Electronic Journal of Linear Algebra

Volume 23 *ELA Volume 23 (2012)* Article 51

2012

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 703-730, August 2012

ELA
http://math.technion.ac.il/iic/ela

ANALYSIS AND REFORMULATION OF LINEAR DELAY DIFFERENTIAL-ALGEBRAIC EQUATIONS*

PHI HA† AND VOLKER MEHRMANN†

Abstract. General linear systems of delay differential-algebraic equations (DDAEs) of arbitrary order are studied in this paper. Under some consistency conditions, it is shown that every linear high-order DAE can be reformulated as an underlying high-order ordinary differential equation (ODE) and that every linear DDAE with single delay can be reformulated as a high-order delay differential equation (DDE). Condensed forms for DDAEs based on the algebraic structure of the system coefficients are derived and these forms are used to reformulate DDAEs as strangeness-free systems, where all constraints are explicitly available. The condensed forms are also used to investigate structural properties of the system like solvability, regularity, consistency and smoothness requirements.

Key words. Delay differential-algebraic equation, Differential-algebraic equation, Regularization, Strangeness-index, Index reduction.

AMS subject classifications. 34A09, 34A12, 65L05, 65H10.

1. Introduction. In this paper, we study general *linear delay differential-algebraic equations (DDAEs)* of the form

$$A_k x^{(k)}(t) + \cdots + A_0 x(t) + A_{-1} x(t-\tau) + \cdots + A_{-\kappa} x^{(\kappa)}(t-\tau) = f(t), \tag{1.1}$$

where the coefficients satisfy $A_i \in \mathbb{C}^{\ell,n}$, $i = k, \ldots, -\kappa$, $A_k \not\equiv 0$, $f : [0,\infty) \to \mathbb{C}^{\ell}$, and where $\tau > 0$ is a single constant delay. We consider the time interval $t \in [0,\infty)$. Note that most of our analysis also carries over to multiple and nonconstant delays but here we restrict ourselves to the constant single delay case.

An important special case of (1.1) is the initial value problem for a first order linear delay differential-algebraic equation with single delay

$$A_1 \dot{x}(t) + A_0 x(t) + A_{-1} x(t-\tau) = f(t), \tag{1.2}$$

where A_1, A_0, $A_{-1} \in \mathbb{C}^{\ell,n}$, $f : [0,\infty) \to \mathbb{C}^{\ell}$.

*Received by the editors on April 12, 2012. Accepted for publication on August 1, 2012. Handling Editor: James G. Nagy.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 703-730, August 2012

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To achieve uniqueness of solutions, for DDAEs one typically has to prescribe initial functions, which for the special case (1.2) take the form

$$x|_{[-\tau,0]} = \phi : [-\tau, 0] \to \mathbb{C}^n. \tag{1.3}$$

Ordinary delay differential equations (DDEs) of the form (1.2), with A_1 being the identity matrix, arise in various applications, see [3, 10] and the references therein. If the states of the physical system are constrained, e.g., by conservation laws or interface conditions, then algebraic equations have to be included and one has to analyze delay differential-algebraic equations (DDAEs). DDAEs may be considered from two different perspectives. On the one hand, they are differential-algebraic equations (DAEs) that involve delayed terms. On the other hand, DDAEs are ordinary delay differential equations (DDEs) subject to constraints that also may involve time-delayed variables. Of course, DDAEs inherit all the difficulties that are associated with both DAEs and DDEs. Their interaction, however, leads to new effects that do not arise in either DAEs or DDEs, as has been pointed out in [2, 6, 8].

Although DDEs are well studied analytically and numerically, see e.g. [3, 10], and a similar maturity has been reached for the simulation and control of DAEs, see e.g. [4, 12, 13], the theoretical understanding and the development of appropriate numerical methods for DDAEs, however, is far from complete even for the case of linear systems with constant coefficients. Only very few results are available, see e.g., [1, 2, 6, 7, 8, 19], and these are mainly for the special case of DAEs, where the delay component is nothing else than an additional part of the inhomogeneity.

The main difficulty so far is the lack of a suitable regularity analysis (via the concept of an index) and a canonical form which allows to investigate structural properties like existence, uniqueness of solutions, consistency and smoothness requirements for the initial function.

In this paper, we derive such a canonical form for the linear constant coefficient case by extending the algebraic approach introduced in [13, 20] and combining it with the behavior approach [18]. Surprisingly, already in order to deal with (1.2), it is necessary to study linear high-order differential-algebraic equations of the form

$$A_k x^{(k)}(t) + \cdots + A_1 \dot{x}(t) + A_0 x(t) = f(t), \tag{1.4}$$

with associated initial conditions of the form

$$x^{(k-1)}(0) = x_0^{(k-1)}, \ldots, \quad \dot{x}(0) = x_0^{(1)}, \quad x(0) = x_0^{(0)}. \tag{1.5}$$

We study the theoretical aspects of (1.4)–(1.5) in Section 3 and then use these to study the general case of DDAEs in Section 4. The analysis is based on reformulation procedures which bring the systems into a *strangeness-free* form and allows also to

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study theoretical aspects like existence and uniqueness of solutions, as well as the consistency and smoothness requirements for the initial functions.

2. Notation and preliminaries. In the following, we denote by $I_n \in \mathbb{C}^{n,n}$ (or I) the identity matrix and by A^T the transpose of a matrix A. For an interval $\mathbb{I} \subset [0,\infty)$, by $C^k(\mathbb{I},\mathbb{C}^n)$ we denote the space of k-times continuously differentiable functions from $\mathbb{I}$ to $\mathbb{C}^n$.

We use the following solution concept for (1.2).

Definition 2.1. A function $x : [0,\infty) \to \mathbb{C}^n$ is called a *(classical) solution* to (1.2) if $x \in C^1([0,\infty),\mathbb{C}^n)$ and x satisfies (1.2) pointwise. An initial function ϕ is called *consistent* with system (1.2) if the associated initial value problem (1.2)–(1.3) has at least one classical solution. System (1.2) is called *solvable* if for every sufficiently smooth f and every consistent initial function ϕ, the associated initial value problem (1.2)–(1.3) has a solution. It is called *regular* if it is solvable and the solution is unique.

Introducing $X_0 := \begin{bmatrix} x_0^{(k-1)} \\ \vdots \\ x_0^{(0)} \end{bmatrix}$ as *initial vector* of the initial value problem consisting of (1.4)–(1.5), Definition 2.1 extends to higher order systems, i.e., an initial vector $X_0 \in \mathbb{C}^{(k+1)\ell}$ is called *consistent* for system (1.4) if the initial value problem (1.4)–(1.5) has a solution, and system (1.4) is called *solvable* if for every sufficiently smooth f and every consistent initial vector X_0, the associated initial value problem (1.4)–(1.5) has a solution.

Systems of differential-algebraic equations arise in real-time simulation and automated physical modeling via software packages such as Dymola [9] or Matlab Simulink [16] in all areas of technology. In general they may be over- or under-determined, i.e., they may contain free variable (controls) and/or redundant equations. Usually then a reformulation procedure is necessary to remove redundant equations, to make the system consistent or to achieve better properties for simulation and control [5, 13].

The same procedure is necessary for delay differential-algebraic equations. To achieve a good reformulation, also for delay differential-algebraic equations one must add hidden constraints to the system and decouple the equations into two subsystems, one which describes the dynamics of the system and one which describes all the constraints, together with redundancy and consistency conditions that allow a regularization.

We will derive such a reformulation and regularization procedure for systems (1.4) and (1.1) in Sections 3, 4.

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To illustrate the difficulties that may arise we present a simple example.

EXAMPLE 2.2. Consider the first order DDAE

$$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}\dot{x}_1(t)\\\dot{x}_2(t)\\\dot{x}_3(t)\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix} \tag{2.1}$$
$$+\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x_1(t-\tau)\\x_2(t-\tau)\\x_3(t-\tau)\end{bmatrix}=\begin{bmatrix}f_1(t)\\f_2(t)\\f_3(t)\end{bmatrix},$$

and the associated non-delay equation

$$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}\dot{x}_1(t)\\\dot{x}_2(t)\\\dot{x}_3(t)\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}=\begin{bmatrix}f_1(t)\\f_2(t)\\f_3(t)\end{bmatrix}, \tag{2.2}$$

in the time interval $[0,\infty)$. Obviously, system (2.2) is under-determined, and $f_3 = 0$ is a consistency condition. If this holds, and if x_1 and f_1, f_2 are continuous, then the solution x is continuous as well. However, if the consistency condition

$$x_3(t-\tau) = f_3(t),\ t \in [0,\tau],$$

holds, then it immediately follows that system (2.1) has the general solution

$$x(t) = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}x(t-\tau)+\begin{bmatrix}0&-1&0\\0&0&0\\0&0&0\end{bmatrix}\dot{x}(t-\tau)+\begin{bmatrix}f_1(t)-\dot{f}_2(t)+\ddot{f}_3(t+\tau)\\f_2(t)-\dot{f}_3(t+\tau)\\f_3(t+\tau)\end{bmatrix},$$

for all $t \in [0,\infty)$. To obtain a unique solution, we have to provide an initial function $\phi = x|_{[-\tau,0]}$, which is consistent if and only if

$$\phi_3(t-\tau) = f_3(t),\ t \in [0,\tau].$$

Note, however, that the inhomogeneity f has to be at least twice continuously differentiable and, since $x(t)$ is differentiated in the delay, further smoothness requirements are necessary for the initial function ϕ, depending on the length of the considered time interval.

