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Representations for the Drazin inverse of block cyclic matrices

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Abstract. A formula for the Drazin inverse of a block \( k \)-cyclic \((k \geq 2)\) matrix \( A \) with nonzeros only in blocks \( A_{i,i+1}, \) for \( i = 1, \ldots, k \) (mod \( k \)) is presented in terms of the Drazin inverse of a smaller order product of the nonzero blocks of \( A, \) namely \( B_i = A_{i,i+1} \cdots A_{i-1,i} \) for some \( i.\) Bounds on the index of \( A \) in terms of the minimum and maximum indices of these \( B_i \) are derived. Illustrative examples and special cases are given.

Key words. Drazin inverse, Block cyclic matrix, Index.

AMS subject classifications. 15A09.

1. Introduction. We consider \( k \)-cyclic \((k \geq 2)\) block real or complex matrices of the form

\[
A = \begin{bmatrix}
0 & A_{12} & 0 & \cdots & 0 \\
0 & 0 & A_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_{k-1,k} \\
A_{k1} & 0 & 0 & \cdots & 0
\end{bmatrix},
\]

where \( A_{12}, \ldots, A_{k1} \) are block submatrices and the diagonal zero blocks are square. It is easily verified that for any matrix \( A \) of the form (1.1), the Moore-Penrose inverse \( A^\dagger \) of \( A \) is given by

\[
A^\dagger = \begin{bmatrix}
0 & 0 & \cdots & 0 & A_{k1}^\dagger \\
0 & 0 & \cdots & 0 & 0 \\
A_{12}^\dagger & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_{k-1,k}^\dagger & 0
\end{bmatrix}.
\]
where $A_{ij}^{\dagger}$ denotes the Moore-Penrose inverse of the block submatrix $A_{ij}$. Note that if each of the blocks $A_{ij}$ is square and invertible, then (1.2) gives the formula for the usual inverse $A^{-1}$ of $A$. We present a block formula for another type of generalized inverse, the Drazin inverse, of matrices of the form (1.1). Unlike the Moore-Penrose inverse, the Drazin inverse is defined only for square matrices.

Let $A$ be a real or complex square matrix. The Drazin inverse of $A$ is the unique matrix $A^D$ satisfying

\begin{align}
A A^D &= A^D A \quad (1.3) \\
A^D A A^D &= A^D \quad (1.4) \\
A^{q+1} A^D &= A^q \quad (1.5)
\end{align}

where $q = \text{index } A$, the smallest nonnegative integer $q$ such that $\text{rank } A^{q+1} = \text{rank } A^q$. If $\text{index } A = 0$, then $A$ is nonsingular and $A^D = A^{-1}$. If $\text{index } A = 1$, then $A^D = A^\#$, the group inverse of $A$. See [1], [2], [6] and references therein for applications of the Drazin inverse.

**Theorem 1.1.** [2, Theorem 7.2.3] Let $A$ be a square matrix with index $A = q$. If $p$ is a nonnegative integer and $X$ is a matrix satisfying $XAX = X$, $AX =XA$, and $A^{q+1}X = X$, then $p \geq q$ and $X = A^D$.

The problem of finding explicit representations for the Drazin inverse of a general $2 \times 2$ block matrix of the form

\begin{equation}
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\end{equation}

in terms of its blocks was posed by Campbell and Meyer in [2], and special cases of this problem were the focus of several recent papers, including [3]–[10], [13], [14] and [15]. In [4] and [13], representations for $2 \times 2$ block matrices matrices of the form (1.6) with $A_{11}$ and $A_{22}$ being square zero diagonal blocks were presented. Such block matrices were called bipartite (or 2-cyclic), and in this article, we extend the results given in [4] to general block $k$-cyclic matrices as defined in (1.1).

