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Positive semidefinite maximum nullity and zero forcing number

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Abstract. The zero forcing number $Z(G)$ is used to study the minimum rank/maximum nullity of the family of symmetric matrices described by a simple, undirected graph $G$. The positive semidefinite zero forcing number is a variant of the (standard) zero forcing number, which uses the same definition except with a different color-change rule. The positive semidefinite maximum nullity and zero forcing number for a variety of graph families are computed. In addition, field independence of the minimum rank of the hypercube is established, by showing there is a positive semidefinite matrix that is universally optimal.

Key words. Zero forcing number, Maximum nullity, Minimum rank, Positive semidefinite, Matrix, Graph.

AMS subject classifications. 05C50, 15A03, 15B57.

1. Introduction. The minimum rank/maximum nullity problem and zero forcing have been studied quite extensively. For many graphs and families of graphs, the maximum nullity and zero forcing number have been determined, and these are reported in the AIM graph catalog [2]. However, a great deal of work remains in determining the positive semidefinite maximum nullity and zero forcing number of many of these graphs and graph families. In Section 3 we use a variety of techniques to compute the positive semidefinite maximum nullity and zero forcing number for many of the graphs in the AIM graph catalog [2], and these results are summarized in Table 3.1. In Section 2 we determine the positive semidefinite maximum nullity and zero forcing number of all hypercubes by constructing a vector representation recursively. Our method produces a universally optimal matrix and establishes field independence of the minimum rank, answering an open question (see [8], [17]).

Every graph discussed in this paper is simple (no loops or multiple edges), undirected, and has a finite nonempty vertex set. Given a graph $G = (V, E)$, color the vertices of $G$ either black or white. This is known as an initial coloring of $G$. Vertices change color according to the positive semidefinite color-change rule: Let $S$ denote the set of black vertices of $G$. Identify the sets of vertices $W_1, \ldots, W_k$ of the $k$ compo-
ments of $G - S$. If $u \in S$ and $w \in W_i$ is the only white neighbor of $u$ in $G[W_i \cup S]$, then change the color of $w$ to black. Given an initial coloring of $G$, the derived set is the set of initial black vertices along with vertices that are colored black after repeated application of the positive semidefinite color-change rule, i.e., until no more changes are possible. A positive semidefinite zero forcing set of $G$ is a subset $B \subseteq V(G)$ such that if initially the vertices of $B$ are colored black and the remaining vertices are colored white, then the derived set is $V(G)$. The positive semidefinite zero forcing number $Z_+(G)$ is defined as the minimum of $|B|$ over all positive semidefinite zero forcing sets $B \subseteq V(G)$. The positive semidefinite zero forcing number is a variant of the (standard) zero forcing number, which uses the same definition except with a different color-change rule: If $u$ is black and $w$ is the only white neighbor of $u$, then change the color of $w$ to black. In [1], it was shown that the zero forcing number is an upper bound for the maximum nullity of a graph. Similarly, it was shown in [3] that the positive semidefinite zero forcing number is an upper bound for the maximum positive semidefinite nullity of a graph.

We denote the set of real symmetric $n \times n$ matrices by $S_n(\mathbb{R})$, and we denote the set of (possibly complex) Hermitian $n \times n$ matrices by $H_n$. Given a matrix $A \in H_n$, the graph of $A$, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Notice that the diagonal of $A$ is ignored in determining $\mathcal{G}(A)$. We denote the set of real symmetric (real positive semidefinite) matrices of a graph by $S(\mathcal{G})$ ($S_+(\mathcal{G})$). The minimum positive semidefinite rank of $G$ and the minimum Hermitian positive semidefinite rank of $G$ are, respectively,

$$mr_+^P(G) = \min \{ \text{rank}(A) : A \in S_n \text{ is positive semidefinite and } \mathcal{G}(A) = G \}$$

and

$$mr_+^C(G) = \min \{ \text{rank}(A) : A \in H_n \text{ is positive semidefinite and } \mathcal{G}(A) = G \}.$$

The maximum positive semidefinite nullity of $G$ and the maximum Hermitian positive semidefinite nullity of $G$ are, respectively,

$$M_+^P(G) = \max \{ \text{null}(A) : A \in S_n \text{ is positive semidefinite and } \mathcal{G}(A) = G \}$$

and

$$M_+^C(G) = \max \{ \text{null}(A) : A \in H_n \text{ is positive semidefinite and } \mathcal{G}(A) = G \}.$$

Note that $mr_+^P(G) + M_+^P(G) = |G|$ and $mr_+^C(G) + M_+^C(G) = |G|$.

