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ON THE SPECTRAL RADIi OF GRAPHS
WITHOUT GIVEN CYCLES

WANLIAN YUAN†, BING WANG†, AND MINGQING ZHAI‡

Abstract. Let $G$ be a graph with $n$ vertices and $\rho(G)$ be the spectral radius of its adjacency matrix. Write $C_l$ for the cycle of order $l$ and let $g_l(n) = \max\{\rho(G) : |V(G)| = n, \text{ neither } C_l \text{ nor } C_{l+1} \text{ is a subgraph of } G\}$. This paper obtains the exact value of $g_5(n)$ with the unique extremal graph.

Key words. Forbidden subgraph, Adjacency matrix, Spectral radius.

AMS subject classifications. 05C50.

1. Introduction. Let $V(G)$ be the vertex set of a graph $G$ and

$$N^d(u) = \{v | v \in V(G), d_G(v, u) = d\},$$

where $d_G(v, u)$ is the distance between two vertices $u$ and $v$. Denote by $d_G(u)$, the degree of $u$. A vertex of degree $k$ is called a $k$-vertex. For a nonempty subset $S$ of $V(G)$, the subgraph induced by $S$ is denoted by $G[S]$. Let $A(G)$ be the adjacency matrix of $G$ and $P(G, \lambda)$ be the characteristic polynomial of $A(G)$. The largest modulus of an eigenvalue of $A(G)$ is called the spectral radius of $G$ and denoted by $\rho(G)$. It is valuable to study the relation between spectral radius and some kinds of subgraphs (such as, clique, path, cycle, complete bipartite subgraph, etc). See [2, 3, 5, 6, 7] for results along these lines.

We use $C_r$, $P_r$, $K_{1,r-1}$ and $K_r$ to denote the cycle, path, star and complete graph of order $r$, respectively. In particular, $K_{1,0} \cong K_1$. For each positive integer $l \geq 3$, V. Nikiforov [7] defined a function $g_l(n)$ as follows.

$$g_l(n) = \max\{\rho(G) : |V(G)| = n, \text{ neither } C_l \text{ nor } C_{l+1} \text{ is a subgraph of } G\}.$$

Favaron, Mahéo and Saclé [4] showed that if a graph $G$ of order $n$ contains neither $C_3$ nor $C_4$, then $\rho(G) \leq \sqrt{n} - 1$. Further, one can find that $g_3(n) = \sqrt{n} - 1$ and $K_{1,n-1}$

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is the unique extremal. In [7], V. Nikiforov gave an estimate for the value of $g_l(n)$ and proposed the following conjecture:

**Conjecture 1.1.** ([7]) Let $k \geq 2$ and $G$ be a graph of sufficiently large order $n$. Let $S_{n,k}$ be the graph obtained by joining each vertex of $K_k$ to $n-k$ isolated vertices and $S_{n,k}^+$ be the graph obtained by adding one edge within the independent set of $S_{n,k}$.

(i) If $G$ does not contain $C_{2k+1}$ and $C_{2k+2}$, then $\rho(G) \leq \rho(S_{n,k})$ with equality if and only if $G \cong S_{n,k}$.

(ii) If $G$ does not contain $C_{2k+2}$, then $\rho(G) \leq \rho(S_{n,k}^+)$ with equality if and only if $G \cong S_{n,k}^+$.

Note that $S_{n,k}^+$ contains neither $C_{2k+2}$ nor $C_{2k+3}$. If Conjecture 1.1 is true, then the exact value of $g_l(n)$ is completely obtained for $l \geq 5$ and sufficiently large $n$. This paper proves the following theorem, which implies the above conjecture is true for $k = 2$.

**Theorem 1.2.** Let $n \geq 6$ and

$$G_n = \{G : |V(G)| = n, \text{ neither } C_5 \text{ nor } C_6 \text{ is a subgraph of } G\}$$

and $G^*$ have maximal spectral radius among all graphs in $G_n$. Then, $G^* \cong S_{n,2}$.

**2. Proof.** A graph is said to be trivial, if its edge set is empty. Straightforward calculation shows that

\[(2.1) \quad \rho^2 - \rho - 2(n-2) = 0.\]

for $\rho = \rho(S_{n,2})$. Let $G^*$ have maximal spectral radius among all graphs in $G_n$. Clearly, $G^*$ is connected. Since $G^*$ is $C_5$-free, for any vertex $u \in V(G^*)$, $G^*[N^1(u)]$ cannot contain $P_3$ as a subgraph. Further, we can observe the following properties.