**2. Drazin inverse formula for block cyclic matrices.** Let $A$ be a block $k$-cyclic matrix of the form given in (1.1). For our Drazin inverse formula we introduce some notation that is also used in writing powers of $A$. For $i = 2, \ldots, k-1$, let $B_i$ be the square matrix defined by

\begin{equation}
B_i = A_{i,i+1} \cdots A_{k-1,k} A_{k-1,k} A_{12} \cdots A_{i-1,i},
\end{equation}

with $B_1 = A_{12} A_{23} \cdots A_{k-1,k} A_{k-1,k}$ and $B_k = A_{k1} A_{12} \cdots A_{k-1,k},$ i.e., subscripts are taken mod $k$. For ease of notation, we define the matrix product

\begin{equation}
A_{i\to j} := A_{i,i+1} A_{i+1,i+2} \cdots A_{j-1,j},
\end{equation}

where $A_{ij}$ denotes the Moore-Penrose inverse of the block submatrix $A_{ij}$. Note that if each of the blocks $A_{ij}$ is square and invertible, then (1.2) gives the formula for the usual inverse $A^{-1}$ of $A$. We present a block formula for another type of generalized inverse, the Drazin inverse, of matrices of the form (1.1). Unlike the Moore-Penrose inverse, the Drazin inverse is defined only for square matrices.

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\end{align}

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\begin{equation}
B_i = A_{i,i+1} \cdots A_{k-1,k} A_{k-1,k} A_{12} \cdots A_{i-1,i},
\end{equation}

with $B_1 = A_{12} A_{23} \cdots A_{k-1,k} A_{k-1,k}$ and $B_k = A_{k1} A_{12} \cdots A_{k-1,k},$ i.e., subscripts are taken mod $k$. For ease of notation, we define the matrix product

\begin{equation}
A_{i\to j} := A_{i,i+1} A_{i+1,i+2} \cdots A_{j-1,j},
\end{equation}
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for \( j \neq i \). Whenever it arises, we use the convention \( A_{i \rightarrow i} = I \), an identity matrix. For example, if \( k = 4 \) then \( A_{2 \rightarrow 3} = A_{23}, A_{3 \rightarrow 2} = A_{34}A_{41}A_{12}, A_{2 \rightarrow 1} = A_{23}A_{34}A_{41}, \) and by \( B_3 = A_{34}A_{41}A_{12}A_{23} \). Observe that \( B_i = A_{i \rightarrow j}A_{j \rightarrow i} \), for any \( j \in \{1, \ldots, k\} \setminus \{i\} \).

**Lemma 2.1.** For \( A \) given in (1.1) and with the notation above, for \( p \geq 0 \),

\[
A^{kp} = \begin{bmatrix}
    B_1^p & 0 & \cdots & 0 \\
    0 & B_2^p & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & B_k^p
\end{bmatrix},
\]

\[
A^{kp+1} = \begin{bmatrix}
    0 & B_1^pA_{1 \rightarrow 2} & 0 & \cdots & 0 \\
    0 & 0 & B_2^pA_{2 \rightarrow 3} & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & B_{k-1}^pA_{k-1 \rightarrow k} \\
    B_k^pA_{k \rightarrow 1} & 0 & 0 & \cdots & 0
\end{bmatrix},
\]

\[
A^{kp+2} = \begin{bmatrix}
    0 & 0 & B_1^pA_{1 \rightarrow 3} & \cdots & 0 \\
    0 & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & B_{k-2}^pA_{k-2 \rightarrow k} \\
    B_{k-1}^pA_{k-1 \rightarrow 1} & 0 & 0 & \cdots & 0 \\
    B_k^pA_{k \rightarrow 2} & 0 & 0 & \cdots & 0
\end{bmatrix},
\]

and so on, until

\[
A^{kp+k-1} = \begin{bmatrix}
    0 & 0 & 0 & \cdots & 0 & B_1^pA_{1 \rightarrow k} \\
    B_2^pA_{2 \rightarrow 1} & 0 & \cdots & 0 & 0 \\
    0 & B_3^pA_{3 \rightarrow 2} & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & B_{k-1}^pA_{k-1 \rightarrow k} & 0 \\
    0 & 0 & \cdots & B_k^pA_{k \rightarrow k-1} & 0
\end{bmatrix}.
\]