If $mr_+^P(G) = mr_+^C(G)$, then we denote the common value $mr_+^P(G) = mr_+^C(G)$ by $mr_+(G)$. Similarly, we denote the common value $M_+^P(G) = M_+^C(G)$ by $M_+(G)$. With
the exception of the Möbius ladder, all of the graphs discussed in this paper have $\mathcal{M}_R^B(G) = \mathcal{M}_C^B(G) = \mathcal{Z}_+(G)$, and all have $M_R^B(G) = M_C^B(G)$.

**Observation 1.1.** If $M_R^B(G) = \mathcal{Z}_+(G)$, then $M^B(G) = \mathcal{Z}_+(G)$ as $M_R^B(G) \leq M_C^B(G) \leq \mathcal{Z}_+(G)$ for every graph $G$.

To discuss field independence of the hypercube, we introduce minimum rank over fields other than $\mathbb{R}$ and $\mathbb{C}$. We denote the set of $n \times n$ symmetric matrices over a field $F$ by $S_n(F)$. The minimum rank over field $F$ of a graph $G$ is

$$\text{mr}^F(G) = \min \{ \text{rank}(A) : A \in S_n(F), \mathcal{G}(A) = G \},$$

and the maximum nullity over $F$ of a graph $G$ is

$$\mathcal{M}^F(G) = \max \{ \text{null}(A) : A \in S_n(F), \mathcal{G}(A) = G \}.$$  

Clearly,

$$\text{mr}^F(G) + \mathcal{M}^F(G) = |G|.$$  

The minimum rank of a graph $G$ is field independent if the minimum rank of $G$ is the same over all fields. A matrix $A \in S_n(F)$ is optimal for a graph $G$ over a field $F$ if $\mathcal{G}(A) = G$ and rank$^F(A) = \text{mr}^F(G)$. If $A$ is an integer matrix, then $A$ can be viewed as a matrix with entries in $\mathbb{Z}_p$ for $p$ a prime, and hence $A \in F^{n \times n}$ where $\mathbb{Z}_p \subseteq F$ or $\mathbb{Z} \subseteq F$. Note that the graph of $A$ may depend on the characteristic of the field (e.g., $2 \equiv 0$ in $\mathbb{Z}_2$). But if all off-diagonal entries are 0, 1, or -1, the graph $\mathcal{G}(A)$ does not depend on the field. A universally optimal matrix is an integer matrix $A$ such that every off-diagonal entry of $A$ is 0, 1, or -1, and rank$^F(A) = \text{mr}^F(\mathcal{G}(A))$ for all fields $F$.

DeAlba et al. [8] explored universally optimal matrices and field independence of the minimum rank for a number of graph families. However, they were unable to verify field independence of the minimum rank of $Q_n$ or to find a universally optimal matrix for $Q_n$. Huang et al. [17] later found universally optimal matrices for a subclass of the class of hypercubes. We improve upon these results by verifying that the minimum rank of the hypercube is field independent and finding a universally optimal matrix for every hypercube.

We will need some additional terminology. The subgraph $G[U]$ of $G = (V, E)$ induced by $U \subseteq V$ is the subgraph with vertex set $U$ and edge set $\{ \{i, j\} \in E : i, j \in U \}$. A subgraph $G'$ of a graph $G$ is a clique if there is an edge between every pair of vertices of $G'$, i.e., $G' \cong K_{|G'|}$. A clique covering of $G$ is a set of subgraphs of $G$ that are cliques and such that every edge of $G$ is contained in at least one clique. The clique covering number of $G$, $\text{cc}(G)$, is the fewest number of cliques in a clique covering of $G$. 
Let $G = (V, E)$ be a graph with ordered set of vertices $V = \{v_1, v_2, \ldots, v_n\}$. We associate a vector $\vec{v}_i \in \mathbb{R}^d$ with each vertex $v_i$ of $G$ (for minimum Hermitian positive semidefinite rank, use $\vec{v}_i \in \mathbb{C}^d$). If two vertices $v_i$ and $v_j$ are adjacent, then $\langle \vec{v}_i, \vec{v}_j \rangle \neq 0$, where $\langle \vec{v}_i, \vec{v}_j \rangle$ denotes the Euclidean inner product. If two vertices $v_i$ and $v_j$ are not adjacent, then $\langle \vec{v}_i, \vec{v}_j \rangle = 0$. We say $\vec{X} = \{\vec{v}_i\}_{i=1}^n$ is a vector representation of $G$. Let

$$X = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}.$$ 

Then $X^T X$ is a positive semidefinite matrix called the Gram Matrix of $\vec{X}$ with respect to the Euclidean inner product. The graph of $X^T X$ has vertices 1, 2, $\ldots$, $n$ corresponding to the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and edges corresponding to nonzero inner products among these vectors, i.e., $\mathcal{G}(X^T X) \cong G$. Furthermore, $\text{mr}^\mathbb{R}_+(G) \leq d$ if and only if there is a vector representation of $G$ in $\mathbb{R}^d$ (and analogously for $\text{mr}^\mathbb{C}_+(G)$ and $\mathbb{C}^d$).

