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## ADDITIVE PRESERVERS OF TENSOR PRODUCT OF RANK ONE HERMITIAN MATRICES\*

MING-HUAT LIM<sup>†</sup>

**Abstract.** Let  $K$  be a field of characteristic not two or three with an involution and  $F$  be its fixed field. Let  $H_m$  be the  $F$ -vector space of all  $m$ -square Hermitian matrices over  $K$ . Let  $\rho_m$  denote the set of all rank-one matrices in  $H_m$ . In the tensor product space  $\bigotimes_{i=1}^k H_{m_i}$ , let  $\bigotimes_{i=1}^k \rho_{m_i}$  denote the set of all decomposable elements  $\bigotimes_{i=1}^k A_i$  such that  $A_i \in \rho_{m_i}$ ,  $i = 1, \dots, k$ . In this paper, additive maps  $T$  from  $H_m \otimes H_n$  to  $H_s \otimes H_t$  such that  $T(\rho_m \otimes \rho_n) \subseteq (\rho_s \otimes \rho_t) \cup \{0\}$  are characterized. From this, a characterization of linear maps is found between tensor products of two real vector spaces of complex Hermitian matrices that send separable pure states to separable pure states. Also classified in this paper are almost surjective additive maps  $L$  from  $\bigotimes_{i=1}^k H_{m_i}$  to  $\bigotimes_{i=1}^l H_{n_i}$  such that  $L(\bigotimes_{i=1}^k \rho_{m_i}) \subseteq \bigotimes_{i=1}^l \rho_{n_i}$  where  $2 \leq k \leq l$ . When  $K$  is algebraically closed and  $K = F$ , it is shown that every linear map on  $\bigotimes_{i=1}^k H_{m_i}$  that preserves  $\bigotimes_{i=1}^k \rho_{m_i}$  is induced by  $k$  bijective linear rank-one preservers on  $H_{m_i}$ ,  $i = 1, \dots, k$ .

**Key words.** Hermitian matrix, Rank-one preserver, Rank-one non-increasing map, Tensor product, Additive map, Separable pure state.

**AMS subject classifications.** 15A03, 15A04, 15A69.

**1. Introduction.** A map from a vector space of matrices over a field to another is called a *rank-one preserver* if it sends rank-one matrices to rank-one matrices and is called *rank-one non-increasing* if it sends rank-one matrices to matrices of rank less than or equal to one. Many authors have studied the structures of linear and additive rank-one preservers and rank-one non-increasing maps between spaces of Hermitian matrices [2], [3], [7]–[11], [16]–[18]. This paper is concerned with linear and additive maps on tensor products of spaces of Hermitian matrices that carry the set of tensor product of rank-one matrices into itself.

Let  $K$  be a field with an automorphism  $\bar{\phantom{a}}$  such that  $\bar{\bar{a}} = a$  for any  $a \in K$ . Let  $F = \{a \in K : \bar{a} = a\}$  be its fixed field. For each positive integer  $m \geq 2$ , let  $H_m$  denote the  $F$ -vector space of all  $m$ -square Hermitian matrices over  $K$ . Consider the tensor product space  $\bigotimes_{i=1}^k H_{m_i}$  over  $F$ . When the automorphism  $\bar{\phantom{a}}$  of  $K$  is of order 2,  $\bigotimes_{i=1}^k H_{m_i}$  can be identified naturally with  $H_{m_1 m_2 \dots m_k}$  via kronecker product of matrices. Similarly, when the automorphism  $\bar{\phantom{a}}$  of  $K$  is the identity,  $\bigotimes_{i=1}^k H_{m_i}$  can

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be identified naturally with a proper subspace of  $H_{m_1 m_2 \dots m_k}$ . Let  $\rho_m$  be the set of all rank-one matrices in  $H_m$  and let  $\bigotimes_{i=1}^k \rho_{m_i}$  denote the set of all decomposable elements  $\bigotimes_{i=1}^k A_i$  such that  $A_i \in \rho_{m_i}$ ,  $i = 1, \dots, k$ . We first characterize additive maps  $\varphi$  from  $H_m \otimes H_n$  to  $H_s \otimes H_t$  such that  $\varphi(\rho_m \otimes \rho_n) \subseteq (\rho_s \otimes \rho_t) \cup \{0\}$  where  $F$  has characteristic not two or three by using the structure of additive maps between tensor products of two vector spaces that preserve decomposable elements and also rank-one non-increasing additive maps between spaces of Hermitian matrices. An additive mapping  $\eta$  from a vector space to another vector space  $V$  is said to be *almost surjective* if  $V$  is linearly spanned by  $\text{Im} \eta$ . In Section 3, we show that if  $\text{char } F \neq 2$  and  $(K, F) \neq (3, 3)$ , and  $\theta$  is an almost surjective additive map from  $\bigotimes_{i=1}^k H_{m_i}$  to  $\bigotimes_{i=1}^l H_{n_i}$  such that  $\theta\left(\bigotimes_{i=1}^k \rho_{m_i}\right) \subseteq \bigotimes_{i=1}^l \rho_{n_i}$  where  $2 \leq k \leq l$ , then  $k = l$ ,  $\theta$  is quasilinear and induced by  $k$  quasilinear rank-one preservers between spaces of Hermitian matrices with respect to the same endomorphism of  $F$ . When  $K$  is algebraically closed and  $K = F$ , we show in Section 4 that every linear map on  $\bigotimes_{i=1}^k H_{m_i}$  that preserves  $\bigotimes_{i=1}^k \rho_{m_i}$  is induced by  $k$  bijective linear rank-one preservers on  $H_{m_i}$ ,  $i = 1, \dots, k$ .

Suppose now that  $K$  is the complex field and  $F$  is the real field. Then  $H_m$  is the real vector space of complex Hermitian matrices. A positive semi-definite matrix of trace one in  $H_m$  is called a *density matrix*. A mixed state in  $\bigotimes_{i=1}^k H_{m_i}$ ,  $k \geq 2$ , is called *separable* if it is a convex combination of elements of the form  $A_1 \otimes \dots \otimes A_k$  where  $A_i$  is a density matrix in  $H_{m_i}$ ,  $i = 1, \dots, k$ . In [5], Friedland et al. determined the group  $G(m_1, \dots, m_k)$  of all bijective linear maps on  $\bigotimes_{i=1}^k H_{m_i}$  that leave invariant the set  $S(m_1, \dots, m_k)$  of all separable states. The bipartite case of this result was first obtained in [1] by a different method. Let  $P_m$  denote the set of all rank-one matrices in  $H_m$  with trace 1. Let  $\bigotimes_{i=1}^k P_{m_i}$  be the set of all separable pure states, i.e., all decomposable elements of the form  $B_1 \otimes \dots \otimes B_k$  where  $B_i \in P_{m_i}$ ,  $i = 1, \dots, k$ . Since  $\bigotimes_{i=1}^k P_{m_i}$  is the set of all extreme points of  $S(m_1, \dots, m_k)$ , it follows that  $G(m_1, \dots, m_k)$  is the set of all bijective linear maps on  $\bigotimes_{i=1}^k H_{m_i}$  that leave invariant the set  $\bigotimes_{i=1}^k P_{m_i}$  (see [1], [5]). Thus, our work concerning linear and additive maps on  $\bigotimes_{i=1}^k H_{m_i}$  that preserve  $\bigotimes_{i=1}^k \rho_{m_i}$  is a generalization the above mentioned results.

We remark that Westwick [19] showed that every linear map on  $\bigotimes_{i=1}^k C^{m_i}$  that preserves the product states, i.e., the non-zero decomposable elements, is induced by  $k$  bijective linear maps on  $C^{m_i}$ ,  $i = 1, \dots, k$ . Here  $C$  is the complex field. The case for  $k = 2$  was proved in [14]. Additive preservers of non-zero decomposable tensors have been studied by several authors [6, 12, 13, 15, 20].

**2. Linear preservers of bipartite separable pure states.** For each positive integer  $m$ , let  $H_m$  denote the  $F$ -vector space of all  $m$ -square Hermitian matrices over  $K$  and  $K^m$  be the set of all  $m \times 1$  column vectors over  $K$ . For each  $m \times n$  matrix

$A = (a_{ij})$  over  $K$  and each endomorphism  $\sigma$  of  $K$ , let  $A^\sigma = (\sigma(a_{ij}))$  and  $A^*$  denote the transpose of the matrix  $(\bar{a}_{ij})$ .

Throughout this paper, all Hermitian matrices we consider are of size at least 2.

In this section, we study additive map from  $H_m \otimes H_n$  to  $H_s \otimes H_t$  such that  $\varphi(\rho_m \otimes \rho_n) \subseteq (\rho_s \otimes \rho_t) \cup \{0\}$ .