**Lemma 2.2.** For all \( i \neq j \), \( B_i^kA_{i \rightarrow j} = A_{i \rightarrow j}B_j^k \).

**Proof.** \( B_i^kA_{i \rightarrow j} = (A_{i \rightarrow j}A_{j \rightarrow i})^kA_{i \rightarrow j} = A_{i \rightarrow j}A_{j \rightarrow i}(A_{i \rightarrow j}A_{j \rightarrow i})^{k-1}A_{i \rightarrow j} = A_{i \rightarrow j}(A_{j \rightarrow i}A_{i \rightarrow j})^k = A_{i \rightarrow j}B_j^k \). \( \Box \)

**Lemma 2.3.** For all \( i \neq j \), \( B_i^D A_{i \rightarrow j} = A_{i \rightarrow j}B_j^D \). Hence, if \( \ell \neq i, j \) satisfies \( A_{i \rightarrow j} = A_{i \rightarrow \ell}A_{\ell \rightarrow j} \), then \( B_i^D A_{i \rightarrow j} = A_{i \rightarrow j}B_j^D = A_{i \rightarrow \ell}B_{\ell}^D A_{\ell \rightarrow j} \).
Proof. \( B_i^D A_1 \rightarrow j = (A_1 \rightarrow j A_j \rightarrow i)^D A_1 \rightarrow j = A_1 \rightarrow j (A_j \rightarrow i A_i \rightarrow j)^D = A_1 \rightarrow j B_j^D \), where the second equality is due to [4, Lemma 2.4]. □

With the above notation, we now give a formula for the Drazin inverse of a \( k \)-cyclic matrix \( A \) given by (1.1).

**Theorem 2.4.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined as in (2.1) and \( A_1 \rightarrow j \) defined in (2.2). Then, for all \( i = 1, \ldots, k \),

\[
A^D = \begin{bmatrix}
0 & 0 & \cdots & 0 & A_1 \rightarrow i B_i^D A_i \rightarrow k \\
A_2 \rightarrow i B_i^D A_1 \rightarrow 1 & 0 & \cdots & 0 & 0 \\
0 & A_3 \rightarrow i B_i^D A_2 \rightarrow 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_{k-1} \rightarrow i B_i^D A_i \rightarrow k-1 & 0
\end{bmatrix}
\] (2.6)

Moreover, if index \( B_i = s_i \), then index \( A \leq ks_i + k - 1 \).

**Proof.** Denote the matrix on the right hand side of (2.6) by \( X \). Performing block multiplication gives

\[
AX = \begin{bmatrix}
A_{12} A_2 \rightarrow i B_i^D A_1 \rightarrow 1 & 0 & \cdots & 0 \\
0 & A_{23} A_3 \rightarrow i B_i^D A_2 \rightarrow 1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{k1} A_1 \rightarrow i B_i^D A_{i-1} \rightarrow k
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A_{12} A_2 \rightarrow i A_1 \rightarrow 1 B_i^D & 0 & \cdots & 0 \\
0 & A_{23} A_3 \rightarrow i A_2 \rightarrow 1 B_i^D & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{k1} A_1 \rightarrow i A_{i-1} \rightarrow k B_i^D
\end{bmatrix}
\]