Example 2.2 shows that further differentiability and consistency conditions may be required for DDAEs. To characterize these for general linear DDAEs with constant coefficients, we will derive a condensed form. For this we need the following preliminary results.

For matrices $Q \in \mathbb{C}^{q,n}$, $P \in \mathbb{C}^{p,n}$, the matrix pair (Q,P) is said to *have no hidden redundancy* if

$$\operatorname{rank}\left(\begin{bmatrix}Q\\P\end{bmatrix}\right) = \mathrm{rank(Q)} + \mathrm{rank(P)}.$$

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LEMMA 2.3. *Suppose that for $Q \in \mathbb{C}^{q,n}$, $P \in \mathbb{C}^{p,n}$, the pair (Q, P) has no hidden redundancy. Then, for any matrix $U \in C^{q,q}$ and any $V \in C^{p,p}$, the pair (UQ, VP) has no hidden redundancy.*

Proof. The proof follows from the observation that a matrix pair has no hidden redundancy if and only if the intersection of the two vector spaces spanned by the rows of the two matrices contains only the vector 0. □

If $\begin{bmatrix} Q \\ P \end{bmatrix}$ is of full row rank for two matrices $Q \in \mathbb{C}^{q,n}$, $P \in \mathbb{C}^{p,n}$, then obviously, the pair (Q, P) has no hidden redundancy. However, the converse is not true as is obvious for $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, since (Q, P) has no hidden redundancy, but $\begin{bmatrix} Q \\ P \end{bmatrix}$ does not have full row rank.

LEMMA 2.4. *For $Q \in \mathbb{C}^{q,n}$, $P \in \mathbb{C}^{p,n}$, there exists*

$$\begin{bmatrix} S & 0 \\ Z_1 & Z_2 \end{bmatrix} \in \mathbb{C}^{q,q+p},$$

where $\begin{bmatrix} S \\ Z_1 \end{bmatrix} \in \mathbb{C}^{q,q}$ is nonsingular and the rows of $S \in \mathbb{C}^{p,q}$ are the rows of a permutation matrix such that

$$\begin{bmatrix} S & 0 \\ Z_1 & Z_2 \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} SQ \\ 0 \end{bmatrix},$$

and (SQ, P) has no hidden redundancy.

Proof. The proof follows by taking $[Z_1, Z_2]$ to be a full rank matrix spanning the left nullspace of $\begin{bmatrix} Q \\ P \end{bmatrix}$ and completing it to a full rank matrix by rows of a permutation matrix so that $\begin{bmatrix} S \\ Z_1 \end{bmatrix}$ is invertible and (SQ, P) has no hidden redundancy. □

Lemma 2.4 will be used later to recursively remove hidden redundancy in the coefficients of linear DAEs and DDAEs.

LEMMA 2.5. *Consider $k + 1$ full row rank matrices $R_0 \in \mathbb{C}^{r_0,n}, \ldots, R_k \in \mathbb{C}^{r_k,n}$, such that none of the matrix pairs*

$$\left(R_j, \begin{bmatrix} R_{j-1} \\ \vdots \\ R_0 \end{bmatrix} \right), \quad j = k, \ldots, 1 \tag{2.3}$$

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has a hidden redundancy. Then, $\begin{bmatrix} R_k \\ \vdots \\ R_0 \end{bmatrix}$ *has full row rank.*

Proof. Since none of the matrix pairs in (2.3) has a hidden redundancy, it follows that

$$\begin{aligned} \operatorname{rank}\left(\begin{bmatrix} R_k \\ \vdots \\ R_0 \end{bmatrix}\right) &= \operatorname{rank}(R_k) + \operatorname{rank}\left(\begin{bmatrix} R_{k-1} \\ \vdots \\ R_0 \end{bmatrix}\right), \\ &= \operatorname{rank}(R_k) + \operatorname{rank}(R_{k-1}) + \operatorname{rank}\left(\begin{bmatrix} R_{k-2} \\ \vdots \\ R_0 \end{bmatrix}\right), \\ &= \cdots \\ &= \operatorname{rank}(R_k) + \operatorname{rank}(R_{k-1}) + \cdots + \operatorname{rank}(R_0), \end{aligned}$$

and since $R_k, \ldots, R_0$ have full row rank, also $\begin{bmatrix} R_k \\ \vdots \\ R_0 \end{bmatrix}$ has full row rank. □

3. Analysis and reformulations of high-order DAEs. It will turn out that the reformulation of DDAEs leads to higher order DAEs. In this section, we therefore study the analysis of high-order DAEs of the form (1.4) and of the initial value problem (1.4)–(1.5), see also [15, 17, 21] and the references therein for previous work on this topic. We will extend these results by combining it with the regularization procedure for DAEs proposed in [20] in a behavior setting [18]. Let

$$M := [A_k, \ldots, A_0] \in \mathbb{C}^{\ell,(k+1)n} \text{ and } X(t) := \begin{bmatrix} x^{(k)}(t) \\ \vdots \\ x(t) \end{bmatrix}.$$

Then M (resp., $X(t)$) is called the *behavior matrix* (resp., *behavior vector*) of system (1.4), which can be written as

$$MX(t) = f(t). \tag{3.1}$$

For notational convenience, in this section we omit the argument t in X, x, f and their derivatives.

Scaling (1.4) with a nonsingular matrix $P \in \mathbb{C}^{\ell,\ell}$, we obtain

$$PMX = P\sum_{i=0}^{k} A_i x^{(i)} = Pf. \tag{3.2}$$

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Since the systems (3.1) and (3.2) have the same solution spaces, we introduce the following definition.

DEFINITION 3.1. Two behavior matrices $M = [A_k, \ldots, A_0]$ and $\tilde{M} = [\tilde{A}_k, \ldots, \tilde{A}_0]$ in $\mathbb{C}^{\ell,(k+1)n}$ are called *(strongly) left equivalent* (denoted by $\overset{\ell}{\sim}$) if there exists a nonsingular matrix $P \in \mathbb{C}^{\ell,\ell}$ such that $\tilde{M} = PM$ or equivalently,

$$\tilde{A}_j = PA_j, \; j = k, \ldots, 0.$$

LEMMA 3.2. *Consider the behavior matrix M of system* (1.4). *Then, M is left equivalent to a matrix*

$$\tilde{M} := \left[\begin{array}{c|c|c|c} A_{k,1} & A_{k-1,1} & \cdots & A_{0,1} \\ & A_{k-1,2} & \cdots & A_{0,2} \\ & & \ddots & \vdots \\ & & & A_{0,k+1} \\ \hline 0 & 0 & \cdots & 0 \end{array}\right], \quad \begin{array}{c} r_1 \\ r_2 \\ \vdots \\ r_{k+1} \\ \hline v \end{array} \tag{3.3}$$

where all the matrices $A_{k-j,j+1}$, $j = k, \ldots, 0$ on the main diagonal have full row rank. The integer numbers $r_1, \ldots, r_{k+1}$, v are the sizes of the block rows.

Proof. We first compress the first block column of M via a QR-decomposition, see [11], to

$$\begin{aligned} M &= [A_k, \ldots, A_0], \\ &\overset{\ell}{\sim} \left[\begin{array}{c|ccc} A_{k,1} & A_{k-1,1} & \cdots & A_{0,1} \\ 0 & A_{k-1,2} & \cdots & A_{0,2} \end{array}\right], \end{aligned}$$

such that $A_{k,1}$ has full row rank. Continuing, by compressing the 2nd block column from the second block row and then inductively the other columns of M, we finally arrive at (3.3). ∎

We call the number

$$r_u := (k+1)r_1 + kr_2 + \cdots + 2r_k + r_{k+1},$$

the *upper rank* of the behavior matrix M. Note, that some of the r_i may vanish and obviously, the upper rank is invariant under left equivalence transformations.