We need the following result proved in [7, 10] for the spaces of symmetric matrices and in [16] for the spaces of Hermitian matrices where  $K \neq F$ .

LEMMA 2.1. *Let  $\eta : H_m \rightarrow H_n$  be a rank-one non-increasing additive map. If  $(K, F) \notin \{(4, 2), (3, 3), (2, 2)\}$ , then one of the following holds:*

- (i)  $\eta(X) = \zeta(X)B$  for some additive functional  $\zeta : H_m \rightarrow F$  and some rank-one matrix  $B \in H_n$ ,
- (ii)  $\eta(X) = \lambda P X^\sigma P^*$  for some  $n \times m$  matrix  $P$  over  $K$ , some nonzero endomorphism  $\sigma$  of  $K$  commuting with the automorphism  $-$ , and some nonzero  $\lambda \in F$ .

An additive subgroup of the tensor product of several vector spaces over a field is called decomposable (pure) if it consists of decomposable (pure) elements.

LEMMA 2.2. *Suppose that  $F$  is of the characteristic not two or three. Let  $\varphi$  be an additive map from  $H_m \otimes H_n$  to  $H_s \otimes H_t$  such that  $\varphi(\rho_m \otimes \rho_n) \subseteq (\rho_s \otimes \rho_t) \cup \{0\}$ . Let  $A$  be a fixed nonzero element in  $H_m$ . Suppose that for each  $B \in \rho_n$ , either  $\varphi(A \otimes B)$  is the zero element or a non-zero decomposable element of the form  $f_B \otimes g_B$  where either  $f_B \in \rho_s$  or  $g_B \in \rho_t$ . Then  $\varphi(A \otimes H_n)$  is a decomposable subgroup of  $H_s \otimes H_t$ .*

*Proof.* Since  $\varphi$  is additive and the additive span of  $\rho_n$  is  $H_n$ , it suffices to show that  $\varphi(A \otimes \rho_n)$  is contained in a decomposable subgroup of  $H_s \otimes H_t$ . Suppose the contrary. Then there nonzero vectors  $u, v \in K^n$  and nonzero scalars  $\alpha, \beta \in F$  such that

$$\varphi(A \otimes \alpha uu^*) = U \text{ and } \varphi(A \otimes \beta vv^*) = V$$

where  $U - V$  is not decomposable. Since  $4\alpha = (1 + \alpha)^2 - (1 - \alpha)^2$  and  $\varphi$  is additive, it follows that there exists  $\delta \in \{1 + \alpha, 1 - \alpha\}$  such that  $\varphi(A \otimes \delta^2 uu^*) \neq 0$ . Let  $y = \delta u$ . Since the map  $\theta : H_m \rightarrow H_{st}$  defined by  $\theta(X) = \varphi(X \otimes uu^*)$  is an additive rank-one non-increasing map, it follows from Lemma 2.1 that  $\varphi(A \otimes yy^*) \in \langle U \rangle \setminus \{0\}$ . Similarly,  $\varphi(A \otimes zz^*) \in \langle V \rangle \setminus \{0\}$  for some  $z \in \langle v \rangle \setminus \{0\}$ . Hence,

$$\varphi(A \otimes yy^*) = f_1 \otimes g_1, \varphi(A \otimes zz^*) = f_2 \otimes g_2$$

for some linearly independent vectors  $f_1, f_2 \in H_s$  and some linearly independent

vectors  $g_1, g_2 \in H_t$ . Let

$$\varphi(A \otimes (y + z)(y + z)^*) = h \otimes k.$$

Let  $G$  be the prime subfield of  $K$ . Since for any  $\lambda \in G \setminus \{0, 1\}$ ,

$$(2.1) \quad \begin{aligned} & \varphi(A \otimes (y + \lambda z)(y + \lambda z)^*) \\ &= (1 - \lambda)f_1 \otimes g_1 + (\lambda^2 - \lambda)f_2 \otimes g_2 + \lambda h \otimes k \\ &= h_\lambda \otimes k_\lambda, \end{aligned}$$

where either  $h_\lambda$  or  $k_\lambda$  has rank less than 2. It follows that

$$(1 - \lambda)f_1 \otimes g_1 + (\lambda^2 - \lambda)f_2 \otimes g_2 = -\lambda h \otimes k + h_\lambda \otimes k_\lambda.$$

Since the left hand side is a rank 2 tensor in  $H_s \otimes H_t$ , it follows that

$$\langle f_1, f_2 \rangle = \langle h, h_\lambda \rangle, \quad \langle g_1, g_2 \rangle = \langle k, k_\lambda \rangle.$$

This shows that

$$h \otimes k = af_1 \otimes g_1 + bf_1 \otimes g_2 + cf_2 \otimes g_1 + df_2 \otimes g_2$$

for some  $a, b, c, d$  in  $F$ . From (2.1), we have for any  $\lambda \in G \setminus \{0, 1\}$ ,

$$\begin{aligned} & \varphi(A \otimes (y + \lambda z)(y + \lambda z)^*) \\ &= ((a - 1)\lambda + 1)f_1 \otimes g_1 + b\lambda f_1 \otimes g_2 + c\lambda f_2 \otimes g_1 + (\lambda d - \lambda + \lambda^2)f_2 \otimes g_2. \end{aligned}$$

Hence,

$$\begin{vmatrix} \lambda(a - 1) + 1 & \lambda b \\ c\lambda & \lambda d - \lambda + \lambda^2 \end{vmatrix} = (d - 1)\lambda + ((a - 1)(d - 1) - bc + 1)\lambda^2 + (a - 1)\lambda^3 = 0$$

for any  $\lambda \in G \setminus \{0, 1\}$ . Since  $|G| \geq 5$ , it follows that  $d = a = 1$ ,  $bc = 1$ . Hence,

$$\begin{aligned} & \varphi(A \otimes (y + \lambda z)(y + \lambda z)^*) \\ &= f_1 \otimes g_1 + b\lambda f_1 \otimes g_2 + c\lambda f_2 \otimes g_1 + \lambda^2 f_2 \otimes g_2 \\ &= (f_1 + c\lambda f_2) \otimes (g_1 + b\lambda g_2) \\ &= h_\lambda \otimes k_\lambda. \end{aligned}$$

Since both  $f_1, f_2$  and  $g_1, g_2$  are linearly independent Hermitian matrices, either  $f_1$  or  $g_1$  is of rank-one, either  $g_1$  or  $g_2$  is of rank-one, and  $|G| \geq 5$ , it is not hard to see that there exists  $\lambda \in G \setminus \{0, 1\}$  such that both  $f_1 + c\lambda f_2$  and  $g_1 + b\lambda g_2$  are of rank at least 2. This is a contradiction since one of  $h_\lambda, k_\lambda$  has rank less than 2. Hence,  $\varphi(A \otimes H_n)$  is a decomposable subgroup of  $H_s \otimes H_t$ .  $\square$

The following result was essentially proved in [6] (see also [4, 12]).

LEMMA 2.3. *Let  $U_1, U_2, V_1, V_2$  be vector spaces over a field. Let  $\theta$  be an additive map from  $U_1 \otimes U_2$  to  $V_1 \otimes V_2$  that sends decomposable elements to decomposable elements. Then one of the following holds:*

1.  $\text{Im } \theta$  is a decomposable subgroup of  $V_1 \otimes V_2$ ,
2. There exist a permutation  $\pi$  on  $\{1, 2\}$  and quasilinear maps  $\theta_i : U_{\pi(i)} \rightarrow V_i$ ,  $i = 1, 2$ , with respect to a endomorphism  $\sigma$  on the underlying field such that

$$\theta(u_1 \otimes u_2) = \theta_1(u_{\pi(1)}) \otimes \theta_2(u_{\pi(2)})$$

for any  $u_1 \in U_1$ ,  $u_2 \in U_2$ .