(by Lemma 2.3)

\[
= \begin{bmatrix}
B_1 B_1^D & 0 & \cdots & 0 \\
0 & B_2 B_2^D & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_k B_k^D
\end{bmatrix}
\]
and by using Lemma \ref{lem3} again

\[
XA = \begin{bmatrix}
A_{1\rightarrow i}B_1^DA_{i\rightarrow k}A_{k1} & 0 & \cdots & 0 \\
0 & A_{2\rightarrow i}B_1^DA_{i\rightarrow 1}A_{12} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & A_{k\rightarrow i}B_1^DA_{i\rightarrow k-1}A_{k-1,k}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
B_1^DA_{1\rightarrow i}A_{i\rightarrow k}A_{k1} & 0 & \cdots & 0 \\
0 & B_2^DA_{2\rightarrow i}A_{i\rightarrow 1}A_{12} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & B_k^DA_{k\rightarrow i}A_{i\rightarrow k-1}A_{k-1,k}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
B_1^DB_1 & 0 & \cdots & 0 \\
0 & B_2^DB_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_k^DB_k
\end{bmatrix}
\begin{bmatrix}
B_1^DB_1 & 0 & \cdots & 0 \\
0 & B_2^DB_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_k^DB_k
\end{bmatrix}
\]

\[
= AX,
\]

since \(B_i^DB_i = B_iB_i^D\) by \ref{lem:1.3}. Also, block-multiplying \(X\) with \(AX\) gives

\[
XAX = X(AX)
= \begin{bmatrix}
0 & 0 & \cdots & 0 & A_{1\rightarrow k}B_1^DB_kB_k^D \\
A_{2\rightarrow i}B_1^DB_iB_i^D & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_{k\rightarrow k-1}B_{k-1}^DB_{k-1}B_{k-1}^D & 0
\end{bmatrix}
\]

\[
= X, \text{ by Lemma } \ref{lem3} \text{ and since } B_i^DB_iB_i^D = B_i^D \text{ by } \ref{lem:1.4}.
\]

Let \(i\) be any integer in \(\{1, \ldots, k\}\) and suppose that index \(B_i = s_i = s\). Then using

\ref{lem:1.3} and Lemma \ref{lem2}

\[
A^{k+s+k}X = A^{k(s+1)}X
= \begin{bmatrix}
0 & 0 & \cdots & 0 & A_{1\rightarrow i}B_i^{s+1}B_i^DA_{i\rightarrow k} \\
A_{2\rightarrow i}B_i^{s+1}B_i^DA_{i\rightarrow 1} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & A_{k\rightarrow i}B_i^{s+1}B_i^DA_{i\rightarrow k} & \cdots & 0 & 0
\end{bmatrix}
\]
Since index $B_i = s$, it follows by (1.5) that $B_i^{s+1}B_i^D = B_i^s$. Thus, using Lemma 2.2 and $A_{\ell \to i}A_{i \to j} = A_{\ell \to j}$ for $\ell \neq j$,

$$
A^{ks+k}X = \begin{bmatrix}
0 & 0 & \cdots & 0 & A_{1 \to k}B_k^s \\
A_{2 \to 1}B_1^s & 0 & \cdots & 0 & 0 \\
0 & A_{3 \to 2}B_2^s & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_{k \to k-1}B_{k-1}^s & 0
\end{bmatrix}
$$

from (2.5) by using Lemma 2.2. By Theorem 1.1, index $\lambda \leq ks + k - 1$ and $X = A^D$. \[\Box\]

Thus, the Drazin inverse of a $k$-cyclic matrix is reduced to calculating the Drazin inverse of the smallest order Drazin inverse of any of the matrix products $B_i$.

**Corollary 2.5.** If $A$ of the form in (1.1) is nonnegative and has at least one $B_iD_i \geq 0$, then $A^D$ is nonnegative.

The following example illustrates Theorem 2.4 and Corollary 2.5.

**Example 2.6.** Let

$$
A = \begin{bmatrix}
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & A_{12} & 0 \\
0 & 0 & A_{23} \\
A_{31} & 0 & 0
\end{bmatrix}.
$$

Then $B_1 = A_{12}A_{23}A_{31} = 1, B_2 = A_{23}A_{31}A_{12} = \frac{1}{2}J_2$ (where $J_2$ is $2 \times 2$ all ones matrix) and $B_3 = A_{31}A_{12}A_{23} = 1$. Note that index $\lambda = 0$ and $B_2^D = B_1^{-1} = 1$. Using Theorem 2.4,

$$
A^D = \begin{bmatrix}
0 & 0 & B_1^D A_{12}A_{23} \\
A_{24}A_{31}B_2^D & 0 & 0 \\
0 & A_{31}B_2^DA_{12} & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0
\end{bmatrix} = A^2.
$$

In fact, $\text{rank } A = \text{rank } A^2$, hence $A^D = A^\# = A^2$ agreeing with Theorem 2.2 in [11].