In the following, without loss of generality, we assume that the behavior matrix M is already in the form $\tilde{M}$. Rewriting system (3.1) block row-wise, we obtain the

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system

$$
\begin{aligned}
A_{k,1}x^{(k)} + A_{k-1,1}x^{(k-1)} + \cdots + A_{1,1}\dot{x} + A_{0,1}x &= f_1,\\
A_{k-1,2}x^{(k-1)} + \cdots + A_{1,2}\dot{x} + A_{0,2}x &= f_2,\\
&\vdots \\
A_{0,k+1}x &= f_{k+1},\\
0 &= f_{k+2}.
\end{aligned} \tag{3.4}
$$

Recall that the diagonal blocks $A_{k,1},\ A_{k-1,2},\dots,A_{0,k+1}$ have full row rank, therefore in system (3.4), for every j with $k \geq j \geq 0$, the $(k+1-j)$-th block row

$$A_{j,k+1-j}x^{(j)} + \cdots + A_{0,k+1-j}x = f_{k+1-j},$$

represents $r_{k+1-j} = \operatorname{rank}(A_{j,k+1-j})$ scalar differential equations of order j. The idea now is to use differential equations of order smaller than j and their derivatives to reduce the number of scalar differential equations of order j. Let us illustrate this idea for the case $j = k$.

If the pair $\left(A_{k,1}, \begin{bmatrix} A_{k-1,2} \\ \vdots \\ A_{0,k+1}\end{bmatrix}\right)$ has hidden redundancy, then Lemma 2.4 implies that there exists a matrix

$$\left[\begin{array}{c|ccc} S_k & 0 & \dots & 0 \\ \hline Z_{k,k} & Z_{k,k-1} & \dots & Z_{k,0}\end{array}\right]$$

such that $\begin{bmatrix} S_k \\ Z_{k,k}\end{bmatrix} \in \mathbb{C}^{r_1,r_1}$ is nonsingular,

$$Z_{k,k}A_{k,1} + [Z_{k,k-1}\dots Z_{k,0}]\begin{bmatrix} A_{k-1,2} \\ \vdots \\ A_{0,k+1}\end{bmatrix} = 0, \tag{3.5}$$

and the matrix pair $\left(S_kA_{k,1}, \begin{bmatrix} A_{k-1,2} \\ \vdots \\ A_{0,k+1}\end{bmatrix}\right)$ has no hidden redundancy.

Scaling the first equation of (3.4) with $\begin{bmatrix} S_k \\ Z_{k,k}\end{bmatrix}$ from the left we get

$$S_kA_{k,1}x^{(k)} + S_kA_{k-1,1}x^{(k-1)} + \cdots + S_kA_{0,1}x = S_kf_1,$$

and

$$Z_{k,k}A_{k,1}x^{(k)} + Z_{k,k}\Big(A_{k-1,1}x^{(k-1)} + \cdots + A_{0,1}x\Big) = Z_{k,k}f_1.$$

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From (3.5), we deduce

$$
\begin{aligned}
Z_{k,k}A_{k,1}x^{(k)} &= -[Z_{k,k-1}\dots Z_{k,0}]\begin{bmatrix} A_{k-1,2}\\ \vdots \\ A_{0,k+1}\end{bmatrix} x^{(k)}\\
&= -\sum_{i=0}^{k-1} Z_{k,i}A_{i,k+1-i}x^{(k)}\\
&= -\sum_{i=0}^{k-1} Z_{k,i}\left(\frac{d}{dt}\right)^{k-i} A_{i,k+1-i}x^{(i)},\\
&= -\sum_{i=0}^{k-1} \left(\frac{d}{dt}\right)^{k-i} Z_{k,i}\left(-\sum_{\ell=0}^{i-1} A_{\ell,k+1-i}x^{(\ell)} + f_{k+1-i}\right).
\end{aligned}
$$

This leads to the systems

$$
S_kA_{k,1}x^{(k)} + S_kA_{k-1,1}x^{(k-1)} + \dots + S_kA_{0,1}x = S_kf_1,
$$

and

$$
\begin{aligned}
&\sum_{i=0}^{k-1} Z_{k,i}\left(\frac{d}{dt}\right)^{k-i}\left(\sum_{\ell=0}^{i-1} A_{\ell,k+1-i}x^{(\ell)} - f_{k+1-i}\right)\\
&+Z_{k,k}\Big(A_{k-1,1}x^{(k-1)} + \dots + A_{0,1}x\Big) = Z_{k,k}f_1.
\end{aligned}
\tag{3.6}
$$

Note that (3.6) is a set of differential equations of order at most $k-1$. Hence, we have reduced the number of scalar differential equations of order k from $r_1 = \operatorname{rank}(A_{k,1})$ to $d_1 := \operatorname{rank}(S_kA_{k,1})$.

Applying the same argument to the block rows numbered $j = k-1,\dots,1$, we obtain the following two lemmas. For notational convenience, we denote by $*$ unspecified matrices.

Lemma 3.3. *Consider the DAE* (1.4) *in its behavior form* (3.1). *Moreover, assume that the behavior matrix M is in the form* (3.3). *Then, there exist matrices S_j, $Z_{j,i}$, $j = k,\dots,1$, $i = j,\dots,0$ of appropriate size such that*

i) the matrices $\begin{bmatrix} S_j\\ Z_{j,j}\end{bmatrix} \in \mathbb{C}^{r_j,r_j}$, $k \geq j \geq 1$ are nonsingular,

ii) for each j with $k \geq j \geq 1$,

$$
Z_{j,j}A_{j,k+1-j} + [Z_{j,j-1}\dots Z_{j,0}]\begin{bmatrix} A_{j-1,k+2-j}\\ \vdots \\ A_{0,k+1}\end{bmatrix} = 0,
$$

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iii) for each j with $k \geq j \geq 1$, the matrix pair

$$\left(S_j A_{j,k+1-j}, \begin{bmatrix} A_{j-1,k+2-j} \\ \vdots \\ A_{0,k+1} \end{bmatrix} \right)$$

has no hidden redundancy.

Proof. For each j with $k \geq j \geq 1$, by applying Lemma 2.4 to the matrix pair

$$\left(A_{j,k+1-j}, \begin{bmatrix} A_{j-1,k+2-j} \\ \vdots \\ A_{0,k+1} \end{bmatrix} \right)$$

we obtain matrices S_j, $Z_{j,i}$, $i = j, \ldots, 0$ that satisfy conditions i)-iii). ◻

Setting

$$\tilde{P} := \operatorname{diag}\left(\begin{bmatrix} S_k \\ Z_{k,k} \end{bmatrix}, \ldots, \begin{bmatrix} S_1 \\ Z_{1,1} \end{bmatrix}, I_{r_{k+1}+v} \right) \in \mathbb{C}^{\ell,\ell},$$

and scaling system (1.4) with $\tilde{P}$ from the left we obtain

$$\left[\begin{array}{c|c|c|c} S_k A_{k,1} & * & \ldots & * \\ Z_{k,k} A_{k,1} & * & \ldots & * \\ \hline & S_{k-1} A_{k-1,2} & \ldots & * \\ & Z_{k-1,k-1} A_{k-1,2} & \ldots & * \\ \hline & & \ddots & \vdots \\ & & & A_{0,k+1} \\ \hline 0 & 0 & \ldots & 0 \end{array}\right] \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} S_k f_1 \\ Z_{k,k} f_1 \\ S_{k-1} f_2 \\ Z_{k-1,k-1} f_2 \\ \vdots \\ f_{k+1} \\ f_{k+2} \end{bmatrix} \tag{3.7}$$

For each j with $k \geq j \geq 1$, we then reduce the number of differential equations of order j by eliminating the block $Z_{j,j} A_{j,k+1-j}$ of (3.7), as in the following lemma.

LEMMA 3.4. *Let matrices S_j, $Z_{j,i}$, $j = k, \ldots, 1$, $i = j, \ldots, 0$, be defined as in* Lemma 3.3. *Then, the DAE* (3.7) *has the same solution set as the DAE*

$$\begin{array}{c} d_1 \\ s_1 \\ \hline d_2 \\ s_2 \\ \hline \vdots \\ d_{k+1} \\ \hline v \end{array} \underbrace{\left[\begin{array}{c|c|c|c} S_k A_{k,1} & * & \ldots & * \\ 0 & * & \ldots & * \\ \hline & S_{k-1} A_{k-1,2} & \ldots & * \\ & 0 & \ldots & * \\ \hline & & \ddots & \vdots \\ & & & A_{0,k+1} \\ \hline 0 & 0 & \ldots & 0 \end{array}\right]}_{\check{M}} \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} S_k f_1 \\ g_{2k} \\ S_{k-1} f_2 \\ g_{2(k-1)} \\ \vdots \\ f_{k+1} \\ f_{k+2} \end{bmatrix}, \tag{3.8}$$

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A publication of the International Linear Algebra Society
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where

$$g_{2j} := \sum_{i=0}^{j} Z_{j,i} \left(\frac{d}{dt}\right)^{j-i} f_{k+1-i}, \;\; j = k, \ldots, 1.$$