**THEOREM 2.4.** *Suppose that  $F$  is of the characteristic not two or three. Let  $\varphi$  be an additive map from  $H_m \otimes H_n$  to  $H_s \otimes H_t$ . Then  $\varphi(\rho_m \otimes \rho_n) \subseteq (\rho_s \otimes \rho_t) \cup \{0\}$  if and only if one of the following holds:*

- (i)  $\varphi(Z) = C \otimes \psi(Z)$  for some  $C \in \rho_s$  and some additive map  $\psi : H_m \otimes H_n \rightarrow H_t$  such that  $\psi(\rho_m \otimes \rho_n) \subseteq \rho_t \cup \{0\}$ ,
- (ii)  $\varphi(Z) = \pi(Z) \otimes D$  for some  $D \in \rho_t$  and some additive map  $\pi : H_m \otimes H_n \rightarrow H_s$  such that  $\pi(\rho_m \otimes \rho_n) \subseteq \rho_s \cup \{0\}$ ,
- (iii)  $\varphi(A \otimes B) = \varphi_1(A) \otimes \varphi_2(B)$ ,  $A \in H_m$ ,  $B \in H_n$ , for some rank-one non-increasing additive maps  $\varphi_1 : H_m \rightarrow H_s$  and  $\varphi_2 : H_n \rightarrow H_t$ ,
- (iv)  $\varphi(A \otimes B) = \varphi_2(B) \otimes \varphi_1(A)$ ,  $A \in H_m$ ,  $B \in H_n$ , for some rank-one non-increasing additive maps  $\varphi_1 : H_m \rightarrow H_t$  and  $\varphi_2 : H_n \rightarrow H_s$ .

*Proof.* The sufficiency part is clear. We now consider the necessity part. Let  $A \in \rho_m$ . Then for any  $B \in \rho_n$ ,  $\varphi(A \otimes B) \in (\rho_s \otimes \rho_t) \cup \{0\}$ . Hence, it follows from Lemma 2.2 that  $\varphi(A \otimes H_n)$  is a decomposable subgroup of  $H_s \otimes H_t$ . Hence,

$$\varphi(A \otimes H_n) \subseteq A_1 \otimes H_t \text{ for some } A_1 \in \rho_s,$$

or

$$\varphi(A \otimes H_n) \subseteq H_s \otimes A_2 \text{ for some } A_2 \in \rho_t.$$

Let  $C \in H_n$  be any nonzero matrix. Since for any  $A \in \rho_m$ ,

$$\varphi(A \otimes C) = 0 \text{ or } \varphi(A \otimes C) = f_A \otimes g_A$$

where either  $f_A \in \rho_s$  or  $g_A \in \rho_t$ , it follows from Lemma 2.2 that  $\varphi(H_m \otimes C)$  is a decomposable subgroup of  $H_s \otimes H_t$ . Hence,  $\varphi$  sends decomposable elements to decomposable elements. The result now follows from Lemma 2.3 and the hypothesis that  $\varphi(\rho_m \otimes \rho_n) \subseteq (\rho_s \otimes \rho_t) \cup \{0\}$ .  $\square$

**REMARK 2.5.**

- (a) Theorem 2.4 is true for any field  $F$  with at least 5 elements if we replace “additive” by “linear”.

- (b) The following result follows from Lemma 2.1. Let  $\eta : H_m \rightarrow H_n$  be a rank-one non-increasing linear map where  $(K, F) \notin \{(4, 2), (3, 3), (2, 2)\}$ . Then one of the following holds:
- (i)  $\eta(X) = \zeta(X)B$  for some linear functional  $\zeta : H_m \rightarrow F$  and some rank-one matrix  $B \in H_n$ ,
  - (ii) there exist an  $n \times m$  matrix  $P$  over  $K$  and a nonzero  $\lambda \in F$  such that  $\eta$  has the form  $A \mapsto \lambda PAP^*$  or  $A \mapsto \lambda PA^t P^*$ .

For the rest of this section,  $H_i$  is the real vector space of  $i$ -square complex Hermitian matrices. Let  $P_i$  denote the set of all rank-one matrices in  $H_i$  with trace 1. Then  $P_m \otimes P_n$  is the set of all bipartite separable pure states in  $H_m \otimes H_n$ .

LEMMA 2.6. [5] *Let  $\eta : H_m \rightarrow H_n$  be a linear map such that  $\eta(P_m) \subseteq P_n$ . Then one of the following holds:*

- (i)  $\eta(X) = (\text{Tr } X)R$  for some  $R \in P_n$ ,
- (ii)  $m \leq n$ , there exists an  $n \times m$  complex matrix  $U$  with  $U^*U = I_m$  such that  $\eta$  has the form  $A \mapsto UAU^*$  or  $A \mapsto UA^t U^*$ .

From Theorem 2.4, we obtain following characterization of linear maps that send bipartite separable pure states to bipartite separable pure states.

COROLLARY 2.7. *Let  $\varphi$  be a linear map from  $H_m \otimes H_n$  to  $H_s \otimes H_t$ . Then  $\varphi(P_m \otimes P_n) \subseteq P_s \otimes P_t$  if and only if one of the following holds:*

- (i)  $\varphi(Z) = C \otimes \psi(Z)$  for some  $C \in P_s$  and some linear map  $\psi : H_m \otimes H_n \rightarrow H_t$  such that  $\psi(P_m \otimes P_n) \subseteq P_t$ ,
- (ii)  $\varphi(Z) = \pi(Z) \otimes D$  for some  $D \in P_t$  and some linear map  $\pi : H_m \otimes H_n \rightarrow H_s$  such that  $\pi(P_m \otimes P_n) \subseteq P_s$ ,
- (iii)  $\varphi(A \otimes B) = \varphi_1(A) \otimes \varphi_2(B)$  for  $A \in H_m, B \in H_n$  where  $\varphi_1 : H_m \rightarrow H_s$  is a linear map such that  $\varphi_1(P_m) \subseteq P_s$ , and  $\varphi_2 : H_n \rightarrow H_t$  is a linear map such that  $\varphi_2(P_n) \subseteq P_t$ ,
- (iv)  $\varphi(A \otimes B) = \varphi_2(B) \otimes \varphi_1(A)$  for  $A \in H_m, B \in H_n$  where  $\varphi_1 : H_m \rightarrow H_t$  is a linear map such that  $\varphi_1(P_m) \subseteq P_t$ , and  $\varphi_2 : H_n \rightarrow H_s$  is a linear map such that  $\varphi_2(P_n) \subseteq P_s$ .

Characterization of linear maps  $\varphi$  on  $H_m \otimes H_n$  such that  $\varphi(P_m \otimes P_n) = P_m \otimes P_n$  was obtained in [1]. The following follows immediately from Corollary 2.7 and Lemma 2.6.

COROLLARY 2.8. *Let  $\varphi$  be a linear map from  $H_m \otimes H_n$  to  $H_s \otimes H_t$  such that  $\varphi(P_m \otimes P_n) \subseteq P_s \otimes P_t$  and the rank of  $\varphi$  is larger than  $\max \{\dim H_s, \dim H_t\}$ . Then one of the following holds:*

- (i)  $m \leq s, n \leq t, \varphi(A \otimes B) = \varphi_1(A) \otimes \varphi_2(B)$  for  $A \in H_m, B \in H_n$ , where

- $\varphi_1 : H_m \rightarrow H_s$  has the form  $A \mapsto UAU^*$  or  $A \mapsto UA^tU^*$  with  $U^*U = I_m$  and  
 $\varphi_2 : H_n \rightarrow H_t$  has the form  $B \mapsto BVV^*$  or  $B \mapsto VB^tV^*$  with  $V^*V = I_n$ ,  
 (ii)  $m \leq t, n \leq s, \varphi(A \otimes B) = \varphi_2(B) \otimes \varphi_1(A)$  for  $A \in H_m, B \in H_n$ , where  
 $\varphi_1 : H_m \rightarrow H_t$  has the form  $A \mapsto UAU^*$  or  $A \mapsto UA^tU^*$  with  $U^*U = I_m$  and  
 $\varphi_2 : H_n \rightarrow H_s$  has the form  $B \mapsto BVV^*$  or  $B \mapsto VB^tV^*$  with  $V^*V = I_n$ .

REMARK 2.9. The corresponding results of Corollaries 2.7 and 2.8 are true for the spaces of real symmetric matrices.

**3. Extension to multipartite systems.** In this section, we shall study almost surjective additive maps from  $\bigotimes_{i=1}^k H_{m_i}$  to  $\bigotimes_{i=1}^l H_{n_i}$  that send  $\bigotimes_{i=1}^k \rho_{m_i}$  to  $\bigotimes_{i=1}^l \rho_{n_i}$  where  $2 \leq k \leq l$  and  $m_i, n_j \geq 2, i = 1, \dots, k, j = 1, \dots, l$ .