**Lemma 2.1.** Let $u$ be a vertex of $G^*$ and $w \in N^2(u)$.

(i) Each component of $G^*[N^1(u)]$ is either a star $K_{1,r}$ for some $r \geq 0$ or a copy of $K_3$.

(ii) If $w$ is adjacent to some nontrivial component of $G^*[N^1(u)]$, then this component is the unique one to which $w$ is adjacent.

(iii) Particularly, if $w$ is adjacent to some $K_{1,r}$-component for $r \geq 2$ or a $K_3$-component, then its neighbor in this component is also unique.

Now we introduce some additional notation. Let $A = A(G^*)$ and $B = (b_{ij})_{n \times n} = A^2 - A - 2(n-2)I$. Given a vertex $u \in V(G^*)$, let $t_u(H)$ be the number of $H$-components of $G^*[N^1(u)]$ and $t'_u(H)$ be the number of vertices in $N^2(u)$ adjacent to an $H$-component of $G^*[N^1(u)]$. Let $F_u$ be the bipartite subgraph induced by the
edges from all the isolated vertices of \(G^*[N^1(u)]\) to \(N^2(u)\) and \(e(F_u)\) be the number of edges in \(F_u\).

**Lemma 2.2.** For any vertex \(u \in V(G^*)\),

\[
\sum_{i=1}^{n} b_{ui} \leq 2 - 2 \sum_{r \geq 0} t_u(K_{1,r}) - 2t'_u(K_3) - 2t''_u(K_{1,0}) + e(F_u).
\]

Equality holds if and only if \(N^3(u) = \emptyset\) and \(N^1(v) \setminus \{w\} = N^1(w) \setminus \{v\}\) for any \(v, w\) within a same \(K_{1,1}\)-component of \(G^*[N^1(u)]\).

**Proof.** Note that the \((i,j)\)-element of \(A^k\) is the number of walks of length \(k\) from the vertex \(i\) to the vertex \(j\) in \(G^*\). Clearly, \(b_{uu} = d_{G^*}(u) - 2(n - 2)\) and \(b_{ui} = 0\) for any \(i \in \cup_{j \geq 3}N^3(u)\). Further, we can observe that

\[
\sum_{i \in N^1(u)} b_{ui} = 2\sum_{r \geq 0} rt_u(K_{1,r}) + 3t_u(K_3) - d_{G^*}(u).
\]

By Lemma 2.1,

\[
\sum_{i \in N^2(u)} b_{ui} \leq 2t'_u(K_{1,1}) + 2t'_u(K_3) + t'_u(K_3) + e(F_u).
\]

Note that

\[
|N^1(u)| = d_{G^*}(u) = \sum_{r \geq 0} (r + 1)t_u(K_{1,r}) + 3t_u(K_3),
\]

and

\[
|N^2(u)| = \sum_{r \geq 0} t'_u(K_{1,r}) + t'_u(K_3)
\]

and

\[
|N^1(u)| + |N^2(u)| \leq n - 1.
\]

We have

\[
\sum_{i=1}^{n} b_{ui} \leq 2\sum_{r \geq 0} rt_u(K_{1,r}) + 3t_u(K_3) - 2(n - 2) + \sum_{i \in N^2(u)} b_{ui}
\]

\[
\leq 2[1 - \sum_{r \geq 0} t_u(K_{1,r}) - |N^2(u)|] + \sum_{i \in N^2(u)} b_{ui}
\]

\[
= 2\{1 - \sum_{r \geq 0} [t_u(K_{1,r}) + t'_u(K_{1,r}) - t''_u(K_3)] - 2t'_u(K_{1,1})
\]

\[
+ \sum_{r \geq 2} t'_u(K_{1,r}) + t'_u(K_3) + e(F_u)
\]

\[
= 2 - 2\sum_{r \geq 0} t_u(K_{1,r}) - \sum_{r \geq 2} t'_u(K_{1,r}) - t''_u(K_3) - 2t'_u(K_{1,0}) + e(F_u).
\]
Equality holds if and only if both equality holds in both (2.2) and (2.3). This implies that $N^3(u) = \emptyset$ and $N^1(v) \setminus \{w\} = N^1(w) \setminus \{v\}$ for any $v, w$ within a same $K_{1,1}$-component of $G^*[N^1(u)]$.  

**Lemma 2.3.** Let $G = (X,Y)$ be a nontrivial bipartite graph of size $e(G)$. If $G$ does not contain a path $P_5$ with both endpoints in $X$, then $e(G) \leq 2|Y| + |X| - 2$. Equality holds if and only if $G \cong K_{2,|Y|}$ or $G \cong K_{1,|X|,1}$.