Proof. For each j with $k \geq j \geq 1$, by inserting

$$Z_{j,j} A_{j,k+1-j} = -\sum_{i=0}^{j-1} Z_{j,i} A_{i,k+1-i},$$

into the equation

$$Z_{j,j} A_{j,k+1-j} x^{(j)} + Z_{j,j} \sum_{i=0}^{j-1} A_{i,k+1-j} x^{(i)} = Z_{j,j} f_{k+1-j},$$

we have

$$\left(-\sum_{i=0}^{j-1} Z_{j,i} A_{i,k+1-i}\right) x^{(j)} + Z_{j,j} \sum_{i=0}^{j-1} A_{i,k+1-j} x^{(i)} = Z_{j,j} f_{k+1-j}$$

or equivalently

$$-\sum_{i=0}^{j-1} Z_{j,i} \left(\frac{d}{dt}\right)^{j-i} A_{i,k+1-i} x^{(i)} + Z_{j,j} \sum_{i=0}^{j-1} A_{i,k+1-j} x^{(i)} = Z_{j,j} f_{k+1-j}. \tag{3.9}$$

Moreover, the $(k+1-i)$-th equation of (3.4) implies that

$$A_{i,k+1-i} x^{(i)} = -\sum_{\ell=0}^{i-1} A_{\ell,k+1-i} x^{(\ell)} + f_{k+1-i}. \tag{3.10}$$

Thus, substituting (3.10) into (3.9) we have

$$\begin{aligned}
-\sum_{i=0}^{j-1} Z_{j,i} \left(\frac{d}{dt}\right)^{j-i} \left(-\sum_{\ell=0}^{i-1} A_{\ell,k+1-i} x^{(\ell)} + f_{k+1-i}\right) + Z_{j,j} \sum_{i=0}^{j-1} A_{i,k+1-j} x^{(i)} \\
= Z_{j,j} f_{k+1-j},
\end{aligned}$$

or equivalently

$$\begin{aligned}
\sum_{i=0}^{j-1} Z_{j,i} \left(\frac{d}{dt}\right)^{j-i} \left(\sum_{\ell=0}^{i-1} A_{\ell,k+1-i} x^{(\ell)}\right) + Z_{j,j} \sum_{i=0}^{j-1} A_{i,k+1-j} x^{(i)} \\
= \sum_{i=0}^{j-1} Z_{j,i} \left(\frac{d}{dt}\right)^{j-i} f_{k+1-i} + Z_{j,j} f_{k+1-j} = g_{2j}.
\end{aligned}$$

Continuing like this inductively, we obtain (3.8). ∎

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From (3.8), we deduce that $r_j = d_j + s_j$, $j = 1, \dots, k+1$, $s_{k+1} = 0$ and therefore the upper rank of the behavior matrix $\breve{M}$ of system (3.8) can be bounded from above via

$$\begin{aligned}
\breve{r}_u &\leqslant (k+1)d_1 + k(s_1 + d_2) + \dots + (s_k + d_{k+1}), \\
&= (k+1)(d_1 + s_1) + k(d_2 + s_2) + \dots + (d_{k+1} + s_{k+1}) - \sum_{i=0}^{k} s_i, \\
&= (k+1)r_1 + kr_2 + \dots + r_{k+1} - \sum_{i=0}^{k} s_i = r_u - \sum_{i=0}^{k} s_i,
\end{aligned}$$

and thus this procedure of passing the system (3.3) to (3.8) has reduced the upper rank. This reduction of the upper rank leads to the following procedure.

Procedure 3.5.
Input: *The DAE* (1.4) *and its behavior form* (3.1).
Begin: *Set* $\alpha = 0$ *and let* $M^0 = M$, $f^0 = f$.
Step 1. *Determine a nonsingular matrix* $P \in \mathbb{C}^{\ell,\ell}$ *(as in* Lemma 3.2) *such that*

$$PM^{\alpha} = \left[\begin{array}{c|c|c|c} A_{k,1} & A_{k-1,1} & \dots & A_{0,1} \\ & A_{k-1,2} & \dots & A_{0,2} \\ & & \ddots & \vdots \\ & & & A_{0,k+1} \\ \hline 0 & 0 & \dots & 0 \end{array}\right], \quad \begin{array}{c} r_1 \\ r_2 \\ \vdots \\ r_{k+1} \\ \hline v \end{array}$$

where all the matrices on the main diagonal have full row rank, and let

$$r_u^{\alpha} := (k+1)r_1 + mr_2 + \dots + 2r_k + r_{k+1},$$

be the upper rank of the behavior matrix M^{α} *in the* α*-th iteration.*

Step 2. *Determine matrices* S_j, $Z_{j,i}$, $j = k, \dots, 1$, $i = j, \dots, 0$ *of appropriate size such that*

i) matrices $\begin{bmatrix} S_j \\ Z_{j,j} \end{bmatrix} \in \mathbb{C}^{r_j,r_j}$, $k \geq j \geq 1$ *are nonsingular,*

ii) for each j *with* $k \geq j \geq 1$,

$$Z_{j,j}A_{j,k+1-j} + [Z_{j,j-1} \dots Z_{j,0}] \begin{bmatrix} A_{j-1,k+2-j} \\ \vdots \\ A_{0,k+1} \end{bmatrix} = 0,$$

iii) for each j *with* $k \geq j \geq 1$, *the matrix pair*

$$\left(S_j A_{j,k+1-j}, \begin{bmatrix} A_{j-1,k+2-j} \\ \vdots \\ A_{0,k+1} \end{bmatrix} \right)$$

has no hidden redundancy.

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Step 3. *Setting*

$$\tilde{P} := \operatorname{diag}\left(\begin{bmatrix} S_k \\ Z_{k,k} \end{bmatrix}, \dots, \begin{bmatrix} S_1 \\ Z_{1,1} \end{bmatrix}, I_{r_{k+1}+v}\right) \in \mathbb{C}^{\ell,\ell},$$

and scaling system (1.4) *with* $\tilde{P}$ *from the left we obtain*

$$\left[\begin{array}{c|c|c|c} S_k A_{k,1} & * & \dots & * \\ Z_{k,k} A_{k,1} & * & \dots & * \\ \hline & S_{k-1} A_{k-1,2} & \dots & * \\ & Z_{k-1,k-1} A_{k-1,2} & \dots & * \\ \hline & & \ddots & \vdots \\ & & & A_{0,k+1} \\ \hline 0 & 0 & \dots & 0 \end{array}\right] \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} S_k f_1 \\ Z_{k,k} f_1 \\ S_{k-1} f_2 \\ Z_{k-1,k-1} f_2 \\ \vdots \\ f_{k+1} \\ f_{k+2} \end{bmatrix}. \tag{3.11}$$

Step 4. *For each j with $k \geq j \geq 1$, we then reduce the number of differential equations of order j by eliminating the block $Z_{j,j} A_{j,k+1-j}$ of* (3.11)*, as in* Lemma 3.3. *In this way, we obtain the system*

$$\underbrace{\left[\begin{array}{c|c|c|c} S_k A_{k,1} & * & \dots & * \\ 0 & * & \dots & * \\ \hline & S_{k-1} A_{k-1,2} & \dots & * \\ & 0 & \dots & * \\ \hline & & \ddots & \vdots \\ & & & A_{0,k+1} \\ \hline 0 & 0 & \dots & 0 \end{array}\right]}_{\breve{M}} \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} S_k f_1 \\ g_{2k} \\ S_{k-1} f_2 \\ g_{2(k-1)} \\ \vdots \\ f_{k+1} \\ f_{k+2} \end{bmatrix}}_{\breve{f}}, \qquad \begin{array}{c} d_1 \\ s_1 \\ \hline d_2 \\ s_2 \\ \hline \vdots \\ d_{k+1} \\ \hline v \end{array}$$

with

$$g_{2j} := \sum_{i=0}^{j} Z_{j,i} \left(\frac{d}{dt}\right)^{j-i} f_{k+1-i}, \; j = k, \dots, 1.$$

Let $s^\alpha := \sum_{i=0}^{k} s_i$, we then increase α by 1, set $M^\alpha = \breve{M}$, $f^\alpha = \breve{f}$, and repeat the process from Step 1.

End.

Since $r_u^{\alpha+1} \leqslant r_u^\alpha - s^\alpha$, Procedure 3.5 terminates after a finite number of iterations, and thus we have the following theorem.

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THEOREM 3.6. *The DAE* (1.4) *has the same solution set as the DAE*

$$
\left[\begin{array}{c|c|c|c}
\hat{A}_{k,1} & \hat{A}_{k-1,1} & \dots & \hat{A}_{0,1} \\
 & \hat{A}_{k-1,2} & \dots & \hat{A}_{0,2} \\
 & & \ddots & \vdots \\
 & & & \hat{A}_{0,k+1} \\ \hline
0 & 0 & \dots & 0
\end{array}\right]
\begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x \end{bmatrix}
=
\begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{k+1} \\ \hat{f}_{k+2} \end{bmatrix}
\tag{3.12}
$$

where $\begin{bmatrix} \hat{A}_{k,1} \\ \vdots \\ \hat{A}_{0,k+1} \end{bmatrix}$ *has full row rank.*

Proof. Clearly, after carrying out Procedure 3.5, we obtain a system of the form (3.12), where $\hat{A}_{k,1}, \dots, \hat{A}_{0,k+1}$ have full row rank and none of the matrix pairs

$$
\left(\hat{A}_{j,k+1-j}, \begin{bmatrix} \hat{A}_{j-1,k+2-j} \\ \vdots \\ \hat{A}_{0,k+1} \end{bmatrix} \right)
$$

$j = k, \dots, 1$ has a hidden redundancy.