**3. Index of $A$ in relation to the indices of the block products.** With $A$ as in (1.1), for $j \geq 0$, by (2.3) and (2.4),

\begin{align*}
(3.1) & \quad \text{rank } A^{kj} = \text{rank } B_i^j + \text{rank } B_2^j + \cdots + \text{rank } B_k^j \\
(3.2) & \quad \text{rank } A^{kj+1} = \text{rank } B_i^jA_{12} + \text{rank } B_2^jA_{23} + \cdots + \text{rank } B_k^jA_{k1}.
\end{align*}
The following rank inequality is used throughout the proof of Lemma 3.2 and can be found in standard linear algebra texts (see, e.g., [12, page 13]).

**Lemma 3.1.** (Frobenius Inequality) If \( U \) is \( m \times n \), \( V \) is \( n \times p \) and \( W \) is \( p \times q \), then

\[
\text{rank } UV + \text{rank } VW \leq \text{rank } V + \text{rank } UVW.
\]

**Lemma 3.2.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined in (1.1), and let \( s = \text{index } B_i \geq 1 \) for some \( i \in \{1, \ldots, k\} \). Then \( \text{rank } A^{k_s - k + 1} < \text{rank } A^{k_s - k} \).

**Proof.** Let \( s = \text{index } B_i \) for some \( i \in \{1, \ldots, k\} \). From (3.2),

\[
\text{rank } A^{k_s - k + 1} = \text{rank } A^{k(s-1)+1} = \text{rank } B_{i}^{s-1} A_{12} + \text{rank } B_{i}^{s-1} A_{23} + \text{rank } B_{i}^{s-1} A_{34} + \cdots + \text{rank } B_{i}^{s-1} A_{k1},
\]

where the terms can be reordered as

\[
\text{rank } B_{i}^{s-1} A_{i,i+1} + \text{rank } B_{i}^{s-1} A_{i+1,i+2} + \cdots + \text{rank } B_{i}^{s-1} A_{k1} + \text{rank } B_{i}^{s-1} A_{12} + \cdots + \text{rank } B_{i}^{s-1} A_{i-1,i},
\]

(3.3)

Using Lemma 2.2, the first two terms in the expression in (3.3) can be written as

\[
\text{rank } A_{i,i+1} B_{i+1}^{s-1} + \text{rank } B_{i+1}^{s-1} A_{i+1,i+2},
\]

and using the Frobenius inequality (Lemma 3.1),

\[
\text{rank } B_{i}^{s-1} A_{i,i+1} + \text{rank } B_{i}^{s-1} A_{i+1,i+2} \leq \text{rank } B_{i+1}^{s-1} + \text{rank } A_{i,i+1} B_{i+1}^{s-1} A_{i+1,i+2} = \text{rank } B_{i+1}^{s-1} + \text{rank } B_{i}^{s-1} A_{i,i+2},
\]

where the equality is again due to Lemma 2.2. Thus,

\[
\text{rank } A^{k_s - k + 1} \leq \text{rank } B_{i+1}^{s-1} + \text{rank } B_{i}^{s-1} A_{i,i+2} + \text{rank } B_{i+2}^{s-1} A_{i+2,i+3} + \cdots + \text{rank } B_{i}^{s-1} A_{i-1,i},
\]

(3.4)

Applying Lemma 2.2 and the Frobenius inequality again to the second and third terms on the right hand side of the inequality in (3.4) gives

\[
\text{rank } B_{i}^{s-1} A_{i,i+2} + \text{rank } B_{i+2}^{s-1} A_{i+2,i+3} \leq \text{rank } B_{i+2}^{s-1} + \text{rank } A_{i+3} B_{i+3}^{s-1}.
\]

