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ON THE NUMERICAL RANGES OF THE WEIGHTED SHIFT OPERATORS WITH GEOMETRIC AND HARMONIC WEIGHTS

ADIYASUREN VANDANJAV AND BATZORIG UNDRAKH

Abstract. In this paper, an exact formula for \( \det (tI_n - (Q_n + Q_n^*)) \) is obtained. This formula yields a simple computation of the numerical ranges of the geometric weighted shift operator \( Q_n \) and the harmonic weighted shift operator \( H_n \) for \( n = 3, 4 \).

Key words. Numerical range, Weighted shift operators, Geometric weights, Harmonic weights.

AMS subject classifications. 15A60, 47A12.

1. Introduction. The numerical range of an \( n \times n \) matrix \( T \) is defined as the set

\[
W(T) = \{ \langle Tx, x \rangle : \|x\| = 1 \}
\]

where \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) denote the standard inner product and its associated norm in \( \mathbb{C}^n \). It is known that \( W(T) \) is a nonempty convex subset of \( \mathbb{C} \); see for example [3]. The numerical radius \( w(T) \) of a matrix \( T \) is given by

\[
w(T) = \sup \{ |\lambda| : \lambda \in W(T) \}.
\]

For its other properties, see [3].

A shift matrix

\[
T = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
a_1 & 0 & 0 & 0 & \cdots & 0 \\
0 & a_2 & 0 & 0 & \cdots & 0 \\
0 & 0 & a_3 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & a_{n-1} & 0
\end{bmatrix},
\]

and a diagonal matrix

\[
U = \text{diag} (1, \exp(i\theta), \exp(2i\theta), \exp(3i\theta), \ldots, \exp((n-1)i\theta))
\]
satisfy the equation

\[ UTU^* = \exp(i\theta)T, \]

and hence, \( \exp(i\theta)W(T) = W(T) \) for \( 0 \leq \theta \leq 2\pi \). For a shift matrix, the numerical radius \( w(T) \) is characterized as the maximum root of the characteristic polynomial

\[ P(x) = \det \left( xI_n - \frac{1}{2}(T + T^*) \right). \]

In [2], the value

\[ M(\theta) = \max \{ \Re(z \exp(-i\theta)) : z \in W(T) \} \]

for a matrix \( T \) is characterized as the maximum eigenvalue of a hermitian matrix

\[ \frac{1}{2}(\exp(i\theta)T + \exp(-i\theta)T^*) \]

(\( 0 \leq \theta \leq 2\pi \)). If \( T \) is a shift matrix, then the numerical range \( W(T) \) is a closed circular disc with center at the origin, and hence, \( w(T) \) is the maximum eigenvalue of a hermitian matrix \( (T + T^*)/2 \).

We consider a weighted shift operator \( A \) on the Hilbert space \( \ell^2(\mathbb{N}) \) defined by

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 \\
a_1 & 0 & 0 & 0 & \ldots & 0 \\
0 & a_2 & 0 & 0 & \ldots & 0 \\
0 & 0 & a_3 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & a_n \\
\end{bmatrix}
\]

(1.2)

where \( \{a_n\} \) is a bounded sequence. The numerical range is also defined for Hilbert space operators. It is known that \( W(A) \) is a circular disk centered at the origin [5]. In particular, if the weights are geometric \( a_n = q^{n-1} \) for some \( 0 < q < 1 \) and \( n \in \mathbb{N} \), then the numerical range of \( T_n \) is closed disc centered at the origin [1]. Furthermore, the authors of [1] found upper and lower bounds for \( w(T) \). However, we do not use their result and we develop a simple and different method to solve it. Consider the following two finite operators

\[
Q_n = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & 0 & \ldots & 0 \\
0 & q & 0 & 0 & \ldots & 0 \\
0 & 0 & q^2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & a_n \\
\end{bmatrix}
\]

(1.3)
where $0 < q < 1$ and

$$H_n = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
0 & \frac{1}{q} & 0 & \ldots & 0 \\
0 & 0 & \frac{1}{q} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & \frac{1}{n-1} & 0
\end{bmatrix}. \tag{1.4}$$

In this paper, we study the numerical ranges of matrices defined in (2.10) and (1.4). We give a general exact formula for $\det (tI_n - (Q_n + Q_n^*))$. Using this exact formula for $n = 3, 4$, we verify that $W(Q_n)$ and $W(H_n)$ are closed disks centered at the origin.