**Proof.** We may assume that $G$ is connected. Now we use induction on $|Y|$. If $|Y| = 1$, then $G \cong K_{1,|X|,1}$ and $e(G) = |X| = 2|Y| + |X| - 2$. Suppose that $|Y| \geq 2$. If $Y$ contains a vertex $v$ of degree one, then $G - v$ is also connected. By the induction hypothesis, $e(G - v) \leq 2(|Y| - 1) + |X| - 2$, and hence, $e(G) \leq 2(|Y| - 1) + |X| - 2 + d_G(v) < 2|Y| + |X| - 2$.

Next suppose that $d_G(v) \geq 2$ for any vertex $v \in Y$. Note that $G$ is connected. If all of the vertices in $Y$ have common neighborhood, then $G \cong K_{|X|,|Y|}$. Since $G$ does not contain a copy of $P_5$ with both endpoints in $X$, $|X| \leq 2$ and the inequality holds. If not all the vertices in $Y$ have common neighborhood, then there are two vertices $u, v \in Y$ with $N^1(u) \neq N^1(v)$ and $N^1(u) \cap N^1(v) \neq \emptyset$. Say $w_1, w_2 \in N^1(u)$ and $w_2, w_3 \in N^1(v)$. Then $w_1uw_2v_3w_3$ is a copy of $P_5$ with $w_1, w_3 \in X$, a contradiction.  

Let $R_k$ be the graph obtained from $k$ copies of $K_4$ by identifying a vertex of them. Let $R_{k,r}$ be the graph obtained from $R_k$ and $S_{r,2}$ by identifying the central vertex of $R_k$ with one of the $(r-1)$-vertices of $S_{r,2}$, where $k \geq 0$ and $r \geq 2$. In particular, $R_{0,2} \cong S_{2,2}$.

**Claim 2.4.** For any vertex $u \in V(G^*)$, if $t_u(K_{1,0}) > 0$, then $\sum_{i=1}^{n} b_{ui} \leq 0$. Equality holds if and only if $G^* \cong R_{k,2}$ for some nonnegative integer $k = \frac{1}{2}(n - 2)$.

**Proof.** First assume that $t_u'(K_{1,0}) > 0$. If $F_u$ contains a copy $P(v, w)$ of $P_5$ with both endpoints $v, w \in N^1(u)$, then $P(v, w) + uw + uw$ is a 6-cycle. Since $G^*$ does not contain $C_6$, by Lemma 2.3 $e(F_u) = 2t_u(K_{1,0}) + t_u(K_{1,0}) - 2$. Thus, by Lemma 2.2

$$\sum_{i=1}^{n} b_{ui} \leq -2 \sum_{r \geq 1} t_u(K_{1,r}) - \sum_{r \geq 2} t_u'(K_{1,r}) - t_u'(K_3) - t_u(K_1) < 0.$$ 

Now suppose that $t_u'(K_{1,0}) = 0$, then $e(F_u) = 0$. By Lemma 2.2 we have

$$\sum_{i=1}^{n} b_{ui} \leq 2 \sum_{r \geq 0} t_u(K_{1,r}) - \sum_{r \geq 2} t_u'(K_{1,r}) + t_u'(K_3).$$

Note that $t_u(K_{1,0}) > 0$. Thus, $\sum_{i=1}^{n} b_{ui} \leq 0$. Equality holds if and only if $N^3(u) = \emptyset$,
\[ \sum_{r \geq 0} t_u(K_{1,r}) = t_u(K_{1,0}) = 1, \text{ and } t'_u(K_3) = 0. \] This implies that \( G^* \cong R_{k,2} \) for some \( k = \frac{4}{3}(n - 2) \).

**Claim 2.5.** For any vertex \( u \in V(G^*) \), \( \sum_{i=1}^{n} b_{ui} \leq 2 \). Moreover,

(i) \( \sum_{i=1}^{n} b_{ui} = 2 \) if and only if \( G^* \cong R_k \) for some positive integer \( k = \frac{4}{3}(n - 1) \);

(ii) \( \sum_{i=1}^{n} b_{ui} = 1 \) if and only if \( G^* \cong R^+_k \) for some positive integer \( k = \frac{4}{3}(n - 2) \), where \( R^+_k \) is the graph obtained from \( R_k \) by adding a pendant edge to some 3-vertex.