Continuing in this manner gives

\[
\text{rank } A^{k_s - k + 1} \leq \text{rank } B_{i+1}^{s-1} + \text{rank } B_{i+2}^{s-1} + \cdots + \text{rank } B_{i}^{s-1} + \text{rank } A_{i-1,i} B_{i-1}^{s-1} A_{i-1,i}.
\]

(3.5)
Using Lemma 2.2, the last term on the righthand side of the inequality in \(3.5\) becomes
\[
\text{rank } B_i^{s+1} A_{i \rightarrow i-1} A_{i-1,i} = \text{rank } B_i^{s+1} B_i = \text{rank } B_i^s < \text{rank } B_i^{s+1},
\]
since index \(B_i = s\). Thus,
\[
\text{rank } A^{ks-k+1} < \text{rank } B_i^{s+1} + \text{rank } B_i^{s-1} + \cdots = \text{rank } A^{k(s-1)} = \text{rank } A^{ks-k},
\]
where the equality follows from (3.1).

**Theorem 3.3.** Let \(A\) be as in (1.1) with associated matrices \(B_i\) defined in (2.1). Then, the following statements hold.

(i) If index \(B_i = 0\) for all \(i = 1, \ldots, k\), then \(A\) is nonsingular and index \(A = 0\).

(ii) If index \(B_i = s_i \geq 1\) for some \(i \in \{1, \ldots, k\}\), then index \(A \geq ks_i - k + 1\).

**Proof.** The first statement follows immediately from (2.3) and (3.1). For the second statement, let index \(B_i = s_i \geq 1\) for some \(i \in \{1, \ldots, k\}\). Then \(\text{rank } A^{ks_i-k+1} < \text{rank } A^{ks_i-k}\), by Lemma 3.2. From the strict inequality, index \(A \geq ks_i - k + 1\).

The next result follows immediately from Theorem 3.3(ii).

**Corollary 3.4.** Let \(A\) be as in (1.1) with associated matrices \(B_i\) defined in (2.1). If index \(A \leq 1\), then index \(B_i \leq 1\) for all \(i = 1, \ldots, k\). That is, if the group inverse \(A^\#\) exists, then the group inverse \(B_i^\#\) exist for all \(i = 1, \ldots, k\).

Note however that the converse to Corollary 3.4 is false (see, e.g., [4, Example 4.3]).

**Remark 3.5.** If \(A\) of the form (1.1) is nonnegative and all matrices with the same +,0 sign pattern as \(A\) that have index 1 have at least one \(B_i^\#\) nonnegative, then these group inverses are nonnegative (Corollary 2.5) and \(A\) is conditionally \(S^2GI\) in the notation of Zhou et al. [15].

**Corollary 3.6.** Let \(A\) be as in (1.1) with associated matrices \(B_i\) defined in (2.1), and let \(s = \min_{1 \leq i \leq k} \text{index } B_i\) and \(s' = \max_{1 \leq i \leq k} \text{index } B_i > 0\). Then \(ks' - k + 1 \leq \text{index } A \leq ks + k - 1\). If \(s' = 0\), then index \(A = 0\).

Corollary 3.6 leads to a result about the indices of \(B_i\) that is of independent interest.

**Theorem 3.7.** Let \(A\) be as in (1.1) with associated matrices \(B_i\) defined in (2.1), and let \(s_\ell = \text{index } B_\ell\) for \(\ell \in \{1, \ldots, k\}\). Then \(|s_i - s_j| \leq 1\) for all \(i, j \in \{1, \ldots, k\}\).
Proof. Let \( s = \min_{1 \leq i \leq k} \text{index } B_i \) and \( s' = \max_{1 \leq i \leq k} \text{index } B_i \), and suppose that \( s' = s + t \) where \( t \geq 0 \). By Corollary 3.6

\[
ks + k - 1 \leq \text{index } A \leq ks + k - 1.
\]

It follows that

\[
k(s + t) - k + 1 \leq \text{index } A \leq ks + k - 1,
\]

or equivalently,

\[
k(t - 2) + 2 \leq 0.
\]

As \( k \geq 2 \), the inequality above is possible only if \( t \leq 1 \). Thus, \( s' - s = t \leq 1 \) and \( |\text{index } B_i - \text{index } B_j| \leq 1 \) for all \( i, j \).