2. Geometric weights.

**Theorem 2.1.** Let

$$f_m = \det (zI_m - (Q_m + Q_m^*)). \tag{2.1}$$

Then we have

$$f_m(z) = z^m + \sum_{k=1}^{\left\lfloor \frac{m}{2} \right\rfloor} (-1)^k z^{m-2k} p^{k(k-1)} \prod_{i=1}^{k} \frac{1-p^{m-2k+i}}{1-p^i}, \tag{2.2}$$

where $p = q^2$, for $m \geq 2$.

**Proof.** Let $q^2 = p$. Assume that $f_0(z) = 1$, $f_1(z) = z$. Then we have $f_2(z) = z^2 - 1$, $f_3(z) = z^3 - z(1 + p)$. Expanding on the last row of the matrix (2.1) leads to the recurrence formula

$$f_{k+2}(z) = zf_{k+1}(z) - p^k f_k(z). \tag{2.3}$$

Now we prove (2.3) by induction method. We prove the formula (2.2) for the case $m = 2n$ and the case $m = 2n + 1$ can be done in an analogous way: $m = 2$ is trivial. Now assume that (2.2) is holds for $m = 2, 3, \ldots, n, n + 1$ then we prove that for $m = n + 2$.

$$f_m(z) = z^{2n} + \sum_{k=1}^{n} (-1)^k z^{2n-2k} p^{k(k-1)} \prod_{i=1}^{k} \frac{1-p^{2n-2k+i}}{1-p^i}, \tag{2.4}$$

$$f_{m+1}(z) = z^{2n+1} + \sum_{k=1}^{n} (-1)^k z^{2n+1-2k} p^{k(k-1)} \prod_{i=1}^{k} \frac{1-p^{2n+1-2k+i}}{1-p^i}. \tag{2.5}$$
On the other hand, we have

\[(2.8)\]

\[f_{m+2} = zf_{m+1}(z) - p^m f_m(z)\]

\[= z^{2n+2} + \sum_{k=1}^{n} (-1)^k z^{2n+2-2k} p^k (k-1) \prod_{i=1}^{k} \frac{1 - p^{2n+1-2k+i}}{1 - p^i} \]

\[= -p^{2n} z^{2n} - p^{2n} \sum_{k=1}^{n} (-1)^k z^{2n-2k} p^k (k-1) \prod_{i=1}^{k} \frac{1 - p^{2n-2k+i}}{1 - p^i}.
\]

Substituting \((2.4)\) and \((2.5)\) into the \((2.3)\), we have

\[(2.7)\]

\[= z^{2n+2} + \sum_{k=1}^{n} (-1)^k z^{2n+2-2k} p^k (k-1) \prod_{i=1}^{k} \frac{1 - p^{2n+1-2k+i}}{1 - p^i} \]

On the other hand, we have

\[(2.7)\]

\[= (-1)^k z^{2n+2-2k} p^k (k-1) \prod_{i=1}^{k} \frac{1 - p^{2n+1-2k+i}}{1 - p^i} \]

\[= (-1)^k z^{2n-2k+2} p^k (k-1) \prod_{i=1}^{k-1} \frac{1 - p^{2n-2k+2+i}}{1 - p^i} \]

\[= (-1)^k z^{2n-2k+2} p^k (k-1) \prod_{i=1}^{k-1} \frac{1 - p^{2n-2k+3} \cdot (1 - p^{2n-2k+4}) \cdots (1 - p^{2n-k+1})}{(1 - p)(1 - p^2) \cdots (1 - p^{n-1})} \]

\[= (-1)^k z^{2n+2-2k} p^k (k-1) \prod_{i=1}^{k} \frac{1 - p^{2n+1-2k+i}}{1 - p^i} \]