Applying Lemma 2.5 to the matrices $\hat{A}_{j,k+1-j}$, $j = 0, \dots, k$, it follows that $\begin{bmatrix} \hat{A}_{k,1} \\ \vdots \\ \hat{A}_{0,k+1} \end{bmatrix}$ has full row rank. ☐

Following the notation in [13] we call (3.12) the *strangeness-free reformulation* of the DAE (1.4).

Obviously, if at $t = 0$ the consistency assumptions

$$
\begin{aligned}
&\left(\frac{d}{dt}\right)^i \left(\hat{A}_{k-1,2}x^{(k-1)}(t) + \dots + \hat{A}_{1,2}x^{(1)}(t) + \hat{A}_{0,2}x(t) - \hat{f}_2(t)\right) = 0, \ i = 0, 1, \\
&\quad \dots \\
&\left(\frac{d}{dt}\right)^i \left(\hat{A}_{0,k+1}x(t) - \hat{f}_{k+1}(t)\right) = 0, \ i = 0, \dots, k, \\
&\hat{f}_{k+2}(t) = 0,
\end{aligned}
\tag{3.13}
$$

hold, then we can differentiate all but the first equation of system (3.12) to obtain an underlying ODE as in the following theorem.

THEOREM 3.7. *Consider the DAE* (1.4) *and assume that the consistency condition* (3.13) *is satisfied. Then,* (1.4) *has the same solution set as the* underlying

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A publication of the International Linear Algebra Society
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ODE

$$\left[\begin{array}{c|c|c|c|c} \hat{A}_{k,1} & \hat{A}^{k-1,1} & \dots & \hat{A}_{1,1} & \hat{A}_{0,1} \\ \hat{A}_{k-1,2} & \hat{A}_{k-1,3} & \dots & \hat{A}_{0,2} & \\ \vdots & \vdots & \iddots & & \\ \hat{A}_{1,k} & \hat{A}_{0,k} & & & \\ \hat{A}_{0,k+1} & 0 & & & \end{array}\right]\begin{bmatrix} x^{(k)} \\ x^{(k-1)} \\ \vdots \\ x^{(1)} \\ x \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2^{(1)} \\ \vdots \\ \hat{f}_k^{(k-1)} \\ \hat{f}_{k+1}^{(k)} \end{bmatrix},$$

where the first column $\begin{bmatrix} \hat{A}_{k,1} \\ \vdots \\ \hat{A}_{0,k+1} \end{bmatrix}$ *has full row rank.*

The following corollary is a direct consequence of Theorem 3.7.

COROLLARY 3.8. *Consider the initial value-problem* (1.4)–(1.5)*, and assume that the function f is sufficiently smooth. Then we have:*

i) Consistency conditions for f and the initial vector $X_0 := \begin{bmatrix} x_0^{(k-1)} \\ \vdots \\ x_0^{(0)} \end{bmatrix}$ *are given by system* (3.13)*.*

ii) System (1.4) *is uniquely solvable if and only if in addition, the matrix*

$$\begin{bmatrix} \hat{A}_{k,1} \\ \vdots \\ \hat{A}_{0,k+1} \end{bmatrix}$$

is square.

Motivated by the strangeness-free reformulation (3.12) of the DAE (1.4), we introduce the following definition.

DEFINITION 3.9. Consider the behavior matrix

$$M = [A_k, \dots, A_0] \in \mathbb{C}^{\ell,(k+1)n},$$

associated with the DAE

$$A_k x^{(k)}(t) + \dots + A_0 x(t) + h(t) = 0,$$

where $A_i \in \mathbb{C}^{\ell,n}$, $i = k, \dots, 0$, and $h : [0, \infty) \to \mathbb{C}^{\ell}$.

The matrix M is called *strangeness-free* if there exists a nonsingular matrix $P \in$

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$\mathbb{C}^{\ell,\ell}$ such that

$$PM = \left[\begin{array}{c|c|c|c} A_{k,1} & A_{k-1,1} & \dots & A_{0,1} \\ & A_{k-1,2} & \dots & A_{0,2} \\ & & \ddots & \vdots \\ & & & A_{0,k+1} \\ \hline 0 & 0 & \dots & 0 \end{array}\right],$$

where

i) each block column contains exactly n columns, and

ii) the matrix $\begin{bmatrix} A_{k,1} \\ A_{k-1,2} \\ \vdots \\ A_{0,k+1} \end{bmatrix}$ has full row rank.

REMARK 3.10. One of the characteristics of the behavior approach [18] is that all the variables are treated equal. This is important, for example, if one wants to know which variables to choose as controls. The behavior approach, however, is not appropriate if for example, the choice of controls is predetermined. In this case a uniform treatment for a general high-order system (with or without delay) is still not available, see [14] and [15] for special cases of first and second order systems without delay.

4. Analysis and reformulation of DDAEs. This section discusses DDAEs with single delay of the form (1.2) and the initial value problem (1.2)–(1.3). Analogous to Section 3, the behavior approach and the algebraic approach will be combined. Consider a behavior formulation of (1.2) as

$$N^0 X^0 = f_0, \tag{4.1}$$

with

$$N^0 = [A_1 \ A_0 \ A_{-1}] \text{ and } X^0(t) = \begin{bmatrix} \dot{x}(t) \\ x(t) \\ x(t-\tau) \end{bmatrix}, \ f_0(t) := f(t).$$

A first remarkable difference between DAEs and DDAEs is that for the DAE (1.4) of order k, after applying the strangeness-free reformulation (Procedure 3.5) the resulting system is still a DAE of order at most k. However, when applying a similar procedure for the DDAE (1.2) then the order of the system may increase, as is illustrated in the following example.

EXAMPLE 4.1. Consider the system

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x(t-\tau) \\ y(t-\tau) \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad t \ge 0.$$

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A publication of the International Linear Algebra Society
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In behavior form, we have

$$N^0 = \left[\begin{array}{cc|cc|cc} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{array}\right], \quad X^0 = \left[\begin{array}{c} \dot{x}(t) \\ \dot{y}(t) \\ \hline x(t) \\ y(t) \\ \hline x(t-\tau) \\ y(t-\tau) \end{array}\right].$$

Differentiating the second equation and inserting it into the first, we get

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t-\tau) \\ y(t-\tau) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t-\tau) \\ \dot{y}(t-\tau) \end{bmatrix} + \begin{bmatrix} f_1 + \dot{f}_2 \\ f_2 \end{bmatrix}.$$

In behavior form we have $N^1 X^1 = \begin{bmatrix} f_1 + \dot{f}_2 \\ f_2 \end{bmatrix}$ with

$$N^1 = \left[\begin{array}{cc|cc|cc|cc} 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \end{array}\right], \quad X^1 = \left[\begin{array}{c} \dot{x}(t) \\ \dot{y}(t) \\ \hline x(t) \\ y(t) \\ \hline x(t-\tau) \\ y(t-\tau) \\ \hline \dot{x}(t-\tau) \\ \dot{y}(t-\tau) \end{array}\right].$$

Thus, the size of the behavior matrix is increased.

The second important difference between DAEs and DDAEs is the strangeness-free reformulation procedure. Let us illustrate this by considering the following example.

Example 4.2. Consider the system

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t-\tau) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}, \quad t \geq 0. \tag{4.2}$$

The associated non-delayed system is

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}, \quad t \geq 0.$$

Using the strangeness-free reformulation in [13] for the non-delayed system, we differentiate the first equation and insert it into the second equation to obtain

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t) = \begin{bmatrix} f(t) \\ g(t) - \dot{f}(t) \end{bmatrix}.$$

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A publication of the International Linear Algebra Society
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Clearly, if $g(t) + \dot{f}(t) = 0$ holds, then we obtain a unique solution $x(t) = f(t)$.

