} \beta_{ii}.$$ 

Recently, Huang [5, Theorem 9] improved this result and gave a new lower bound for $\tau(A \circ B^{-1})$, that is

$$\tau(A \circ B^{-1}) \geq \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}}.$$

In this paper, for two nonsingular $M$-matrices $A$ and $B$, we give a new lower bound for $\tau(A \circ B^{-1})$; some examples are given to illustrate our result.

2. Some lemmas and the main result. In order to prove our result, we first give some lemmas.

**Lemma 2.1.** [4, Lemma 5.1.2] Let $A, B \in \mathbb{C}^{n \times n}$ and suppose that $D \in \mathbb{C}^{n \times n}$ and $E \in \mathbb{C}^{n \times n}$ are diagonal matrices, then

$$D(A \circ B)E = (DAE) \circ (DBE) = (AE) \circ (DB) = A \circ (DBE).$$
Lemma 2.2. [5, Lemma 8] Let $B = (b_{ij}) \in M_n$ be irreducible, and let $y = (y_i)$ be a positive vector such that $J_B y = \rho(J_B) y$. Then for $B^{-1} = (\beta_{ij})$, we have
\[
|\beta_{ji}| \leq \rho(J_B) \frac{y_j}{y_i}, \quad i \neq j,
\]
and
\[
\beta_{ii} \geq \frac{1}{b_{ii}(1 + \rho^2(J_B))}.
\]

Lemma 2.3. [3, Theorem 6.4.7] Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. Then all the eigenvalues of $A$ lie in the region:
\[
\bigcup_{i,j=1 \atop i \neq j}^n \left\{ z \in \mathbb{C} : |z - a_{ii}| |z - a_{jj}| \leq \sum_{k \neq i} |a_{ki}| \sum_{k \neq j} |a_{kj}| \right\}.
\]

By the definition of $J_A$, we have
\[
\rho(J_A^T) = \rho(D^{-1} N^T) = \rho(N D^{-1}) = \rho(D^{-1} (ND^{-1}) D) = \rho(D^{-1} N) = \rho(J_A).
\]

Theorem 2.4. Let $A = (a_{ij}), B \in \mathbb{R}^{n \times n}$ be two nonsingular $M$-matrices and let $B^{-1} = (\beta_{ij})$. Then
\[
\tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left( a_{ii} \beta_{ii} - a_{jj} \beta_{jj} \right)^2 \right. \\
\left. + 4 a_{ii} a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right\}^{\frac{1}{2}}.
\]

Proof. It is evident that (2.1) is an equality for $n = 1$.

We next assume that $n \geq 2$.

If $A \circ B^{-1}$ is irreducible, then $A$ and $B$ are irreducible. Then $J_A$ and $J_B$ are also irreducible and nonnegative, so there exists a positive vector $u = (u_i)$ such that $J_A^T u = \rho(J_A^T) u$. Note that $\rho(J_A^T) = \rho(J_A)$, so we have
\[
\sum_{j \neq i} \frac{|a_{ji}| u_j}{u_i} = a_{ii} \rho(J_A).
\]
Let \( \hat{A} = (\hat{a}_{ij}) = \hat{U}A\hat{U}^{-1} \) and \( \hat{B}^{-1} = (\hat{b}_{ij}) = \hat{V}B^{-1}\hat{V}^{-1} \) in which \( \hat{U} \) and \( \hat{V} \) are the nonsingular diagonal matrices \( \hat{U} = \text{diag}(u_1, u_2, \ldots, u_n) \) and \( \hat{V} = \text{diag}\left(\frac{1}{v_1}, \frac{1}{v_2}, \ldots, \frac{1}{v_n}\right) \). Then, we have

\[
\hat{A} = (\hat{a}_{ij}) = \hat{U}A\hat{U}^{-1}
\]

\[
= \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
  \frac{1}{u_1} & 1 & \cdots & 0 \\
  1 & \frac{1}{u_2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & \frac{1}{u_n}
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_n
\end{bmatrix}
\]

and

\[
\hat{B}^{-1} = (\hat{b}_{ij}) = \hat{V}B^{-1}\hat{V}^{-1}
\]

\[
= \begin{bmatrix}
  \frac{1}{v_1} & 1 & \cdots & 0 \\
  1 & \frac{1}{v_2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & \frac{1}{v_n}
\end{bmatrix}
\begin{bmatrix}
  \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\
  \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn}
\end{bmatrix}
\begin{bmatrix}
  v_1 \\
  v_2 \\
  \vdots \\
  v_n
\end{bmatrix}
\]