Let  $X = x_1 \otimes \dots \otimes x_m$  and  $Y = y_1 \otimes \dots \otimes y_m$  be two nonzero decomposable elements in the tensor product  $\bigotimes_{i=1}^m W_i$  of  $m$  vector spaces  $W_1, \dots, W_m$  over  $F$ . Then each  $x_i$  is called a *factor* of  $X$ . If  $x_i$  and  $y_i$  are linearly independent for at most one  $i$ , then we write  $X \sim Y$ . It is known that  $X - Y$  is decomposable if and only if  $X \sim Y$ . If  $X \sim Y$  and  $X, Y$  are linearly independent, then we say that  $X$  and  $Y$  are *adjacent*. It is easily seen that an additive subgroup  $S$  of  $\bigotimes_{i=1}^m W_i$  is decomposable if and only if  $X \sim Y$  for any non-zero elements  $X, Y \in S$ , and every decomposable subgroup of  $\bigotimes_{i=1}^m W_i$  is of the form

$$(3.1) \quad w_1 \otimes \dots \otimes w_{i-1} \otimes G \otimes w_{i+1} \otimes \dots \otimes w_m$$

for some  $i$ , some nonzero vectors  $w_j \in W_j, j \neq i$  and some subgroup  $G$  of  $W_i$ . The nonzero vectors  $w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_m$  are called *factors* of the decomposable subgroup. A decomposable subgroup of  $\bigotimes_{i=1}^m W_i$  is said to be of *type- $i$*  if it is of the form (3.1). If  $G = W_i$ , then it is called a *maximal decomposable subgroup of type- $i$* .

A rank-one non-increasing map is said to be *degenerate* if its image consists of matrices of rank less than or equal to one.

For the following five Lemmas, we assume that  $\text{char } F \neq 2$  and  $(K, F) \neq (3, 3)$ .

LEMMA 3.1. Let  $\theta$  be an additive map from  $\bigotimes_{i=1}^k H_{m_i}$  to  $\bigotimes_{i=1}^l H_{n_i}$  such that  $\theta\left(\bigotimes_{i=1}^k \rho_{m_i}\right) \subseteq \bigotimes_{i=1}^l \rho_{n_i}$ . If  $A, B \in \bigotimes_{i=1}^k \rho_{m_i}$  and  $A \sim B$ , then  $\theta(A) \sim \theta(B)$ .

*Proof.* Without loss of generality, we may assume that

$$A = A_1 \otimes \dots \otimes A_{k-1} \otimes \alpha uu^*, \quad B = A_1 \otimes \dots \otimes A_{k-1} \otimes \beta vv^*$$

for some  $A_1 \otimes \dots \otimes A_{k-1}$  in  $\bigotimes_{i=1}^{k-1} \rho_{m_i}$ , some nonzero vectors  $u, v \in K^{m_k}$  and some nonzero scalars  $\alpha, \beta \in F$ . Suppose the contrary that  $\theta(A) - \theta(B)$  is not decomposable. Using the same argument as in the first paragraph of the proof of Lemma 2.2, we see



that there exist

$$C = A_1 \otimes \cdots \otimes A_{k-1} \otimes yy^* \text{ where } y \in \langle u \rangle \setminus \{0\}$$

and

$$D = A_1 \otimes \cdots \otimes A_{k-1} \otimes zz^* \text{ where } z \in \langle v \rangle \setminus \{0\}$$

such that  $\theta(C) - \theta(D)$  is not decomposable. We have

$$\theta(C) = \alpha u_1 u_1^* \otimes \cdots \otimes u_l u_l^*, \quad \theta(D) = b v_1 v_1^* \otimes \cdots \otimes v_l v_l^*$$

for some nonzero vectors  $u_i, v_i \in K^{n_i}$ ,  $i = 1, \dots, l$ , some nonzero  $a, b \in F$ . Since  $\theta(C) - \theta(D)$  is not decomposable, we may assume without loss of generality that both  $u_1 u_1^*$ ,  $v_1 v_1^*$  and  $u_2 u_2^*$ ,  $v_2 v_2^*$  are linearly independent. Let  $\pi$  be the natural additive map from  $H_{m_k}$  to  $\bigotimes_{i=1}^l H_{n_i}$  that is induced by the restriction map of  $\theta$  to  $A_1 \otimes \cdots \otimes A_{k-1} \otimes H_{m_k}$ . Then  $\pi$  is a rank-one preserver from  $H_{m_k}$  to  $H_{n_1 \dots n_l}$  which is non-degenerate. By Lemma 2.1, there exist an  $n \times m_k$  matrix  $U$  over  $K$  where  $n = n_1 \dots n_l$ , a non-zero endomorphism  $\sigma$  of  $K$  commuting with  $-$  and a nonzero scalar  $c \in F$  such that

$$\pi(X) = cUX^\sigma U^*$$

for all  $X \in H_{m_k}$ . Let

$$H = A_1 \otimes \cdots \otimes A_{k-1} \otimes (y+z)(y+z)^*.$$

Then  $\theta(H) = d w_1 w_1^* \otimes \cdots \otimes w_l w_l^*$  for some nonzero  $d \in F$  and for some nonzero vectors  $w_i \in K^{n_i}$ ,  $i = 1, \dots, l$ . Since

$$\theta(C) = cU(yy^*)^\sigma U^*, \quad \theta(D) = cU(zz^*)^\sigma U^*$$

and

$$\theta(H) = cU((y+z)(y+z)^*)^\sigma U^*,$$

it follows that

$$\langle Uy^\sigma \rangle = \langle u_1 \otimes \cdots \otimes u_l \rangle, \quad \langle Uz^\sigma \rangle = \langle v_1 \otimes \cdots \otimes v_l \rangle$$

and

$$\langle U(y+z)^\sigma \rangle = \langle w_1 \otimes \cdots \otimes w_l \rangle.$$

Hence,

$$w_1 \otimes \cdots \otimes w_l = \lambda u_1 \otimes \cdots \otimes u_l + \mu v_1 \otimes \cdots \otimes v_l$$

for some nonzero scalars  $\lambda, \mu \in F$ . This yields a contradiction since both  $u_1, v_1$  and  $u_2, v_2$  are linearly independent vectors. Hence,  $\theta(A) \sim \theta(B)$ .  $\square$

LEMMA 3.2. *Let  $\theta$  be an almost surjective additive map from  $\bigotimes_{i=1}^k H_{m_i}$  to  $\bigotimes_{i=1}^l H_{n_i}$  such that  $\theta\left(\bigotimes_{i=1}^k \rho_{m_i}\right) \subseteq \bigotimes_{i=1}^l \rho_{n_i}$ . Let  $M$  be any maximal decomposable subgroup of  $\bigotimes_{i=1}^k H_{m_i}$  whose factors are rank-one matrices. Then  $\theta(M)$  is a decomposable subgroup of  $\bigotimes_{i=1}^l H_{n_i}$  whose factors are rank-one matrices such that  $\dim(\theta(M)) \geq 2$ .*

*Proof.* By Lemma 3.1, we see that  $\theta(M)$  is a decomposable subgroup of  $\bigotimes_{i=1}^l H_{n_i}$  whose factors are rank-one matrices. Suppose that

$$\theta(M) \subseteq \langle J_1 \otimes \cdots \otimes J_l \rangle$$

for some  $J_i \in \rho_{n_i}$ ,  $i = 1, \dots, l$ . Without loss of generality, we may assume that  $M = A_1 \otimes \cdots \otimes A_{k-1} \otimes H_{m_k}$  for some  $A_1 \otimes \cdots \otimes A_{k-1}$  in  $\bigotimes_{i=1}^{k-1} \rho_{m_i}$ . Let  $\Omega := X_1 \otimes \cdots \otimes X_k$  be any element of  $\bigotimes_{i=1}^k \rho_{m_i}$ . Then there exists a chain of elements  $\Omega_1, \dots, \Omega_s$  in  $\bigotimes_{i=1}^k \rho_{m_i}$  such that  $\Omega_1 = A_1 \otimes \cdots \otimes A_{k-1} \otimes X_k$ ,  $\Omega_s = \Omega$  and  $\Omega_1 \sim \Omega_2, \dots, \Omega_{s-1} \sim \Omega_s$ . In view of Lemma 3.1 and the hypothesis that  $k \leq l$ , we see that  $J_i$  is a factor of  $T(\Omega)$  for some  $i$ . Let

$$V = J_1 \otimes H_{n_2} \otimes \cdots \otimes H_{n_l} + H_{n_1} \otimes J_2 \otimes H_{n_3} \otimes \cdots \otimes H_{n_l} + \cdots + H_{n_1} \otimes \cdots \otimes H_{n_{l-1}} \otimes J_l$$

Then  $\theta\left(\bigotimes_{i=1}^k \rho_{m_i}\right) \subseteq V$  and this implies that  $\text{Im}(\theta) \subseteq V$ . Any decomposable element of the form  $D_1 \otimes \cdots \otimes D_k$  where  $D_i, J_i$  are linearly independent does not belong to  $V$  and hence  $\theta$  is not almost surjective, a contradiction. Hence,  $\dim(\theta(M)) \geq 2$ .  $\square$