Performing the same steps for system (4.2), we obtain that

$$\begin{bmatrix}0\\0\end{bmatrix}\dot{x}(t) + \begin{bmatrix}1\\0\end{bmatrix}x(t) + \begin{bmatrix}1\\0\end{bmatrix}x(t-\tau) + \begin{bmatrix}0\\-1\end{bmatrix}\dot{x}(t-\tau) = \begin{bmatrix}f(t)\\g(t)-\dot{f}(t)\end{bmatrix}. \tag{4.3}$$

The second equation of (4.3) not only gives the consistency condition

$$\dot{x}(t-\tau) + g(t) - \dot{f}(t) = 0,\ t \in [0,\tau],$$

but also the constraint

$$\dot{x}(t) + g(t+\tau) - \dot{f}(t+\tau) = 0,\ t \geq 0,$$

and one obtains the system

$$\begin{bmatrix}0\\1\end{bmatrix}\dot{x}(t) + \begin{bmatrix}1\\0\end{bmatrix}x(t) + \begin{bmatrix}1\\0\end{bmatrix}x(t-\tau) = \begin{bmatrix}f(t)\\-g(t+\tau)+\dot{f}(t+\tau)\end{bmatrix}. \tag{4.4}$$

Thus, the step that passes from system (4.3) to (4.4) changes nothing but the inhomogeneity and we can proceed like this without ever terminating. This shows that DDAEs require a different reformulation procedure which terminates after a finite number of steps.

Considering system (4.2) again, we can proceed as follows. Replacing the first equation of (4.2) by its derivative gives

$$\begin{bmatrix}1\\1\end{bmatrix}\dot{x}(t) + \begin{bmatrix}0\\0\end{bmatrix}x(t) + \begin{bmatrix}0\\0\end{bmatrix}x(t-\tau) + \begin{bmatrix}1\\0\end{bmatrix}\dot{x}(t-\tau) = \begin{bmatrix}\dot{f}(t)\\g(t)\end{bmatrix}.$$

Subtracting the first equation from the second we get

$$\begin{bmatrix}0\\1\end{bmatrix}\dot{x}(t) + \begin{bmatrix}0\\0\end{bmatrix}x(t) + \begin{bmatrix}0\\0\end{bmatrix}x(t-\tau) + \begin{bmatrix}1\\0\end{bmatrix}\dot{x}(t-\tau) = \begin{bmatrix}\dot{f}(t)-g(t)\\g(t)\end{bmatrix}.$$

Shifting the time in the first equation by τ, we obtain

$$\begin{bmatrix}1\\1\end{bmatrix}\dot{x}(t) + \begin{bmatrix}0\\0\end{bmatrix}x(t) + \begin{bmatrix}0\\0\end{bmatrix}x(t-\tau) = \begin{bmatrix}\dot{f}(t+\tau)-g(t+\tau)\\g(t)\end{bmatrix},$$

and subtracting the second equation from the first yields

$$\begin{bmatrix}0\\1\end{bmatrix}\dot{x}(t) + \begin{bmatrix}0\\0\end{bmatrix}x(t) + \begin{bmatrix}0\\0\end{bmatrix}x(t-\tau) = \begin{bmatrix}\dot{f}(t+\tau)-g(t+\tau)-g(t)\\g(t)\end{bmatrix}.$$

If the consistency condition in the first equation is satisfied, then we have a unique solution $x(t)$.

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A publication of the International Linear Algebra Society
Volume 23, pp. 703-730, August 2012

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Motivated by Example 4.2, we propose a new procedure to treat the system (1.2) in the behavior form (4.1). The idea is to replace nontrivial scalar DDEs in system (1.2) by (appropriately chosen) derivatives. Since in this way the order of the system may be increased, we study directly general DDAEs of the form (1.1). Setting

$$N := \left[A_k \quad \dots A_0 \;\middle|\; A_{-\kappa} \quad \dots \quad A_{-1} \right] =: [N_+ \quad N_-],$$
$$X_+(t) := \begin{bmatrix} x^{(k)}(t) \\ \vdots \\ x(t) \end{bmatrix},\; X_-(t-\tau) := \begin{bmatrix} x^{(\kappa)}(t-\tau) \\ \vdots \\ x(t-\tau) \end{bmatrix},$$

we have the behavior form of (1.1) given by

$$[N_+ \quad N_-] \begin{bmatrix} X_+(t) \\ X_-(t-\tau) \end{bmatrix} = f(t). \tag{4.5}$$

Set $r := \operatorname{rank}(N_-)$ and $d := \ell - r$ and perform a column compression of N_- as in the following lemma.

Lemma 4.3. *Consider the DDAE* (1.1) *in its behavior form* (4.5)*. Then there exists a nonsingular matrix $P \in \mathbb{C}^{\ell,\ell}$ such that by scaling system* (4.5) *with P from the left, we obtain the system*

$$\begin{bmatrix} F & G \\ H & 0 \end{bmatrix} \begin{bmatrix} X_+(t) \\ X_-(t-\tau) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \quad \begin{matrix} r \\ d \end{matrix} \tag{4.6}$$

where G has full row rank.

Proof. First we determine a matrix $P_2 \in \mathbb{C}^{d,\ell}$ whose rows span the left nullspace of N_-, i.e., $P_2 N_- = 0$ and then we complement P_2 as $P := \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ to a nonsingular matrix. Then

$$P[N_+ \quad N_-] = \begin{bmatrix} P_1 N_+ & P_1 N_- \\ P_2 N_+ & 0 \end{bmatrix} =: \begin{bmatrix} F & G \\ H & 0 \end{bmatrix},$$

and $G = P_1 N_-$ has full row rank. □

Since in (4.6) G has full row rank, we see that the behavior system (4.5) has r nontrivial scalar delay differential equations, and d scalar differential equations.

Since typically the matrix $\begin{bmatrix} F \\ H \end{bmatrix}$ is not strangeness-free, then a first idea would be carry out the strangeness-free formulation (Procedure 3.5) for the DAE

$$\begin{bmatrix} F \\ H \end{bmatrix} X_+(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}, \tag{4.7}$$

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A publication of the International Linear Algebra Society
Volume 23, pp. 703-730, August 2012

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where

$$g(t) := \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} := -\begin{bmatrix} G \\ 0 \end{bmatrix} X_-(t-\tau) + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}.$$

However, as pointed out in Example 4.2, this may not lead to a procedure that terminates in a finite number of steps.

In order to overcome this difficulty, we propose the following approach. Since the order of the DAE $HX_+(t) = f_2(t)$ is at most k, we replace the first equation of system (4.6) by its $(k+1)$-st derivative and obtain the system

$$\begin{bmatrix} F & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_+(t) \\ X_+(t) \end{bmatrix} + \begin{bmatrix} G \\ 0 \end{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_-(t-\tau) = \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} f_1(t) \\ f_2(t) \end{bmatrix}. \tag{4.8}$$

To guarantee that system (4.8) has the same solution set as (4.7), we must require that the following consistency condition holds at $t = 0$

$$\left(\frac{d}{dt}\right)^j \Big(FX_+(t) + GX_-(t-\tau) - f_1(t)\Big) = 0, \; j = 0, \ldots, k+1. \tag{4.9}$$

Thus, we have shown the following lemma.

Lemma 4.4. *Consider system* (4.7) *and assume that the consistency condition* (4.9) *is satisfied at $t = 0$. Then system* (4.7) *has the same solution set as the DDAE* (4.8).

Setting

$$\begin{bmatrix} \breve{f}_1(t) \\ \breve{f}_2(t) \end{bmatrix} := -\begin{bmatrix} G \\ 0 \end{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_-(t-\tau) + \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} f_1(t) \\ f_2(t) \end{bmatrix},$$

we can apply Procedure 3.5 to the DAE

$$\begin{bmatrix} F & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_+(t) \\ X_+(t) \end{bmatrix} = \begin{bmatrix} \breve{f}_1(t) \\ \breve{f}_2(t) \end{bmatrix}$$

and thus immediately obtain the following lemma.

Lemma 4.5. *Consider the DDAE* (1.1) *in its behavior form* (4.5) *and assume that the consistency condition* (4.9) *is satisfied at $t = 0$. Then, system* (1.1) *has the same solution set as the DDAE*

$$\begin{array}{c} r-s \\ s \\ d-v \\ v \end{array} \begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & 0 \\ 0 & \tilde{H} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_+(t) \\ X_+(t) \end{bmatrix} + \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \\ 0 \\ 0 \end{bmatrix} \tilde{X}_-(t-\tau) = \begin{bmatrix} \tilde{f}_1(t) \\ \tilde{f}_2(t) \\ \tilde{f}_3(t) \\ \tilde{f}_4(t) \end{bmatrix}, \tag{4.10}$$

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A publication of the International Linear Algebra Society
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where $\tilde{X}_-(t-\tau) = \begin{bmatrix} x^{(\tilde{\kappa})}(t-\tau) \\ \vdots \\ x(t-\tau) \end{bmatrix}$ *for some* $\tilde{\kappa} \in \mathbb{N}$ *and where the matrix* $\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & \tilde{H} \end{bmatrix}$ *is strangeness-free and of full row rank.*

Since the second equation in system (4.10) is a DAE of the variable $x(t-\tau)$, we can shift it to obtain a DAE for $x(t)$. We summarize this and Lemmas 4.3, 4.4, 4.5, in the following theorem.