Also

\[(2.8)\]

\[= z^{2n} \cdot p^0 \cdot \frac{1 - p^{2n}}{1 - p} - p^{2n} z^{2n} - p^{2n} z^{2n} = -z^{2n} \left( \frac{1 - p^{2n}}{1 - p} + p^{2n} \right) \]

\[= -z^{2n} \left( \frac{1 - p^{2n+1}}{1 - p} \right), \]

and

\[-p^{2n} \cdot (-1)^n \cdot z^0 \cdot p^{n(n-1)} \prod_{i=1}^{n} \frac{1 - p^i}{1 - p} = -p^{2n} (-1)^n \cdot p^{n^2-n} = (-1)^{n+1} \cdot p^{n^2+n} \]

\[(2.9)\]

\[= (-1)^{n+1} z^{2n+2-2(n+1)} \cdot p^{n(n+1)} \prod_{i=1}^{n+1} \frac{1 - p^i}{1 - p^i} \]

From \((2.7), (2.8)\) and \((2.9)\), it follows

\[f_{m+2}(z) = z^{2n+2} + \sum_{k=1}^{n+1} (-1)^k z^{2n+2-2k} p^k (k-1) \prod_{i=1}^{k} \frac{1 - p^{2n+2-2k+i}}{1 - p^i} \]
Hence, \((2.2)\) is proved. \(\square\)

Now we give a simple proof of a well known result; see for example [4].

**Theorem 2.2.** Let \(S\) be the shift matrix

\[
S = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & 0 & 1 & 0
\end{bmatrix},
\]

then the numerical range of \(S\) is a closed disc with centered at origin and \(w(S) = \cos \left( \frac{\pi}{n+1} \right) \).

**Proof.** In Theorem 2.1, we set \(q = 1\), Then we have

\[
f_n(z) = z^n + \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k C_{n-k}^k z^{n-2k}.
\]

Recalling the Chebyshev polynomials of the second kind, \(U_n(x)\), we have

\[
U_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k C_{n-k}^k (2x)^{n-2k} = \prod_{k=1}^{n} \left( x - \cos \left( \frac{k\pi}{n+1} \right) \right).
\]

If we substitute \(x = \frac{2}{n} z\), then we have

\[
U_n \left( \frac{2}{n} z \right) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k C_{n-k}^k (z)^{n-2k} = \prod_{k=1}^{n} \left( \frac{z}{2} - \cos \left( \frac{k\pi}{n+1} \right) \right)
\]

\[
= \frac{1}{2^n} \prod_{k=1}^{n} \left( z - 2 \cos \left( \frac{k\pi}{n+1} \right) \right).
\]

Now from \((2.11)\) and \((2.12)\), it follows

\[
f_n(z) = \frac{1}{2^n} \prod_{k=1}^{n} \left( z - 2 \cos \left( \frac{k\pi}{n+1} \right) \right)\).
\]

Hence, as we mentioned Section 1 and from \([2]\), we have

\[w(S) = \cos \left( \frac{\pi}{n+1} \right)\).
and the numerical range of $S$ is circular disc with centered at origin. □

**Proposition 2.3.** Let $Q_3$ be the operator in $\mathbb{C}^3$ defined by the matrix

$$Q_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & q & 0 \end{bmatrix}, \quad 0 < q < 1. \tag{2.13}$$

Then the numerical range of $Q_3$ is a closed disk centered at the origin and the radius is $\frac{\sqrt{1+q^2}}{2}$, i.e.,

$$W(Q_3) = \mathcal{D}(0; \frac{\sqrt{1+q^2}}{2}). \tag{2.14}$$

**Proof.** Setting $m = 3$ in (2.2) yields $f_3(z) = z^3 - z(1 + p)$. The maximum root of the equation $f_3(z) = 0$ is $\sqrt{1 + q^2}$. Then, as we mentioned above in Section 1 (see [2]), it is easy to see that $W(Q_3) = \mathcal{D}(0; \frac{\sqrt{1+q^2}}{2})$. □