The next result gives tight bounds on \( \text{index } A \) in terms of the minimum index of the block products \( B_i \). The proof is immediate from Corollary 3.6 and Theorem 3.7.

**Theorem 3.8.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined in (2.1), and let \( s = \min_{1 \leq i \leq k} \text{index } B_i \). Then, exactly one of the following holds:

(i) \( \text{index } B_i = s \) for all \( i = 1, \ldots, k \), or

(ii) \( \text{index } B_i = s + 1 \) for some \( i = 1, \ldots, k \).

If (i) holds, then \( ks - k + 1 \leq \text{index } A \leq ks + k - 1 \). If (ii) holds, then \( ks + 1 \leq \text{index } A \leq ks + k - 1 \).

The above result generalizes bounds found in [4, Section 3] and shows that if \( k = 2 \) and (ii) holds, then \( \text{index } A = 2s + 1 \).

We now give examples that illustrate Theorem 3.8.

**Example 3.9.** Let \( A \) be the matrix in Example 2.6. Using the notation in Theorem 3.8, \( s = 0 = \text{index } B_1 = \text{index } B_3 \) and \( \text{index } B_2 = 1 = s + 1 \). Applying the result with \( k = 3 \) gives the bounds \( 1 \leq \text{index } A \leq 2 \). Since \( \text{rank } A = \text{rank } A^2 \), \( \text{index } A = 1 = ks + 1 \), which is the lower bound of Theorem 3.8, case (ii).

**Example 3.10.** Let

\[
A = \begin{bmatrix}
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]


\[
= \begin{bmatrix}
0 & A_{12} & 0 \\
0 & 0 & A_{23} \\
A_{31} & 0 & 0
\end{bmatrix}
\]
Then $B_1 = 3, B_2 = \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$ and $B_3 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. Note that $\text{index } B_1 = 0$ and $B_1^{-1} = \frac{1}{3}$. Using Theorem 2.4,

$$A^D = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \end{bmatrix}.$$ 

Using the notation in Theorem 3.8, $s = 0 = \text{index } B_1$ and $\text{index } B_2 = \text{index } B_3 = 1 = s + 1$. Applying the theorem with $k = 3$ gives the bounds $1 \leq \text{index } A \leq 2$. It can be computed that $\text{index } A = 2 = ks + k - 1$, which is the upper bound of Theorem 3.8 case (ii).

**Example 3.11.** Let

$$A = \begin{bmatrix} 0 & B & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ I & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $B$ is a square matrix and $I$ is an identity matrix of the same order as $B$. Note that $B_i = B$ for all $i$. Suppose that $\text{index } B = s$. Then $\text{index } A = ks$, the midpoint of the interval $[ks - k + 1, ks + k - 1]$ in Theorem 3.8 case (i), and from Theorem 2.4

$$A^D = \begin{bmatrix} 0 & 0 & \cdots & 0 & B^D B \\ B^D & 0 & \cdots & 0 & 0 \\ 0 & B^D B & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B^D B & 0 \end{bmatrix}.$$
Example 3.12. Let

\[ A = \begin{bmatrix}
0 & F & 0 & \cdots & 0 \\
0 & 0 & F & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & F \\
F & 0 & 0 & \cdots & 0
\end{bmatrix}, \]

where \( F \) is a square matrix. Then index \( A = \text{index } F \) and \( B_i = F^k \) for \( i = 1, \ldots, k \).