2. Hypercube. The $n$-cube $Q_n$, $n \geq 1$, is defined as the repeated Cartesian product of $n$ complete graphs on two vertices. Specifically, $Q_1 = K_2$ and $Q_n = Q_{n-1} \square K_2$ for $n \geq 2$. The $n$-cube is often referred to as the $n$th hypercube. If $V(K_2) = \{0, 1\}$, then the vertex set of $Q_n$ can be viewed as the set of $n$-tuples $(v_1, v_2, \ldots, v_n)$, where $v_i \in \{0, 1\}$. Moreover, two $n$-tuples share an edge if they differ in exactly one coordinate. The hypercubes $Q_3$ and $Q_4$ are shown in Figure 2.1

![Fig. 2.1. The hypercubes $Q_3$ and $Q_4.$](image)

Mitchell et al. [20] determined the minimum positive semidefinite rank of $Q_3$. In the next theorem, we determine the positive semidefinite maximum nullity and zero forcing number of all hypercubes. We construct a vector representation recursively. Note that the maximum nullity, maximum positive semidefinite nullity, zero forcing number, and positive semidefinite zero forcing number of the hypercube are all equal, and our method produces a universally optimal matrix, thereby establishing field independence. The technique of vector representation has not previously been used to establish field independence of minimum rank or to find a universally optimal
matrix. We begin by illustrating the construction by example.

**Example 2.1.** Let

\[
X_3 = \begin{bmatrix}
X_2 & C'_2 \\
C_2 & X'_2
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & -1 & 0 & 1 & 0 & 1 & 1
\end{bmatrix},
\]

where the set of column vectors of \(X_2\) is a vector representation of \(Q_2\) and \(X'_2\) is obtained by interchanging the first and second columns of \(X_2\) as well as the third and fourth columns of \(X_2\). Furthermore,

\[
C_2 = \begin{bmatrix}
C_1 \\
0 \\
C'_1
\end{bmatrix}
\quad \text{and} \quad
C'_2 = \begin{bmatrix}
C'_1 \\
0 \\
C_1
\end{bmatrix},
\]

where \(C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}\) and \(C'_1 = \begin{bmatrix} -1 & 0 \end{bmatrix}\). Then the set of column vectors of \(X_3\) is a vector representation of \(Q_3\) because

\[
X'^T_3 X_3 =
\begin{bmatrix}
1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\
1 & 3 & 0 & 1 & 0 & 1 & 0 & 0 \\
-1 & 0 & 3 & 1 & 0 & 0 & -1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 3 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & -1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & -1 & 1 & 3
\end{bmatrix},
\]

**Definition 2.2.** Let \(C_n = C_{n-1} \oplus C'_{n-1}\) and \(C'_n = C'_{n-1} \oplus C_{n-1}\), where \(C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}\) and \(C'_1 = \begin{bmatrix} -1 & 0 \end{bmatrix}\). Let

\[
X_n = \begin{bmatrix}
X_{n-1} & C'_{n-1} \\
C_{n-1} & X'_{n-1}
\end{bmatrix},
\]

where \(X_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}\) and the set of column vectors of \(X_{n-1}\) is a vector representation of \(Q_{n-1}\) and \(X'_{n-1} = X_{n-1} P_{n-1}\), where \(P_{n-1} = S_2 \oplus \cdots \oplus S_2\) and \(S_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\).

**Observation 2.3.** If the set of column vectors of \(X_n\) is a vector representation of \(Q_n\) and \(X'_n = X_n P_n\), where \(P_n = S_2 \oplus \cdots \oplus S_2\) and \(S_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\), then \(G((X'_n)^T X'_n) = Q_n\).