**Proof.** According to Claim 2.4, we may assume that \( t_u(K_{1,0}) = 0 \). So \( t'_u(K_{1,0}) = e(F_u) = 0 \). By Lemma 2.2, we have

\[ \sum_{i=1}^{n} b_{ui} \leq 2 - 2 \sum_{r \geq 0} t_u(K_{1,r}) - \sum_{r \geq 2} t'_u(K_{1,r}) - t'_u(K_3). \]

Hence, \( \sum_{i=1}^{n} b_{ui} \leq 2 \). Moreover, \( \sum_{i=1}^{n} b_{ui} = 2 \) if and only if \( \sum_{r \geq 0} t_u(K_{1,r}) = 0 \) and \( t'_u(K_3) = 0 \). This implies that \( G^* \cong R_k \) for some \( k = \frac{4}{3}(n - 1) \).

Similar to the above, \( \sum_{i=1}^{n} b_{ui} = 1 \) if and only if \( \sum_{r \geq 0} t_u(K_{1,r}) = 0 \) and \( t'_u(K_3) = 1 \). This implies that \( G^* \cong R^+_k \) for some \( k = \frac{4}{3}(n - 2) \).

The following lemma is an immediate consequence of Rayleigh’s theorem applied for the adjacency matrices.

**Lemma 2.6.** Let \( G \) be a connected graph in \( G_n \). If \( G \) has two cut vertices, then there exists a connected graph \( G' \in G_n \) such that \( \rho(G) < \rho(G') \).

**Theorem 2.7.** If \( n \geq 6 \), then \( \sum_{i=1}^{n} b_{ui} \leq 0 \) for any \( u \in V(G^*) \), and hence, \( \rho(G^*) \leq \rho(S_{n,2}) \).

**Proof.** According to Claim 2.5 if \( \sum_{i=1}^{n} b_{ui} = 1 \), then \( G^* \cong R^+_k \) for some \( k \geq 2 \). Now, \( G^* \) has two cut vertices. Since \( G^* \) has maximal spectral radius, by Lemma 2.6 we get a contradiction.

If \( \sum_{i=1}^{n} b_{ui} = 2 \), then \( G^* \cong R_k \), where \( n = |V(R_k)| = 3k + 1 \geq 7 \). Straightforward calculation shows that

\[ \rho(R_k) = 1 + \sqrt{n} < \frac{1 + \sqrt{1 + 8(n - 2)}}{2} = \rho(S_{n,2}) \]
for \( n \geq 7 \), a contradiction. Thus, \( \sum_{i=1}^{n} b_{ui} \leq 0 \). Let \( X \) be a positive eigenvector of 
\( A(G^*) \) corresponding to \( \rho = \rho(G^*) \) such that \( \sum_{i=1}^{n} x_i = 1 \). Then
\[
\rho^2 - \rho - 2(n - 2) = \sum_{i=1}^{n} [\rho^2 - \rho - 2(n - 2)]x_i = \sum_{i=1}^{n} (\sum_{j=1}^{n} b_{ij})x_j \leq 0.
\]
By (2.1), \( \rho(G^*) \leq \rho(S_{n,2}) \).

Note that \( S_{n,2} \in G_n \). Theorem 2.7 implies \( g_5(n) = \rho(S_{n,2}) = \frac{1 + \sqrt{1 + 8(n - 2)}}{2} \). Next we shall consider the uniqueness of the extremal graph \( G^* \).

Let \( T_{k,r} \) be the graph obtained from \( R_k \) and \( S_{r,2} \) by identifying the central vertex of \( R_k \) with one of 2-vertices of \( S_{r,2} \), where \( k \geq 0 \) and \( r \geq 3 \). Particularly, \( T_{0,r} \cong S_{r,2} \) and \( T_{k,3} \cong R_{k,3} \).

**Lemma 2.8.** For any two positive integers \( k, r \) with \( 3k + r = n \geq 6 \), \( G^* \) is not isomorphic to either \( R_{k,r} \) or \( T_{k,r} \).

**Proof.** Assume to the contrary that \( G^* \cong R_{k,r} \) for some \( k \geq 1 \) and \( r \geq 2 \). If \( n = 6 \), then \( k = 1 \) and \( r = 3 \). Straightforward calculation shows that \( \rho(R_{1,3}) = 3.2618 < \rho(S_{6,2}) \), a contradiction.