LEMMA 3.3. *Let  $\theta$  be an almost surjective additive map from  $\bigotimes_{i=1}^k H_{m_i}$  to  $\bigotimes_{i=1}^l H_{n_i}$  such that  $\theta\left(\bigotimes_{i=1}^k \rho_{m_i}\right) \subseteq \bigotimes_{i=1}^l \rho_{n_i}$  where  $k \leq l$ . If  $M_1$  and  $M_2$  are maximal decomposable subgroups of the same type in  $\bigotimes_{i=1}^k H_{m_i}$ , then  $\theta(M_1)$  and  $\theta(M_2)$  are decomposable subgroups of the same type in  $\bigotimes_{i=1}^l H_{n_i}$ .*

*Proof.* Without loss of generality, we may assume that

$$M_1 = B \otimes H_{m_k} \quad \text{and} \quad M_2 = C \otimes H_{m_k}$$

for some  $B, C \in \bigotimes_{i=1}^{k-1} \rho_{m_i}$ . Suppose that  $k \geq 3$ . Since there is a chain of decomposable elements  $N_1, \dots, N_t$  in  $\bigotimes_{i=1}^{k-1} \rho_{m_i}$  such that  $B \sim N_1, \dots, N_i \sim N_{i+1}, \dots, N_t \sim C$ , we may assume without loss of generality that  $B \sim C$ . We may also assume that

$$B = A \otimes Y, \quad C = A \otimes Z, \quad A \in \bigotimes_{i=1}^{k-2} \rho_{m_i}, \quad Y, Z \in \rho_{m_{k-1}}.$$

Suppose that  $\theta(M_1)$  and  $\theta(M_2)$  are decomposable subgroups of different types. We may assume that

$$\theta(M_1) \subseteq J_1 \otimes H_{n_2} \otimes D \quad \text{and} \quad \theta(M_2) \subseteq H_{n_1} \otimes J_2 \otimes E$$

for some  $J_1 \in \rho_{n_1}$ ,  $J_2 \in \rho_{n_2}$  and for some  $D, E \in \bigotimes_{i=3}^l \rho_{n_i}$ . By Lemma 3.2,

$$\dim\langle\theta(M_i)\rangle \geq 2, \quad i = 1, 2.$$

Hence, there exist vectors  $K_1, K_2 \in \rho_{m_k}$  such that

$$\theta(B \otimes K_1) = J_1 \otimes W_2 \otimes D \text{ and } \theta(C \otimes K_2) = W_1 \otimes J_2 \otimes E,$$

where  $J_i, W_i$  are linearly independent,  $i = 1, 2$ . Since

$$C \otimes K_1 \sim B \otimes K_1 \text{ and } C \otimes K_1 \in M_2,$$

it follows from Lemma 3.1 that  $D, E$  are linearly dependent. Let  $D = J_3 \otimes \cdots \otimes J_l$ . For any  $X \in \rho_{m_k}$ , let  $M_X = A \otimes H_{m_{k-1}} \otimes X$ . Since

$$\theta(M_X) \cap \theta(M_i) \neq \{0\}, \quad i = 1, 2,$$

and  $\theta(M_X)$  is a decomposable subgroup, it follows that

$$(3.2) \quad \theta(M_X) \subseteq J_1 \otimes \cdots \otimes J_{i-1} \otimes H_{n_i} \otimes J_{i+1} \otimes \cdots \otimes J_l$$

for some  $i = 1, \dots, l$ . Let  $Y = Y_1 \otimes \cdots \otimes Y_k \in \bigotimes_{i=1}^k \rho_{m_i}$ . Since there exist  $Z_1, \dots, Z_s \in \bigotimes_{i=1}^k \rho_{m_i}$  for some positive integer  $s \leq k - 2$  such that

$$Y \sim Z_1, \dots, Z_i \sim Z_{i+1}, \dots, Z_s \sim A \otimes Y_{k-1} \otimes Y_k,$$

it follows that  $Y$  and  $A \otimes Y_{k-1} \otimes Y_k$  have at least 2 common factors. Since

$$A \otimes Y_{k-1} \otimes Y_k \in M_{Y_k}$$

it follows from that (3.2) that  $J_i$  is a factor of  $\theta(Y)$  for some  $i$ . This shows that  $\theta$  is not almost surjective, a contradiction. Hence,  $\theta(M_1)$  and  $\theta(M_2)$  are decomposable subgroups of the same type. If  $k = 2$ , we see from the above proof that the result is also true.  $\square$

For each positive integer  $k$ , let  $[k]$  denote the set  $\{1, \dots, k\}$ .

LEMMA 3.4. *Let  $\theta$  be an almost surjective additive map from  $\bigotimes_{i=1}^k H_{m_i}$  to  $\bigotimes_{i=1}^l H_{n_i}$  such that  $\theta\left(\bigotimes_{i=1}^k \rho_{m_i}\right) \subseteq \bigotimes_{i=1}^l \rho_{n_i}$  where  $k \leq l$ . Then  $k = l$  and  $\theta$  sends maximal decomposable subgroups of distinct types in  $\bigotimes_{i=1}^k H_{m_i}$  to decomposable subgroups of distinct types in  $\bigotimes_{i=1}^l H_{n_i}$ .*

*Proof.* Let  $J$  be the subset of  $[l]$  consisting of all positive integers  $j$  such that  $\theta(M)$  is of type- $j$  for some maximal decomposable subgroup  $M$  of  $\bigotimes_{i=1}^k H_{m_i}$ . Without loss of generality, we may assume that  $J = \{1, \dots, t\}$ . Suppose that  $t < l$ . Let  $X_1 \otimes \cdots \otimes X_k \in \bigotimes_{i=1}^k \rho_{m_i}$  and

$$\theta(X_1 \otimes \cdots \otimes X_k) = Y_1 \otimes \cdots \otimes Y_l.$$

Consider any decomposable element  $A_1 \otimes \cdots \otimes A_k \in \rho_{m_i}$ . In view of Lemma 3.3, we have

$$\xi(H_{m_1} \otimes X_2 \otimes \cdots \otimes X_k) \text{ is of type } \leq t.$$

Since

$$A_1 \otimes X_2 \otimes \cdots \otimes X_k \in H_{m_1} \otimes X_2 \otimes \cdots \otimes X_k,$$

it follows that

$$(3.3) \quad \theta(A_1 \otimes X_2 \otimes \cdots \otimes X_k) \in W \otimes Y_{t+1} \otimes \cdots \otimes Y_l$$

where  $W = \bigotimes_{i=1}^t H_{n_i}$ . Similarly since

$$A_1 \otimes A_2 \otimes X_3 \otimes \cdots \otimes X_k \in A_1 \otimes H_{m_2} \otimes X_3 \otimes \cdots \otimes X_k,$$

and

$$\theta(A_1 \otimes H_{m_2} \otimes X_3 \otimes \cdots \otimes X_k) \text{ is of type } \leq t,$$

it follows from (3.3) that

$$\xi(A_1 \otimes A_2 \otimes X_3 \otimes \cdots \otimes X_k) \in W \otimes Y_{t+1} \otimes \cdots \otimes Y_l.$$

Continuing the process, we see that

$$\xi(A_1 \otimes \cdots \otimes A_k) \in W \otimes Y_{t+1} \otimes \cdots \otimes Y_l.$$

This is a contradiction since  $\theta$  is an almost surjective. Hence,  $t = l$ . This shows that  $k = l$ . Hence,  $\theta$  sends maximal decomposable subgroups of distinct types in  $\bigotimes_{i=1}^k H_{m_i}$  to decomposable subgroups of the distinct types in  $\bigotimes_{i=1}^l H_{n_i}$ .  $\square$

LEMMA 3.5. *Let  $\varphi_1, \varphi_2$  be two non-degenerate additive rank-one preservers from  $H_m$  to  $H_n$  such that  $\varphi_1(A), \varphi_2(A)$  are linearly dependent for any rank-one matrix  $A$  in  $H_m$ . Then  $\varphi_1, \varphi_2$  are linearly dependent.*