Setting index \( A = \ell \) and index \( B_i = s \) gives \( s = \lceil \frac{\ell}{k} \rceil \). Thus, index \( A \) can take any value in the interval \([ks-k+1, ks]\), which is half the range given in Theorem 3.8 case (i).

Examples 5.11 and 6.12 have \( B_i \), and thus index \( B_i \), the same for all \( i \). The following result determines index \( A \) in this case, and the necessary and sufficient conditions reduce to the result of [3, Theorem 3.5] for \( k = 2 \).

Theorem 3.13. Let \( A \) be a block \( k \)-cyclic matrix of the form in (1.7) with associated matrices \( B_i \) defined in (2.1), and suppose that \( s = \min \{ \text{index } B_i \} \geq 1 \).

Then rank \( A = ks \) if and only if

(i) index \( A = ks \) for all \( i = 1, \ldots, k \), and

(ii) rank \( B_j \leq \sum_{i=1}^{k} \text{rank } B_i \) for some \( j \in \{1, \ldots, k\} \).

If (i) holds, then rank \( B_i \) for all \( i, j = 1, \ldots, k \). If (i) holds but (ii) does not hold, then index \( A < ks \).

Proof. Suppose that index \( A = ks \). Then rank \( A^{ks} \leq \text{rank } A^{ks-1} \). It follows, using (2.5), (2.6) and (5.1), that \( \sum_{i=1}^{k} \text{rank } B_i^{s} \leq \sum_{i=1}^{k} \text{rank } A^{s-1} A_{i-s-1} = 1 \). Thus, rank \( B_j^{s} \leq \text{rank } A_{1-j-1}^{s-1} \) for some \( j \in \{1, \ldots, k\} \), hence (ii) holds. Suppose on the contrary that (i) does not hold. Then, for some \( j \in \{1, \ldots, k\} \), index \( B_j = s + 1 \) (by Theorem 3.8). Thus, rank \( B_j^{s+1} \geq \text{rank } A^{s+1} \). Hence, by (3.5) rank \( A^{ks} = \sum_{i=1}^{k} \text{rank } B_i^{s} \geq \sum_{i=1}^{k} \text{rank } B_i^{s+1} = \text{rank } A^{k(s+1)} \). This implies that rank \( A^{ks} > \text{rank } A^{ks+k}, \) so index \( A > ks \), a contradiction. Hence, (i) and (ii) must hold.

For the reverse implication, suppose that (i) and (ii) hold. Then rank \( A^{ks} = \sum_{i=1}^{k} \text{rank } B_i^{s} \leq \sum_{i=1}^{k} \text{rank } A^{s-1} A_{i-s-1} = 1 \). Thus, index \( A \geq ks \). Note that since rank \( B_i^{s} \geq \text{rank } B_i^{s+1} \) and rank \( B_i^{s+1} \geq \text{rank } A^{s+1} \) by Lemma 2.2, it follows using (i) that rank \( B_i^{s} \geq \text{rank } B_i^{s+1} = \text{rank } A^{s+1} = \text{rank } A^{s+1} = \text{rank } A^{s+1} = \text{rank } A^{s+1} = \text{rank } A^{s+1} = \text{rank } A^{s+1}, \) using (3.1) and (3.2). Hence, rank \( A^{ks} = \text{rank } A^{ks+1} \), and so index \( A \leq ks \). This
proves that index \( A = ks \). The last two statements of the theorem follow from the proof above.

The result of Theorem 3.13 is illustrated by Example 3.11 since rank \( B^s_2 < \) rank \( B^{-1}_2 \cdot A_{2\rightarrow 1} = B^{-1}_2 \), it follows that rank \( A = ks \). Example 3.12 also illustrates Theorem 3.13 since rank \( A^{ks} = rank A^{ks+1} \) and rank \( F^{ks} < rank F^{k(s-1)} F^{k-1} \) = rank \( F^{ks-1} \) if and only if index \( F = index A = ks \); otherwise index \( A < ks \).

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REFERENCES