**Theorem 2.4.** \(M_+(Q_n) = Z_+(Q_n) = 2^{n-1}\) and \(mr_+(Q_n) = 2^{n-1}\), and the set of column vectors of \(X_n\) is a vector representation of \(Q_n\).
Proof. The proof is by inductively constructing a vector representation of $Q_n$ in $\mathbb{R}^{2^n-1}$. We can extend the pattern illustrated in Example 2.1 in general. Let $C_n$ and $C'_n$ be defined as in Definition 2.2. Since $C_1^n C_1$ and $(C'_1^n)^T C'_1$ are diagonal, so are $C_1^n C_n = C_1^n C_{n-1} \oplus (C'_1^n)^T C'_n$ and $(C'_1^n)^T C'_n = (C'_1^n)^T C_{n-1} \oplus C_1^n C_{n-1}$. Next, we give an equivalent formulation of $C'_n$. Let $P_n = S_2 \oplus \cdots \oplus S_2$, where $S_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We show by induction that $C'_n = -C_n P_n$. Clearly, $C'_1 = -C_1 P_1$. Suppose $C'_{n-1} = -C_{n-1} P_{n-1}$. Then

$$C'_n = C'_{n-1} \oplus C_{n-1}$$
$$= -C_{n-1} P_{n-1} \oplus C_{n-1}$$
$$= -(C_{n-1} \oplus -C_{n-1} P_{n-1}) P_n$$
$$= -(C_{n-1} \oplus C'_{n-1}) P_n$$
$$= -C_n P_n.$$

Let $X_n = \begin{bmatrix} X_{n-1} & C'_{n-1} \\ C_{n-1} & X'_{n-1} \end{bmatrix}$, as in Definition 2.2. First, we show by induction that $X_n^T C'_n + C_n^T X_n' = D_2 \oplus \cdots \oplus D_2$, where $D_2 := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Observe that $X_n^T C'_n + C_n^T X_n' = D_2.$ Assume $X_{n-1}^T C'_n + C_n^T X_{n-1}' = D_2 \oplus \cdots \oplus D_2$. Then

$$X_{n}^T C'_n + C_n^T X_n' = X_{n}^T C'_n + C_n^T X_n P_n$$
$$= \begin{bmatrix} X_{n-1}^T & C_{n-1}^T \\ (C_{n-1})^T & X'_{n-1} \end{bmatrix} \begin{bmatrix} C_{n-1} & 0 \\ 0 & C_{n-1} \end{bmatrix} + \begin{bmatrix} C_{n-1}^T & 0 \\ 0 & (C_{n-1})^T \end{bmatrix} \begin{bmatrix} X_{n-1} & C_{n-1} \\ C_{n-1}^T & X'_{n-1} \end{bmatrix} P_n$$
$$= \begin{bmatrix} X_{n-1}^T C_{n-1} C_{n-1} & C_{n-1}^T C_{n-1} \end{bmatrix} + \begin{bmatrix} C_{n-1} & 0 \\ 0 & C_{n-1} \end{bmatrix} \begin{bmatrix} X_{n-1} & C_{n-1} \\ C_{n-1}^T & X'_{n-1} \end{bmatrix} P_n$$
$$= \begin{bmatrix} X_{n-1}^T C_{n-1} C_{n-1} + C_{n-1}^T C_{n-1} \end{bmatrix} + \begin{bmatrix} C_{n-1} & 0 \\ 0 & C_{n-1} \end{bmatrix} \begin{bmatrix} X_{n-1} & C_{n-1} \\ C_{n-1}^T & X'_{n-1} \end{bmatrix} P_n$$
$$= \begin{bmatrix} X_{n-1}^T C_{n-1} + C_{n-1}^T X'_{n-1} \\ 0 \end{bmatrix} + \begin{bmatrix} C_{n-1} & 0 \\ 0 & C_{n-1} \end{bmatrix} \begin{bmatrix} X_{n-1} & C_{n-1} \\ C_{n-1}^T & X'_{n-1} \end{bmatrix} P_n$$
$$= \begin{bmatrix} X_{n-1}^T C_{n-1} + C_{n-1}^T X'_{n-1} \\ 0 \end{bmatrix} + \begin{bmatrix} C_{n-1} & 0 \\ 0 & C_{n-1} \end{bmatrix} \begin{bmatrix} X_{n-1} & C_{n-1} \\ C_{n-1}^T & X'_{n-1} \end{bmatrix} P_n$$
$$= (D_2 \oplus \cdots \oplus D_2) \oplus (D_2 \oplus \cdots \oplus D_2).$$

Finally, we show by induction that the set of column vectors of $X_n$ is a vector
representation of $Q_n$.