*Proof.* By Lemmas 2.1, there exist  $n \times m$  matrices  $U, V$  over  $K$ , non-zero endomorphisms  $\sigma, \tau$  of  $K$  commuting with  $-$  and non-zero scalars  $a, b \in F$  such that

$$\varphi_1(X) = aUX^\sigma U^*, \quad \varphi_2(X) = bVX^\tau V^*$$

for all  $X \in H_m$ . Since for any non-zero vector  $x \in K^n$ ,  $U(xx^*)^\sigma U^*, V(xx^*)^\tau V^*$  are linearly dependent, it follows that  $Ux^\sigma = c_x Vx^\tau$  for some nonzero  $c_x \in K$ . Since  $\varphi_2$  is non-degenerate, there exist  $y, z \in K^n$  such that  $Vy^\tau, Vz^\tau$  are linearly independent vectors. Since

$$Uy^\sigma = c_y Vy^\tau, \quad Uz^\sigma = c_z Vz^\tau$$

and

$$U(y^\sigma + z^\sigma) = c_{y+z}V(y+z)^\tau,$$

we have

$$c_{y+z}V(y+z)^\tau = c_yVy^\tau + c_zVz^\tau.$$

Hence,  $c_y = c_z = c_{z+y}$ . Now for any nonzero  $w \in K^n$ , we have either  $Vy^\tau, Vw^\tau$  or  $Vz^\tau, Vw^\tau$  are linearly independent. Hence,  $c_w = c_y$ . This implies that there is a fixed element  $c \in K \setminus \{0\}$  such that  $Ux^\sigma = cVx^\tau$  for any non-zero  $x \in K^n$ . For any non-zero scalar  $d \in K$  and any non-zero  $x \in K^n$ , we have

$$U(dx)^\sigma = cV(dx)^\tau.$$

Hence,  $\sigma(d) = \tau(d)$ . Thus,  $\varphi_1(X) = ab^{-1}cc^*\varphi_2(X)$  for any rank-one matrix  $X \in H_m$ . This shows that  $\varphi_1, \varphi_2$  are linearly dependent.  $\square$

Let  $\sigma$  be an endomorphism of  $F$ . An additive map  $T$  from a vector space  $V$  over  $F$  to a vector space  $W$  over  $F$  is said to be  $\sigma$ -quasilinear if  $T(cv) = \sigma(c)T(v)$  for any  $c \in F, v \in V$ . If in addition  $\sigma$  is an automorphism of  $F$ , then  $T$  is called semilinear.

**THEOREM 3.6.** *Suppose that  $\text{char } F \neq 2$  and  $(K, F) \neq (3, 3)$ . Let  $\theta$  be an almost surjective additive map from  $\bigotimes_{i=1}^k H_{m_i}$  to  $\bigotimes_{i=1}^l H_{n_i}$  such that  $\theta\left(\bigotimes_{i=1}^k \rho_{m_i}\right) \subseteq \bigotimes_{i=1}^l \rho_{n_i}$  where  $k \leq l$ . Then  $k = l$ , and there exist a permutation  $\tau$  on  $[k]$  and almost surjective  $\sigma$ -quasilinear rank-one preservers  $\theta_i$  from  $H_{m_{\tau(i)}}$  to  $H_{n_i}$  such that*

$$\theta(A_1 \otimes \cdots \otimes A_k) = \theta_1(A_{\tau(1)}) \otimes \cdots \otimes \theta_k(A_{\tau(k)})$$

for any  $A_i \in H_{m_i}, i = 1, \dots, k$ .

*Proof.* We see from Lemma 3.3 that for each  $i \in [k]$ , maximal decomposable subgroups of  $\bigotimes_{i=1}^k H_{m_i}$  of type- $i$  are mapped to decomposable subgroups of type- $j_i$  for some  $j_i$ . By Lemma 3.4,  $[l] = [k] = \{j_1, \dots, j_k\}$ . Let  $\alpha$  be the permutation on  $[k]$  such that  $\alpha(j_i) = i, i = 1, \dots, k$ . Let  $S_\alpha$  be the canonical isomorphism from  $\bigotimes_{i=1}^k H_{n_i}$  to  $\bigotimes_{i=1}^k H_{n_{\alpha(i)}}$  such that

$$S_\alpha(A_1 \otimes \cdots \otimes A_k) = A_{\alpha(1)} \otimes \cdots \otimes A_{\alpha(k)}$$

for any  $A_i \in H_{n_i}, i = 1, \dots, k$ . Let  $\xi = S_\alpha \circ \theta$ . It follows that  $\xi$  maps every maximal decomposable subgroups of type- $j$  to decomposable subgroups of type- $j, j = 1, \dots, k$ .

From Lemma 3.3, we see that for any  $C \in \bigotimes_{i=1}^{k-1} \rho_{m_i}, \xi(C \otimes H_{m_k}) \subseteq \tilde{C} \otimes H_{n_k}$  for some  $\tilde{C} \in \bigotimes_{i=1}^{k-1} \rho_{n_i}$  and hence there is a additive rank-one preserver  $\pi_C$  from  $H_{m_k}$

to  $H_{n_k}$  such that  $\xi(C \otimes X) = \tilde{C} \otimes \pi_C(X)$  for any  $X \in H_{m_k}$ . By Lemma 3.2,  $\pi_C$  is non-degenerate. Let  $D, E \in \bigotimes_{i=1}^{k-1} \rho_{m_i}$  be linearly independent. Then we have

$$\begin{aligned}\xi(D \otimes X) &= \tilde{D} \otimes \pi_D(X), \\ \xi(E \otimes X) &= \tilde{E} \otimes \pi_E(X)\end{aligned}$$

for some  $\tilde{D}, \tilde{E} \in \bigotimes_{i=1}^{k-1} \rho_{n_i}$  and for any  $X \in \rho_{m_k}$ . Suppose that  $D \sim E$ . Since  $D \otimes X, E \otimes X$  belong to a maximal decomposable subgroup  $M$  of type- $s$  for some  $s < k$ , it follows that  $\xi(M)$  a decomposable subgroup of type- $s$ . This shows that  $\pi_D(X), \pi_E(X)$  are linearly dependent for any  $X \in \rho_{m_k}$ . Now, suppose that  $D, E$  are not adjacent decomposable elements. Then there exists a chain of elements  $\Omega_1, \dots, \Omega_s$  in  $\bigotimes_{i=1}^{k-1} \rho_{m_i}$  such that  $\Omega_1 = D, E = \Omega_s$ , and  $\Omega_{i-1}, \Omega_i$  are adjacent for  $i = 1, \dots, s - 1$ . We conclude from the previous case that  $\pi_D(X), \pi_E(X)$  must be linearly dependent for any  $X \in \rho_{m_k}$ . In view of Lemma 3.5,  $\pi_D, \pi_E$  are linearly dependent. This shows that there exist a non-degenerate additive rank-one preserver  $\pi$  from  $H_{m_k}$  to  $H_{n_k}$  and an additive map  $\eta$  from  $\bigotimes_{i=1}^{k-1} H_{m_i}$  to  $\bigotimes_{i=1}^{k-1} H_{n_i}$  such that

$$\xi(C \otimes X) = \eta(C) \otimes \pi(X)$$

for any  $C \in \bigotimes_{i=1}^{k-1} \rho_{m_i}$  and any  $X \in H_{m_k}$ . Note that both  $\pi$  and  $\eta$  are almost surjective. Also we see that  $\eta\left(\bigotimes_{i=1}^{k-1} \rho_{m_i}\right) \subseteq \bigotimes_{i=1}^{k-1} \rho_{n_i}$ . If  $k > 2$ , by repeating the process we see that there are  $\sigma_i$ -quasilinear rank-one preservers  $\pi_i$  from  $H_{m_i}$  to  $H_{n_i}$ , such that

$$\xi(A_1 \otimes \dots \otimes A_k) = \pi_1(A_1) \otimes \dots \otimes \pi_k(A_k)$$

for any  $A_i \in H_{m_i}, i = 1, \dots, k$  where  $\pi = \pi_k$ . For any  $\lambda \in F \setminus \{0\}$  and any  $A_i \in H_{m_i}, i = 1, \dots, k$ ,

$$\begin{aligned}\xi((\lambda A_1) \otimes \dots \otimes A_k) &= \pi_1(\lambda A_1) \otimes \dots \otimes \pi_k(A_k) \\ &= \xi(A_1 \otimes \lambda A_2 \otimes \dots \otimes A_k) \\ &= \pi_1(A_1) \otimes \pi_2(\lambda A_2) \otimes \dots \otimes \pi_k(A_k).\end{aligned}$$

Hence,  $\sigma_1(\lambda) = \sigma_2(\lambda)$  for any  $\lambda \in F \setminus \{0\}$ . Thus,  $\sigma_1 = \sigma_2$ . Similarly,  $\sigma_1 = \sigma_i$  for  $2 < i \leq k$ . Since  $\xi = S_\alpha \circ \theta$ , we obtain our required conclusion.  $\square$

**COROLLARY 3.7.** *Suppose that  $\text{char } F \neq 2$  and  $(K, F) \neq (3, 3)$ . Let  $\theta$  be an surjective additive map from  $\bigotimes_{i=1}^k H_{m_i}$  onto itself such that  $\theta\left(\bigotimes_{i=1}^k \rho_{m_i}\right) \subseteq \bigotimes_{i=1}^k \rho_{m_i}$ . Then there exist a permutation  $\tau$  on  $[k]$  with  $m_{\tau(i)} = m_i, i = 1, \dots, k$ , and bijective  $\sigma$ -semilinear rank-one preservers  $\theta_i$  from  $H_{m_{\tau(i)}}$  to  $H_{m_i}$  such that*

$$\theta(A_1 \otimes \dots \otimes A_k) = \theta_1(A_{\tau(1)}) \otimes \dots \otimes \theta_k(A_{\tau(k)})$$

for any  $A_i \in H_{m_i}, i = 1, \dots, k$ .