Also let \( W = \hat{V}\hat{U} \). Then, \( W \) is nonsingular. From Lemma 2.1, we have

\[
(VU)(A \circ B^{-1})(VU)^{-1} = VU(A \circ B^{-1})U^{-1}V^{-1} = (UAU^{-1}) \circ (VB^{-1}V^{-1}) = \hat{A} \circ \hat{B}^{-1}.
\]

Thus, we have \( \tau(A \circ B^{-1}) = \tau(\hat{A} \circ \hat{B}^{-1}) \) and

\[
\hat{A} \circ \hat{B}^{-1} = (c_{ij}) = \begin{bmatrix}
  a_{11}\beta_{11} & a_{12}\beta_{12} & \cdots & a_{1n}\beta_{1n} \\
  a_{21}\beta_{21} & a_{22}\beta_{22} & \cdots & a_{2n}\beta_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1}\beta_{n1} & a_{n2}\beta_{n2} & \cdots & a_{nn}\beta_{nn}
\end{bmatrix}.
\]
We next consider the minimum eigenvalue of $\hat{A} \circ \hat{B}^{-1}$. Let $\tau(\hat{A} \circ \hat{B}^{-1}) = \lambda$, so that $0 < \lambda < a_{ii} \beta_{ii}, \forall i \in N$. Thus, by Lemma 2.3, there is a pair $(i, j)$ of positive integers with $i \neq j$ such that

$$|\lambda - a_{ii}\beta_{ii}| |\lambda - a_{jj}\beta_{jj}| \leq \sum_{k \neq i} |c_{ki}| \sum_{k \neq j} |c_{kj}|.$$ 

Observe that

$$\sum_{k \neq i} |c_{ki}| \sum_{k \neq j} |c_{kj}| = \left(\sum_{k \neq i} \left|\frac{a_{ki} \beta_{ik} u_{ik} v_{ik}}{u_{ik} v_{ik}}\right|\right) \left(\sum_{k \neq j} \left|\frac{a_{kj} \beta_{jk} u_{jk} v_{jk}}{u_{jk} v_{jk}}\right|\right) \leq \left(\sum_{k \neq i} \left|\frac{a_{ki} u_{ik}}{u_{ik}}\right| \rho(J_B) \beta_{ii}\right) \left(\sum_{k \neq j} \left|\frac{a_{kj} u_{jk}}{u_{jk}}\right| \rho(J_B) \beta_{jj}\right) = a_{ii} a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B).$$

Thus, we have

$$|\lambda - a_{ii} \beta_{ii}| |\lambda - a_{jj} \beta_{jj}| \leq a_{ii} a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B).$$

Then, we have

$$\lambda \geq \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4 a_{ii} a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{\frac{1}{2}} \right\}.$$ 

That is,

$$\tau(A \circ B^{-1}) \geq \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4 a_{ii} a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{\frac{1}{2}} \right\} \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4 a_{ii} a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{\frac{1}{2}} \right\}. $$

Now, assume that $A \circ B^{-1}$ is reducible. It is known that a matrix in $Z_n$ is a nonsingular $M$-matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [1]). If we denote by $D = (d_{ij})$ the $n \times n$ permutation matrix with $d_{11} = d_{23} = \cdots = d_{n-1,n} = d_{n1} = 1$, then both $A - tD$ and $B - tD$ are irreducible nonsingular $M$-matrices for any chosen positive real number $t$, sufficiently small such that all the leading principal minors of both $A - tD$ and $B - tD$ are positive. Now we substitute $A - tD$ and $B - tD$ for $A$ and $B$, respectively in the previous case, and then letting $t \to 0$, the result follows by continuity. $\blacksquare$
Theorem 2.5. Let $A = (a_{ij}), B \in \mathbb{R}^{n \times n}$ be two nonsingular $M$-matrices and let $B^{-1} = (\beta_{ij})$. Then

$$
\min_{i \neq j} \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{\frac{1}{2}} \right\} 
\geq \frac{1 - \rho(J_A) \rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}}.
$$