**Proposition 2.4.** Let $Q_4$ be the operator in $\mathbb{C}^4$ defined by the matrix

$$Q_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q^2 & 0 \end{bmatrix}, \quad 0 < q < 1. \tag{2.15}$$

Then the numerical range of $Q_3$ is a closed disk centered at the origin and radius is

$$\frac{1}{2} \sqrt{\frac{1}{2} \left((1 + q^2 + q^4) + \sqrt{(1-q^2+q^4)(1+3q^2+q^4)}\right)},$$

i.e.,

$$W(Q_4) = \mathcal{D}(0; \frac{1}{2} \sqrt{\frac{1}{2} \left((1 + q^2 + q^4) + \sqrt{(1-q^2+q^4)(1+3q^2+q^4)}\right)}).$$

**Proof.** Setting $m = 4$ in (2.2) yields $f_4(z) = z^4 - z^2(1 + p + p^2) + p^2$. The maximum root of the equation $f_4(z) = 0$ is

$$\sqrt{\frac{1}{2} \left((1 + q^2 + q^4) + \sqrt{(1-q^2+q^4)(1+3q^2+q^4)}\right)}.$$

Then, as we mentioned above in Section 1 (see [2]), it is easy to see that

$$W(Q_4) = \mathcal{D}(0; \frac{1}{2} \sqrt{\frac{1}{2} \left((1 + q^2 + q^4) + \sqrt{(1-q^2+q^4)(1+3q^2+q^4)}\right)}).$$
3. Harmonic weights. In this section, we find $W(H_n)$ for $n = 3, 4$. We have

$$H_n + H_n^* = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & \ldots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} & \ldots & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & \frac{1}{n-1} \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

and let

$$P_n(x) = \det (xI_n - (H_n + H_n^*)).$$

We can assume that $P_0(x) = 1$, $P_1(x) = x$. Then we have

$$P_2(x) = 4(x^2 - 1), P_3(x) = 9(4x^3 - 5x).$$

Expanding on the last row of the matrix 3.2 leads to the recurrence formula

$$P_n(x) = n^2 (xP_{n-1} - P_{n-2} (x)), \quad n \geq 2.$$ 

Now we find the numerical range of $H_n$ for $n = 3, 4$ by using the recurrence formula 3.3.

**Proposition 3.1.** In $\mathbb{C}^3$ let $H_3$ be the operator defined by the matrix

$$H_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}.$$ 

Then the numerical range of $H_3$ is a closed disk centered at the origin and radius is $\frac{\sqrt{5}}{4}$, i.e.,

$$W(H_3) = \overline{D} \left( 0; \frac{\sqrt{5}}{4} \right).$$

**Proof.** In 3.3, we set $n = 3$. Then we have $P_3(x) = 9(4x^3 - 5x)$. The maximum root of the equation $P_3(x) = 0$ is $\frac{\sqrt{5}}{2}$. Then as we mentioned above in Section 1 (see [2]), it is easy to see that $W(H_3) = \overline{D} \left( 0; \frac{\sqrt{5}}{4} \right)$. \qed
**Proposition 3.2.** In $\mathbb{C}^4$, let $H_4$ be the operator defined by the matrix

\[
H_4 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 \\
0 & 0 & 1/2 & 0 \\
\end{bmatrix}.
\]

Then, the numerical range of $H_4$ is a closed disk centered at the origin with radius equal to $\sqrt{\frac{49 + 5\sqrt{73}}{72}}$, i.e.,

\[
W(H_4) = \mathbb{D} \left( 0; \frac{1}{2} \sqrt{\frac{49 + 5\sqrt{73}}{72}} \right).
\]

**Proof.** In (3.3), we set $n = 4$. Then we have $P_4(x) = 16(36x^4 - 49x^2 + 4)$, The maximum root of the equation $P_4(x) = 0$ is $\frac{1}{2} \sqrt{\frac{49 + 5\sqrt{73}}{72}}$. Then, as we mentioned above in Section 1 (see [2]), it is easy to see that

\[
W(H_4) = \mathbb{D} \left( 0; \frac{1}{2} \sqrt{\frac{49 + 5\sqrt{73}}{72}} \right). \quad \square
\]

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