*Proof.* By Theorem 3.6, there exist a permutation  $\tau$  on  $[k]$  and almost surjective  $\sigma$ -quasilinear rank-one preservers  $\theta_i$  from  $H_{m_{\tau(i)}}$  to  $H_{m_i}$  such that

$$\theta(A_1 \otimes \cdots \otimes A_k) = \theta_1(A_{\tau(1)}) \otimes \cdots \otimes \theta_k(A_{\tau(k)})$$

for any  $A_i \in H_{m_i}$ ,  $i = 1, \dots, k$ . Hence,  $\theta$  is  $\sigma$ -quasilinear. Since every surjective quasilinear mapping from a finite dimensional vector space onto itself is semilinear, it follows that  $\theta$  is semilinear. This shows that  $\sigma$  is surjective, and hence, each  $\theta_i$  is bijective  $\sigma$ -semilinear. This completes the proof.  $\square$

**COROLLARY 3.8.** *Let  $\theta$  be an almost surjective additive map from  $\bigotimes_{i=1}^k H_{m_i}$  to  $\bigotimes_{i=1}^l H_{n_i}$  such that  $\theta\left(\bigotimes_{i=1}^k \rho_{m_i}\right) \subseteq \bigotimes_{i=1}^l \rho_{n_i}$  where  $k \leq l$ . Suppose that  $F$  is the real field. Then  $k = l$ , and there exist a permutation  $\tau$  on  $[k]$  and bijective linear rank-one preservers  $\theta_i$  from  $H_{m_{\tau(i)}}$  to  $H_{n_i}$  such that*

$$\theta(A_1 \otimes \cdots \otimes A_k) = \theta_1(A_{\tau(1)}) \otimes \cdots \otimes \theta_k(A_{\tau(k)})$$

for any  $A_i \in H_{m_i}$ ,  $i = 1, \dots, k$ .

*Proof.* The result follows from Theorem 3.6 and the fact that identity map is the only non-zero endomorphism of the real field.  $\square$

Since Lemma 3.1 to Lemma 3.5 are true if we replace “additive” by “linear” and the hypothesis on the fields by  $(K, F) \notin \{(4, 2), (3, 3), (2, 2)\}$ , we see from the proof of Theorem 3.6 that the following result is valid.

**THEOREM 3.9.** *Suppose that  $(K, F) \notin \{(4, 2), (3, 3), (2, 2)\}$ . Let  $\theta$  be a surjective linear map from  $\bigotimes_{i=1}^k H_{m_i}$  to  $\bigotimes_{i=1}^l H_{n_i}$  such that  $\theta\left(\bigotimes_{i=1}^k \rho_{m_i}\right) \subseteq \bigotimes_{i=1}^l \rho_{n_i}$  where  $k \leq l$ . Then  $k = l$ , and there exist a permutation  $\tau$  on  $[k]$  and bijective linear rank-one preservers  $\theta_i$  from  $H_{m_{\tau(i)}}$  to  $H_{n_i}$  such that*

$$\theta(A_1 \otimes \cdots \otimes A_k) = \theta_1(A_{\tau(1)}) \otimes \cdots \otimes \theta_k(A_{\tau(k)})$$

for any  $A_i \in H_{m_i}$ ,  $i = 1, \dots, k$ .

**4. Symmetric matrices over algebraically closed fields.** For each positive integer  $m \geq 2$ , let  $S_m$  be the vector space of all  $m$ -square symmetric matrices over the field  $K$ . In this section, we shall show that if  $K$  is algebraically closed, then every linear map  $\theta$  on  $\bigotimes_{i=1}^k S_{m_i}$  such that  $\theta\left(\bigotimes_{i=1}^k \rho_{m_i}\right) \subseteq \bigotimes_{i=1}^k \rho_{m_i}$  is induced by  $k$  bijective rank-one preservers on spaces of symmetric matrices.

The following result is known for the case where  $\text{char } K \neq 2$  and  $n = m$ , see [8].

**LEMMA 4.1.** *Let  $\eta : S_n \rightarrow S_m$  be a linear rank-one preserver where  $K$  is quadratically closed. Then  $n \leq m$  and there exists an  $m \times n$  matrix  $U$  of rank  $n$  such that*

$$\eta(A) = UAU^t$$

for all  $A \in S_n$ .

*Proof.* Suppose that the rank of  $\eta$  is 1. Let  $x, y$  be linearly independent vectors in  $K^n$ . Then

$$\eta(xx^t) = uu^t, \quad \eta(yy^t) = cuu^t$$

and

$$\eta((x+y)(x+y)^t) = duu^t$$

for some nonzero  $c, d \in K$  and some nonzero vector  $u \in K^n$ . Hence,

$$\eta(xy^t + yx^t) = (d - c - 1)uu^t.$$

For any  $\lambda \in K$ ,

$$\eta((x + \lambda y)(x + \lambda y)^t) = (1 + \lambda(d - c - 1) + c\lambda^2)uu^t.$$

Since  $K$  is quadratically closed, there exists  $g \in K$  such that

$$1 + g(d - c - 1) + cg^2 = 0.$$

Hence,

$$\eta((x + gy)(x + gy)^t) = 0,$$

a contradiction. The result now follows from Lemma 2.1.  $\square$

LEMMA 4.2. *Let  $K$  be algebraically closed. Then there does not exist a linear map  $\varphi$  from  $S_m \otimes S_n$  to  $S_m$  such that  $\varphi(\rho_m \otimes \rho_n) \subseteq \rho_m$  where  $m \geq n$ .*

*Proof.* Suppose the contrary that such  $\varphi$  exists. Let  $A, B$  be linearly independent vectors in  $\rho_n$ . Since the restriction map of  $\varphi$  to  $S_m \otimes A$  is a linear rank-one preserver, it follows from Lemma 4.1 that there exists an invertible  $m$ -square matrix  $P$  such that

$$\varphi(X \otimes A) = PXP^t$$

for all  $X \in S_m$ . Similarly, there exists an invertible  $m$ -square matrix  $Q$  such that

$$\varphi(X \otimes B) = QXQ^t$$

for all  $X \in S_m$ . Since  $K$  is algebraically closed, it follows that  $P^{-1}Q$  has a non-zero eigenvalue  $\lambda \in K$ . Let  $u \in K^m$  be an eigenvector of  $P^{-1}Q$  corresponding to  $\lambda$ . Then  $Q(u) = \lambda P(u)$ . Hence,

$$\varphi(uu^t \otimes (\lambda^2 A - B)) = \lambda^2 P(uu^t)P^t - Q(uu^t)Q^t = 0.$$



Since the restriction map of  $\varphi$  to  $uu^t \otimes S_n$  is a linear rank-one preserver, it follows from Lemma 4.1 that this restriction map is injective. Since  $\lambda^2 A - B \neq 0$ , we obtain a contradiction. This completes the proof.  $\square$

LEMMA 4.3. *Let  $K$  be algebraically closed. Let  $\theta$  be a linear map from  $\bigotimes_{i=1}^k S_{m_i}$  to  $\bigotimes_{i=1}^l S_{n_i}$  such that  $\theta\left(\bigotimes_{i=1}^k \rho_{m_i}\right) \subseteq \bigotimes_{i=1}^l \rho_{n_i}$ . Let  $M$  be any maximal decomposable subspace of  $\bigotimes_{i=1}^k S_{m_i}$  whose factors are rank-one matrices. Then  $\theta(M)$  is a decomposable subspace of  $\bigotimes_{i=1}^l S_{n_i}$  whose factors are rank-one matrices and  $\dim \theta(M) = \dim M$ .*

*Proof.* Suppose that  $M$  is of type  $i$ . Since  $\theta$  is linear and  $K$  is algebraically closed, we see from the proof of Lemma 3.1 that  $\theta(A) \sim \theta(B)$  whenever  $A, B \in M$ . Hence,  $\theta(M)$  is a decomposable subspace of  $\bigotimes_{i=1}^l S_{n_i}$  of type  $j$  for some  $j$  whose factors are rank-one matrices. Let  $\pi$  be the natural linear map from  $S_{m_i}$  to  $S_{n_j}$  that is induced by the restriction map of  $\theta$  to  $M$ . Then  $\pi$  is a rank-one preserver from  $S_{m_i}$  to  $S_{n_j}$ . By Lemma 4.1,  $\pi$  is injective, and hence,  $\dim \theta(M) = \dim M$ . This completes the proof.  $\square$