THEOREM 4.6. *Consider the DDAE* (1.1) *in its behavior form* (4.5). *Moreover, assume that the consistency condition* (4.9) *is satisfied at* $t = 0$ *and that*

$$\left(\tilde{G}_2\tilde{X}_-(t-\tau) - \tilde{f}_2(t)\right)\Big|_{t\in[0,\tau]} = 0, \tag{4.11}$$

holds. Then, system (1.1) *has the same solution set as the DDAE*

$$\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & \tilde{H} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_+(t) \\ X_+(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tilde{G}_2 \\ 0 \end{bmatrix} \tilde{X}_+(t) + \begin{bmatrix} \tilde{G}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tilde{X}_-(t-\tau) + \tag{4.12}$$
$$= \begin{bmatrix} \tilde{f}_1(t) \\ \tilde{f}_3(t) \\ \tilde{f}_2(t+\tau) \\ \tilde{f}_4(t) \end{bmatrix}, \quad \begin{matrix} r-s \\ d-v \\ s \\ v \end{matrix}$$

where

$$\tilde{X}_+(t) := \begin{bmatrix} x^{(\tilde{\kappa})}(t) \\ \vdots \\ x(t) \end{bmatrix}.$$

In (4.12), *the matrix* $\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & \tilde{H} \end{bmatrix}$ *is strangeness-free and of full row rank.*

By passing from system (4.6) to (4.12), we have reduced the number of scalar delay differential equations from r (in system (4.6)) to $r-s$ (in system (4.12)). However, the number of system equations (number of rows) is still ℓ.

DEFINITION 4.7. The step that passes system (4.6) to (4.12) is called a *reformulation step.* The natural numbers r, d, s, v are called *characteristic invariants* of system (1.1) and of its behavior form (4.6).

Setting $k^{new} := \max\{2k+1, \tilde{\kappa}\}$, $\kappa^{new} := \tilde{\kappa}$, we can bring system (4.10) into behavior form and perform a new reformulation step. Since the number of nontrivial DDEs decreases every time that we perform a reformulation step, this process terminates after finitely many steps.

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A publication of the International Linear Algebra Society
Volume 23, pp. 703-730, August 2012

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We summarize the discussion above in the following procedure.

PROCEDURE 4.8.
Input: *A DDAE of the form* (1.1).
Begin
Set $i = 0$ *and*

$$N^0 := \left[A_k \quad \dots A_0 \quad \middle| \quad A_{-\kappa} \quad \dots \quad A_{-1}\right] =: [N_+^0 \quad N_-^0],$$

$$X_+^0(t) := \begin{bmatrix} x^{(k)}(t) \\ \vdots \\ x(t) \end{bmatrix}, \; X_-^0(t-\tau) := \begin{bmatrix} x^{(\kappa)}(t-\tau) \\ \vdots \\ x(t-\tau) \end{bmatrix},$$

$$X^0(t) = \begin{bmatrix} X_+^0(t) \\ X_-^0(t-\tau) \end{bmatrix}, \; f^0(t) = f(t),$$

and $k^0 = k$, $\kappa^0 = \kappa$, $r^0 = \mathrm{rank}(N_-^0)$, $d^0 = \ell - r^0$.
Step 1. *Determine a nonsingular matrix* $P \in \mathbb{C}^{\ell,\ell}$ *such that by scaling* P *from the left of the behavior system*

$$N^i X^i(t) = f^i(t),$$

we obtain

$$\begin{bmatrix} F & G \\ H & 0 \end{bmatrix} \begin{bmatrix} X_+(t) \\ X_-(t-\tau) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \quad \begin{matrix} r^i \\ d^i \end{matrix} \tag{4.13}$$

where G *has full row rank.*
If $\begin{bmatrix} F \\ H \end{bmatrix}$ *is strangeness-free and has full row rank* ***then STOP***
else *proceed to* ***Step 2.***
Step 2. *Check the consistency conditions*

$$\frac{d^j}{dt^j}\Big(FX_+(t) + GX_-(t-\tau) - f_1(t)\Big) = 0, \; j = 0, \dots, k+1,$$

at $t = 0$. *If they are satisfied, then transform the behavior system* (4.13) *into*

$$\begin{bmatrix} F & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_+(t) \\ X_+(t) \end{bmatrix} + \begin{bmatrix} G \\ 0 \end{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_-(t-\tau) = \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} f_1(t) \\ f_2(t) \end{bmatrix}. \tag{4.14}$$

Step 3. *Apply Procedure 3.5 to system* (4.14) *to obtain*

$$\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & 0 \\ 0 & \tilde{H} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_+(t) \\ X_+(t) \end{bmatrix} + \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \\ 0 \\ 0 \end{bmatrix} \tilde{X}_-(t-\tau) = \begin{bmatrix} \tilde{f}_1(t) \\ \tilde{f}_2(t) \\ \tilde{f}_3(t) \\ \tilde{f}_4(t) \end{bmatrix}, \quad \begin{matrix} r^i - s^i \\ s^i \\ d^i - v^i \\ v^i \end{matrix} \tag{4.15}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 703-730, August 2012

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where

$$\tilde{X}_-(t-\tau) = \begin{bmatrix} x^{(\tilde{\kappa})}(t-\tau) \\ \vdots \\ x(t-\tau) \end{bmatrix},$$

for some $\tilde{\kappa} \in \mathbb{N}$, *and the matrix* $\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & \tilde{H} \end{bmatrix}$ *is strangeness-free and of full row rank.*

Step 4. *Check the consistency condition*

$$\left(\tilde{G}_2 \tilde{X}_-(t-\tau) - \tilde{f}_2(t) \right)\Big|_{t\in[0,\tau]} = 0.$$

If it is satisfied, then shift the second equation of system (4.15) *and permute the second and the third block rows to get*

$$(4.16) \qquad \begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & \tilde{H} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{d}{dt}\right)^{k+1} X_+(t) \\ X_+(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tilde{G}_2 \\ 0 \end{bmatrix} \tilde{X}_+(t) + \begin{bmatrix} \tilde{G}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tilde{X}_-(t-\tau)$$
$$= \begin{bmatrix} \tilde{f}_1(t) \\ \tilde{f}_3(t) \\ \tilde{f}_2(t+\tau) \\ \tilde{f}_4(t) \end{bmatrix}, \quad \begin{matrix} r^i - s^i \\ d^i - v^i \\ s^i \\ v^i \end{matrix}$$

where

$$\tilde{X}_+(t) := \begin{bmatrix} x^{(\tilde{\kappa})}(t) \\ \vdots \\ x(t) \end{bmatrix}, \ \tilde{X}_-(t-\tau) = \begin{bmatrix} x^{(\tilde{\kappa})}(t-\tau) \\ \vdots \\ x(t-\tau) \end{bmatrix}.$$

Step 5. *Reorganize system* (4.16) *in the form*

$$\tilde{A}_{k_1} x^{(k_1)}(t) + \cdots + \tilde{A}_0 x(t) + \tilde{A}_{-1} x(t-\tau) + \cdots + \tilde{A}_{-\kappa_1} x^{(\kappa_1)}(t-\tau) = \tilde{f}_1(t),$$

where $\tilde{A}_j \in \mathbb{C}^{\ell,n}$, $j = -\kappa_1, \ldots, k_1$, $\tilde{A}_{k_1} \neq 0$, $\tilde{A}_{-\kappa_1} \neq 0$, $k_1 \leqslant \max\{2k+1, \tilde{\kappa}\}$, $\kappa_1 \leqslant \tilde{\kappa}$.
Set

$$N^{i+1} := \left[\tilde{A}_{k_1} \quad \ldots \tilde{A}_0 \ \middle| \ \tilde{A}_{-\kappa_1} \quad \ldots \quad \tilde{A}_{-1} \right] := [N_+^{i+1} \quad N_-^{i+1}],$$
$$X_+^{i+1}(t) := \begin{bmatrix} x^{(k_1)}(t) \\ \vdots \\ x(t) \end{bmatrix}, \ X_-^{i+1}(t-\tau) := \begin{bmatrix} x^{(\kappa_1)}(t-\tau) \\ \vdots \\ x(t-\tau) \end{bmatrix},$$
$$X^{i+1}(t) := \begin{bmatrix} X_+^{i+1}(t) \\ X_-^{i+1}(t-\tau) \end{bmatrix}, \ f^{i+1}(t) := \tilde{f}_1(t),$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 703-730, August 2012

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then from (4.16) *we conclude that the number of nontrivial scalar delay differential equations of system*

$$[N_+^{i+1} \quad N_-^{i+1}] \begin{bmatrix} X_+^{i+1}(t) \\ X_-^{i+1}(t-\tau) \end{bmatrix} = f^{i+1}(t),$$

is

$$\operatorname{rank}(N_-^{i+1}) = \operatorname{rank}(\tilde{G}_1) \leqslant r^i - s^i.$$

Set

$$r^{i+1} := \operatorname{rank}(N_-^{i+1}) \leqslant r^i - s^i, \;\; d^{i+1} = \ell - r^{i+1} \geq d^i + s^i,$$

and repeat the process from ***Step 1***.
End

Definition 4.9. Consider the DDAE (1.1) in its behavior form (4.6) and the sequence (r^i, d^i, s^i, v^i), $i \in \mathbb{N}$ of characteristic invariants generated by Procedure 4.8. Then, we call

$$\omega = \min\{i \in \mathbb{N}^0 | \; s^i = 0\}$$

the *delay index* of (1.1).