Next let \( n \geq 7 \). By Theorem 2.7 \( \sum_{i=1}^{n} b_{ui} \leq 0 \) for any \( u \in V(G^*) \). Let \( v \) be a 3-vertex in some \( K_4 \)-copy of \( R_{k,r} \). Clearly, \( \sum_{r \geq 0} t_v(K_{1,r}) = 0 \) and \( t_v(K_3) = 1 \). Moreover, \( t'_v(K_3) = n - 4 \geq 3 \). According to Lemma 2.2 \( \sum_{i=1}^{n} b_{vi} \leq 2 - 3 < 0 \). Let \( X \) be a positive eigenvector of \( A(G^*) \) corresponding to \( \rho = \rho(G^*) \) such that \( \sum_{i=1}^{n} x_i = 1 \). Thus,
\[
\rho^2 - \rho - 2(n - 2) = \sum_{i=1}^{n} [\rho^2 - \rho - 2(n - 2)]x_i = \sum_{i=1}^{n} (\sum_{j=1}^{n} b_{ij})x_j = \sum_{j=1}^{n} (\sum_{i=1}^{n} b_{ij})x_j < 0.
\]
By (2.1), \( \rho(G^*) < \rho(S_{n,2}) \), a contradiction. So \( G^* \) is not isomorphic to \( R_{k,r} \). Similarly, we can prove that \( G^* \) is not isomorphic to \( T_{k,r} \) for any \( k \geq 1 \) and \( r \geq 3 \).

**Proof of Theorem 1.2.** Suppose that \( G^* \) is not isomorphic to \( S_{n,2} \). By Theorem 2.7 \( \sum_{i=1}^{n} b_{ui} \leq 0 \) for any \( u \in V(G^*) \). Similar to the proof of Lemma 2.8 it suffices to show that there exists a vertex \( v \in V(G^*) \) such that \( \sum_{i=1}^{n} b_{vi} < 0 \).

Select a vertex \( v \in V(G^*) \) arbitrarily. Note that \( G^* \) is not isomorphic to \( R_{k,r} \) for any \( k \geq 1 \) and \( r \geq 2 \). If \( t_v(K_{1,0}) > 0 \), then by Claim 2.4 \( \sum_{i=1}^{n} b_{vi} < 0 \).
Next let \( t_v(K_{1,0}) = 0 \). Then by Lemma 2.2
\[
(2.4) \quad \sum_{i=1}^{n} b_{vi} \leq 2 - 2 \sum_{r \geq 1} t_v(K_{1,r}) - \sum_{r \geq 2} t'_v(K_{1,r}) - t'_v(K_3).
\]

If \( t_v(K_{1,1}) > 0 \), then \( \sum_{i=1}^{n} b_{vi} \leq 0 \) with equality if and only if \( \sum_{r \geq 1} t_v(K_{1,r}) = t_v(K_{1,1}) = 1 \) and \( t'_v(K_3) = 0 \). So the equality implies that \( G^* \cong T_k, r \) for some \( k \geq 1 \) and \( r \geq 3 \) with \( 3k + r = n \), a contradiction. Thus, \( \sum_{i=1}^{n} b_{vi} < 0 \).

Now let \( t_v(K_{1,0}) = t_v(K_{1,1}) = 0 \). Then (2.4) becomes
\[
\sum_{i=1}^{n} b_{vi} \leq 2 - \sum_{r \geq 2} [2t_v(K_{1,r}) + t'_v(K_{1,r})] - t'_v(K_3).
\]

If \( \sum_{r \geq 2} t_v(K_{1,r}) > 0 \), then \( \sum_{i=1}^{n} b_{vi} \leq 0 \), with equality if and only if \( \sum_{r \geq 2} t_v(K_{1,r}) = 1 \) and \( \sum_{r \geq 2} t'_v(K_{1,r}) = t'_v(K_3) = 0 \). So the equality implies that \( G^* \cong R_k, r \) for some \( k \geq 1 \) and \( r \geq 2 \) with \( 3k + r = n \), a contradiction. Thus, \( \sum_{i=1}^{n} b_{vi} < 0 \).

Finally, let \( \sum_{r \geq 0} t_v(K_{1,r}) = 0 \). Then \( t_v(K_3) > 0 \). Since \( G^* \) is not isomorphic to \( S_{n,2} \), \( t'_v(K_3) > 0 \). Note that \( \sum_{i=1}^{n} b_{vi} \leq 2 - t'_v(K_3) \). If \( t'_v(K_3) > 2 \), then \( \sum_{i=1}^{n} b_{vi} < 0 \). It remains the case \( t'_v(K_3) \in \{1, 2\} \). Now, if \( t_v(K_3) > 1 \), then \( G^* \) has at least two cut vertices, a contradiction by Lemma 2.6. So \( t_v(K_3) = 1 \). Since \( n \geq 6 \), \( t'_v(K_3) = 2 \) and \( G^* \cong R_{1,3} \). This also induces a contradiction. \( \square \)

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