LEMMA 4.4. *Let  $K$  be algebraically closed. Let  $\theta$  be a linear map from  $\bigotimes_{i=1}^k S_{m_i}$  to  $\bigotimes_{i=1}^l S_{n_i}$  such that  $\theta\left(\bigotimes_{i=1}^k \rho_{m_i}\right) \subseteq \bigotimes_{i=1}^l \rho_{n_i}$ . If  $M_1$  and  $M_2$  are maximal decomposable subspaces of the same type in  $\bigotimes_{i=1}^k S_{m_i}$ , then  $\theta(M_1)$  and  $\theta(M_2)$  are decomposable subspaces of the same type in  $\bigotimes_{i=1}^l S_{n_i}$ .*

*Proof.* We may assume that  $M_1 = B \otimes S_{m_k}$  and  $M_2 = C \otimes S_{m_k}$  where  $B, C \in \bigotimes_{i=1}^{k-1} \rho_{m_i}$  are adjacent. Suppose that  $\theta(M_1)$  and  $\theta(M_2)$  are decomposable subspaces of different types. We may assume that

$$\theta(M_1) \subseteq J_1 \otimes S_{n_2} \otimes D \text{ and } \theta(M_2) \subseteq S_{n_1} \otimes J_2 \otimes E$$

for some  $J_1 \in \rho_{n_1}$ ,  $J_2 \in \rho_{n_2}$  and for some  $D, E \in \bigotimes_{i=3}^l \rho_{n_i}$  if  $3 \leq l$ . By Lemma 4.1, there exist injective linear rank-one preservers  $\xi : S_{m_k} \rightarrow S_{n_2}$  and  $\psi : S_{m_k} \rightarrow S_{n_1}$  such that

$$\theta(B \otimes X) = J_1 \otimes \xi(X) \otimes D \text{ and } \theta(C \otimes X) = \psi(X) \otimes J_2 \otimes E$$

for any  $X \in \rho_{m_k}$ . There exist two linearly independent rank-one matrices  $U_1, U_2$  in  $S_{m_k}$  such that  $\xi(U_i), J_2$  are linearly independent,  $i = 1, 2$ . Since

$$\theta(B \otimes U_i) \sim \theta(C \otimes U_i)$$

and  $\xi(U_i), J_2$  are linearly independent, it follows that  $\psi(U_i), J_1$  are linearly dependent for  $i = 1, 2$ . This is a contradiction since  $\psi$  is an injective linear map. Hence,  $\theta(M_1)$  and  $\theta(M_2)$  are decomposable subspaces of the same type.  $\square$

THEOREM 4.5. *Let  $K$  be algebraically closed. Let  $\theta$  be a linear map from*

$\bigotimes_{i=1}^k S_{m_i}$  to  $\bigotimes_{i=1}^k S_{n_i}$  such that  $\theta\left(\bigotimes_{i=1}^k \rho_{m_i}\right) \subseteq \bigotimes_{i=1}^k \rho_{n_i}$  where  $m_i = n_i$ ,  $m_i \geq m_{i+1}$ ,  $i = 1, \dots, k-1$  and  $n_{k-1} \geq n_k$ . Then there exist a permutation  $\sigma$  on  $[k]$  and  $k-1$  bijective linear rank-one preservers  $\theta_i$  from  $S_{m_i}$  to  $S_{n_{\sigma(i)}}$ ,  $i = 1, \dots, k-1$ , and a linear rank-one preserver  $\theta_k$  from  $S_{m_k}$  to  $S_{n_{\sigma(k)}}$  such that

$$\theta(A_1 \otimes \cdots \otimes A_k) = \theta_{\pi(1)}(A_{\pi(1)}) \otimes \cdots \otimes \theta_{\pi(k)}(A_{\pi(k)})$$

for any  $A_i \in S_{m_i}$ ,  $i = 1, \dots, k$  where  $\pi = \sigma^{-1}$ .

*Proof.* We shall show that  $\theta$  sends maximal decomposable subspaces of distinct types in  $\bigotimes_{i=1}^k S_{m_i}$  to decomposable subspaces of distinct types in  $\bigotimes_{i=1}^k S_{n_i}$ . Suppose the contrary. Let  $s$  be the smallest positive integer such that there exists another positive integer  $t$  with the property that maximal decomposable subspaces of types  $s$  and  $t$  are mapped to decomposable subspaces of type  $h$  for some positive integer  $h$ . Let  $X_1 \otimes \cdots \otimes X_k \in \bigotimes_{i=1}^k \rho_{m_i}$  and

$$\theta(X_1 \otimes \cdots \otimes X_k) = Y_1 \otimes \cdots \otimes Y_k.$$

Let  $N$  be the subspace of  $\bigotimes_{i=1}^k S_{m_i}$  spanned by the set of all non-zero decomposable elements whose  $i$ -factor is  $X_i$  for  $i \notin \{s, t\}$ . From the proof of Lemma 3.4, we see that

$$\theta(N) \subseteq Y_1 \otimes \cdots \otimes Y_{h-1} \otimes S_{n_h} \otimes Y_{h+1} \otimes \cdots \otimes Y_k.$$

Since  $m_i = n_i$ ,  $m_i \geq m_{i+1}$ ,  $i = 1, \dots, k-1$ , and  $n_{k+1} \geq n_k$ , it follows from Lemma 4.3 that  $m_s = n_h$ . Hence,

$$\theta(N) = Y_1 \otimes \cdots \otimes Y_{h-1} \otimes S_{n_h} \otimes Y_{h+1} \otimes \cdots \otimes Y_k.$$

This shows that there exists a linear map  $\varphi$  from  $S_{m_s} \otimes S_{m_t}$  to  $S_{m_s}$  such that  $\varphi(\rho_{m_s} \otimes \rho_{m_t}) = \rho_{m_s}$  where  $m_s \geq m_t$ . This is a contradiction to Lemma 4.2. Hence,  $\theta$  sends maximal decomposable subspaces of distinct types in  $\bigotimes_{i=1}^k S_{m_i}$  to decomposable subspaces of distinct types in  $\bigotimes_{i=1}^k S_{n_i}$ . Using this fact, Lemmas 4.1, 4.3, 4.4 and the same argument as in the proof of Theorem 3.6, we obtain the required result.  $\square$

For each permutation  $\alpha$  on  $[l]$ , let  $P_\alpha$  denote the canonical isomorphism on  $\bigotimes_{i=1}^l S_{n_i}$  such that

$$P_\alpha(X_1 \otimes \cdots \otimes X_l) = X_{\alpha(1)} \otimes \cdots \otimes X_{\alpha(l)}$$

for any  $X_i \in S_{n_i}$ ,  $i = 1, \dots, l$ .

**COROLLARY 4.6.** *Let  $K$  be algebraically closed. Let  $\theta$  be a linear map from  $\bigotimes^k S_m$  to  $\bigotimes^l S_m$  such that  $\theta\left(\bigotimes^k \rho_m\right) \subseteq \bigotimes^l \rho_m$ . Then  $k \leq l$  and there exist  $B \in \bigotimes^{l-k} \rho_m$  if  $k < l$ , a permutation  $\alpha$  on  $[l]$ , and bijective linear rank-one preservers  $\theta_1, \dots, \theta_k$  on  $S_m$  such that*

$$(P_\alpha \circ \theta)(A_1 \otimes \cdots \otimes A_k) = \theta_1(A_1) \otimes \cdots \otimes \theta_k(A_k) \otimes B$$

for any  $A_1, \dots, A_k \in S_m$  where  $B$  is deleted if  $k = l$ .

*Proof.* From the proof of Theorem 4.5, we see that maximal decomposable subspaces of distinct types in  $\bigotimes^k S_m$  are mapped to decomposable subspaces of distinct types in  $\bigotimes^l S_m$ . This shows that from  $k \leq l$ . If  $k = l$ , the result follows from Theorem 4.5. If  $k < l$ , we see from the proof of Lemma 3.4 that there exist  $B \in \bigotimes^{l-k} \rho_m$ , and a permutation  $\alpha$  on  $[l]$  such that

$$\text{Im}(P_\alpha \circ \theta) \subseteq \left( \bigotimes^k S_m \right) \otimes B$$

and maximal decomposable subspaces of type- $i$  are mapped under  $P_\alpha \circ \theta$  to decomposable subspaces of type- $i$ . It is now clear that the result follows from Theorem 4.5.  $\square$

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