Proof. Without loss of generality, for $i \neq j$, assume that

$$a_{ii} \beta_{ii} - a_{ii} \beta_{ii} \rho(J_A) \rho(J_B) \leq a_{jj} \beta_{jj} - a_{jj} \beta_{jj} \rho(J_A) \rho(J_B).$$

Thus, (2.2) is equivalent to

$$a_{jj} \beta_{jj} \rho(J_A) \rho(J_B) \leq a_{ii} \beta_{ii} \rho(J_A) \rho(J_B) + a_{jj} \beta_{jj} - a_{ii} \beta_{ii}$$

From (2.1) and (2.3), we have

$$\frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{\frac{1}{2}} \right\}$$

$$\geq \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{\frac{1}{2}} \right\}$$

$$= \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{\frac{1}{2}} \right\}$$

$$= \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{jj} \beta_{jj} - a_{ii} \beta_{ii} + 2a_{ii} \beta_{ii} \rho(J_A) \rho(J_B))^2 \right]^{\frac{1}{2}} \right\}$$

$$= \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{jj} \beta_{jj} - a_{ii} \beta_{ii} + 2a_{ii} \beta_{ii} \rho(J_A) \rho(J_B))^2 \right]^{\frac{1}{2}} \right\}$$

$$= a_{ii} \beta_{ii} - a_{jj} \beta_{jj} \rho(J_A) \rho(J_B)$$

$$\geq \frac{1 - \rho(J_A) \rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}}$$

Thus, we have

$$\tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{\frac{1}{2}} \right\}$$

$$\geq \frac{1 - \rho(J_A) \rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}}. \quad \Box$$
Remark 2.6. Theorem 2.5 shows that the result of Theorem 2.4 is better than the result of Theorem 9 in [5].

3. Examples.

Example 3.1. Let
\[
A = \begin{bmatrix}
1 & -0.5 & 0 & 0 \\
-0.5 & 1 & -0.5 & 0 \\
0 & -0.5 & 1 & -0.5 \\
0 & 0 & -0.5 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
4 & -1 & -1 & -1 \\
-2 & 5 & -1 & -1 \\
0 & -2 & 4 & -1 \\
-1 & -1 & -1 & 4
\end{bmatrix}.
\]

Then
\[
A \circ B^{-1} = \begin{bmatrix}
0.4 & -0.1 & 0 & 0 \\
-0.1167 & 0.3667 & -0.1 & 0 \\
0 & -0.1167 & 0.4 & -0.1 \\
0 & 0 & -0.1 & 0.4
\end{bmatrix}.
\]

By calculating with Matlab 7.0, we have \(\rho(J_A) = 0.809, \rho(J_B) = 0.7652\), and \(\tau(A \circ B^{-1}) = 0.2148\). By Theorem 9 in [5], we have
\[
\tau(A \circ B^{-1}) \geq \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}} = 0.048.
\]

By our Theorem 2.4, we have
\[
\tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{\frac{1}{2}} \right\} = 0.1524.
\]
which approaches the real value 0.2148. This numerical example shows that the result in Theorem 2.4 is better than that in Theorem 9 in [5] in some cases.

Example 3.2. Let
\[
A = \begin{bmatrix}
2 & -2 \\
-1 & 2
\end{bmatrix}, \quad B = \begin{bmatrix}
2 & -0.5 \\
-0.5 & 1
\end{bmatrix}.
\]

Then
\[
A \circ B^{-1} = \begin{bmatrix}
1.7142 & -0.5714 \\
-0.2857 & 2.2858
\end{bmatrix}.
\]
By calculating with Matlab 7.0, we have $\rho(J_A) = 0.7071$, $\rho(J_B) = 0.3536$, and $\tau(A \circ B^{-1}) = 1.0144$. By Theorem 9 in [5], we have

$$\tau(A \circ B^{-1}) \geq \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} a_{ii} = 0.6666.$$

By our Theorem 2.4, we have

$$\tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2}\left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A)\rho^2(J_B) \right]^{\frac{1}{2}} \right\} = 1.0144.$$

It is a surprise to see that our bound is the minimum eigenvalue of $A \circ B^{-1}$. This numerical example shows that the bound of Theorem 2.4 is sharp.

REFERENCES