Theorem 4.10. *Consider the DDAE* (1.2) *and let* ω *be the delay-index of* (1.1). *Moreover, suppose that the consistency conditions* (4.9) *at* $t = 0$, *and* (4.11) *of all reformulation steps* $1, \dots, \omega$ *are satisfied. Then,* (1.1) *has the same solution set as the following DDAE*

$$N_+^\omega X_+^\omega(t) + N_-^\omega X_-^\omega(t) = f^\omega(t), \tag{4.17}$$

with delay-index 0, *where*

$$N_+^\omega = \begin{bmatrix} F_1^\omega & F_2^\omega \\ 0 & H^\omega \\ 0 & 0 \end{bmatrix}, \; N_-^\omega = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}, \; f^\omega = \begin{bmatrix} f_1^\omega \\ f_2^\omega \\ f_3^\omega \end{bmatrix} \begin{array}{c} r^\omega \\ d^\omega \\ v^\omega \end{array} \tag{4.18}$$

$$X^\omega = [x^{(k_\omega)}(t)^T, \dots, x(t)^T, \; x^{(\kappa_\omega)}(t-\tau)^T, \dots, x(t-\tau)^T]^T = \begin{bmatrix} X_+^\omega \\ X_-^\omega \end{bmatrix}.$$

In (4.18), *the matrix* $\begin{bmatrix} F_1^\omega & F_2^\omega \\ 0 & H^\omega \end{bmatrix}$ *is strangeness-free and of full row rank.*

Since (1.2) is a special case of system (1.1), we can apply Theorem 4.10 to study (1.2). Due to Definition 3.9, the fact that matrix $\begin{bmatrix} F_1^\omega & F_2^\omega \\ 0 & H^\omega \end{bmatrix}$ is strangeness-free and

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A publication of the International Linear Algebra Society
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of full row rank implies that there exist a nonsingular matrix $P \in \mathbb{C}^{\ell,\ell}$ such that by scaling system (4.17) with P from the left, we obtain

$$\left[\begin{array}{c|c|c|c} \hat{A}_{k_\omega,1} & \hat{A}_{k_\omega-1,1} & \dots & \hat{A}_{0,1} \\ & \hat{A}_{k_\omega-1,2} & \dots & \hat{A}_{0,2} \\ & & \ddots & \vdots \\ & & & \hat{A}_{0,k_\omega+1} \\ \hline 0 & 0 & \dots & 0 \end{array}\right] \begin{bmatrix} x^{(k_\omega)}(t) \\ x^{(k_\omega-1)}(t) \\ \vdots \\ x(t) \end{bmatrix} + \left[\begin{array}{ccc} * & \dots & * \\ * & \dots & * \\ \vdots & & \vdots \\ * & \dots & * \\ \hline 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x^{(\kappa^\omega)}(t-\tau) \\ \vdots \\ x(t-\tau) \end{bmatrix} = \left[\begin{array}{c} \hat{f}_1(t) \\ \hat{f}_2(t) \\ \vdots \\ \hat{f}_{k_\omega+1}(t) \\ \hline \hat{f}_{k_\omega+2}(t) \end{array}\right] \tag{4.19}$$

and

$$\tilde{A}_{k_\omega} := \begin{bmatrix} \hat{A}_{k_\omega,1} \\ \hat{A}_{k_\omega-1,2} \\ \vdots \\ \hat{A}_{0,k_\omega+1} \end{bmatrix}$$

has full row rank. Rewriting (4.19) block row-wise in behavior form as

$$\left[\begin{array}{c} N_1^\omega \\ N_2^\omega \\ \vdots \\ N_{k_\omega+1}^\omega \\ \hline 0 \end{array}\right] X^\omega = \left[\begin{array}{c} \hat{f}_1(t) \\ \hat{f}_2(t) \\ \vdots \\ \hat{f}_{k_\omega+1}(t) \\ \hline \hat{f}_{k_\omega+2}(t) \end{array}\right], \tag{4.20}$$

we get the following consistency conditions at $t = 0$

$$\begin{aligned} N_1^\omega X^\omega - f_1(t) &= 0, \\ \frac{d^i}{dt^i}\Big(N_2^\omega X^\omega - f_2(t)\Big) &= 0, \;\; i = 0, 1, \\ &\vdots \\ \frac{d^i}{dt^i}\Big(N_{k_\omega+1}^\omega X^\omega - f_{k_\omega+1}(t)\Big) &= 0, \;\; i = 0, \dots, k_\omega + 1, \\ f_{k_\omega+2}(t) &= 0. \end{aligned} \tag{4.21}$$

Therefore, similar to Theorem 3.7, we obtain the following theorem, which stresses that every DDAE of the form (1.2) contains an underlying high-order DDE.

THEOREM 4.11. *Consider the DDAE* (1.2). *Let* ω *be its delay-index and assume that* (4.19) *is the delay-index* 0 *formulation of* (1.2). *Moreover, assume that the*

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A publication of the International Linear Algebra Society
Volume 23, pp. 703-730, August 2012

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consistency conditions (4.9) *at* $t = 0$*, and* (4.11) *of all reformulation steps* $1, \ldots, \omega$ *are satisfied. Furthermore, suppose that the consistency condition* (4.21) *is also satisfied. Then,* (1.2) *has the same solution set as the DDE*

$$\tilde{A}_{k_\omega} x^{(k_\omega)}(t) + \cdots + \tilde{A}_0 x(t) + \tilde{A}_{-1} x(t-\tau) + \cdots + \tilde{A}_{-\kappa_\omega - k_\omega} x^{(\kappa_\omega + k_\omega)}(t-\tau) + \tilde{f}^{\omega}(t) = 0,$$

where $\tilde{A}_{k_\omega}$ *has full row rank.*

Proof. The proof follows by differentiating the j-th equation of system (4.20) $j-1$ times for each $2 \leqslant j \leqslant k_\omega + 1$. □

Based on Theorem 4.11 we have the following solvability result.

Corollary 4.12. *Consider the DDAE* (1.2)*. Moreover, assume that the function* f *is sufficiently smooth.*

i) The DDAE (1.2) *is solvable if and only if the consistency conditions* (4.9) *at* $t = 0$*, and* (4.11) *of all reformulation steps* $1, \ldots, \omega$ *are satisfied and also the consistency condition* (4.21) *is satisfied.*
ii) The initial value problem (1.2)–(1.3) *is uniquely solvable if and only if in addition the matrix* $\tilde{A}_{k_\omega}$ *is square.*

We illustrate our result by the following example.

Example 4.13. Consider the first order DDAE

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{A_1} \dot{x}(t) + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_0} x(t) + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_{-1}} x(t-\tau) = \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}}_{f}, \; t \geq 0.$$

Clearly, the matrix pencil $\lambda A_1 - A_0$ is singular, and hence it if we insert $x|_{[-\tau,0]} = \phi$ to determine $x|_{[0,\tau]}$ from the DAE

$$A_1 \dot{x}(t) + A_0 x(t) = -A_{-1}\phi(t) + f(t), \; t \in (0, \tau),$$

we obtain an under-determined system, and therefore, $x|_{[0,\tau]}$ is not uniquely determined. However, using Procedure 4.8, we finally arrive at

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dddot{x}(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{bmatrix},$$

where

$$\begin{bmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{bmatrix} = \begin{bmatrix} \dddot{f}_1(t+\tau) - \dddot{f}_2(t+2\tau) + \ddot{f}_3(t+2\tau) \\ \ddot{f}_2(t+\tau) - \dot{f}_3(t+\tau) \\ f_3(t) \end{bmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 703-730, August 2012
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The underlying third order DAE then is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \dddot{x}(t) = \begin{bmatrix} g_1(t) \\ \dot{g}_2(t) \\ \ddot{g}_3(t) \end{bmatrix}, \ t \geq 0,$$

and it determines a unique solution $x|_{[0,\tau]}$.

5. Conclusion. In this paper, we have presented the theoretical analysis for a class of delay differential-algebraic equations (DDAEs). We have proved that under some consistency conditions every DDAE with single delay can be reformulated as a high order DDE. We have also introduced an appropriate delay-index for nontrivial DDAEs and constructed strangeness-free reformulations that can be used to investigate solvability, consistency and smoothness requirements.

Acknowledgment. The authors would like to thank two anonymous referees for their valuable comments and remarks that improved the paper.

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