$$X^T_n X_n = \begin{bmatrix} X^T_{n-1} & C^T_{n-1} \\ (C'_{n-1})^T & (X'_{n-1})^T \end{bmatrix} \begin{bmatrix} X_{n-1} & C'_{n-1} \\ C_{n-1} & X_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} X^T_{n-1}X_{n-1} + C^T_{n-1}C_{n-1} & X^T_{n-1}C'_{n-1} + C^T_{n-1}X'_{n-1} \\ (C'_{n-1})^T X_{n-1} + (X'_{n-1})^T C_{n-1} & (C'_{n-1})^T C'_{n-1} + (X'_{n-1})^T X'_{n-1} \end{bmatrix}.$$  

By the induction hypothesis, the set of column vectors of $X_{n-1}$ is a vector representation of $Q_{n-1}$, so $G(X^T_{n-1}X_{n-1}) = Q_{n-1}$. Because $X'_{n-1} = X_{n-1}P_{n-1}$ and the set of column vectors of $X_{n-1}$ is a vector representation of $Q_{n-1}$, $G((X'_{n-1})^T X'_{n-1}) = Q_{n-1}$ by Observation 2.3. Since $C^T_{n-1}C_{n-1}$ and $(C'_{n-1})^T C'_{n-1}$ are diagonal, $X^T_{n-1}X_{n-1} + C^T_{n-1}C_{n-1} \in \mathcal{S}_+(Q_{n-1})$ and $(X'_{n-1})^T X'_{n-1} + (C'_{n-1})^T C'_{n-1} \in \mathcal{S}_+(Q_{n-1})$. Furthermore, $X'_{n-1}C_{n-1} + C'_{n-1}X'_{n-1} = D_2 \oplus \cdots \oplus D_2$. Thus, $G(X^T_n X_n) = Q_n$.

We have $M^2_+(Q_n) \leq Z_+(Q_n) \leq Z(Q_n) = 2^{n-1}$ by [1], so $2^{n-1} = 2^n - 2^{n-1} \leq m^2_+(Q_n) \leq 2^{n-1}$ as $X_n$ is a representation of $Q_n$ in $\mathbb{R}^{2^{n-2}}$. \[ \square \]

**Theorem 2.5.** The minimum rank of $Q_n$ is field independent and $X^T_n X_n$ is a universally optimal matrix for $Q_n$ for the representation $X_n$ in Definition 2.2.

**Proof.** Given a graph $G$, $M^F(G) \leq Z(G)$ for any field $F$, and $Z(Q_n) = 2^{n-1}$ by [2]. Thus, if we assume $X^T_n X_n \in \mathcal{S}(F, Q_n)$, $2^n - 2^{n-1} = 2^n - 1 \leq m^F(Q_n) \leq \text{rank}^F(X^T_n X_n) \leq 2^{n-1}$. Hence, $m^F(Q_n) = 2^{n-1} = m^R(Q_n)$, establishing field independence of the minimum rank of $Q_n$. To show $X^T_n X_n$ is a universally optimal matrix for $Q_n$ and thus that $Q_n$ is field independent, it suffices to show that every off-diagonal entry of $X^T_n X_n$ is 0, 1, or -1. The proof is by induction. The off-diagonal entries of $X^T_n X_1$ are 1. Suppose every off-diagonal entry of $X^T_{n-1} X_{n-1}$ is 0, 1, or -1. Then the off-diagonal entries of $X^T_{n-1} X_{n-1} + C^T_{n-1}C_{n-1}$ and $(C'_{n-1})^T C'_{n-1} + (X'_{n-1})^T X'_{n-1}$ are 0, 1, or -1 as $C^T_{n-1}C_{n-1}$ and $(C'_{n-1})^T C'_{n-1}$ are diagonal matrices. We have already established in the proof of Theorem 2.4 that $X^T_{n-1} C_{n-1} + C^T_{n-1} X_{n-1} = D_2 \oplus \cdots \oplus D_2$. Thus, the off-diagonal entries of $X^T_n X_n$ are 0, 1, or -1. \[ \square \]

**3. Graph families.** In this section, we determine the positive semidefinite maximum nullity and zero forcing number of a variety of graph families. The results are summarized in Table 3.1. Many of these graph families appear in a graph catalog developed through the American Institute of Mathematics workshop “Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns” [2]. We use a variety of known results to determine the maximum positive semidefinite nullity.

The vertex connectivity $\kappa(G)$ of a connected graph $G$ is the minimum size of $S \subseteq V(G)$ such that $G - S$ is disconnected or a single vertex. The following result
is especially useful when the vertex connectivity and the positive semidefinite zero forcing number of a graph agree.

**Theorem 3.1.** [18][19] For a graph $G$, $M^+_G(G) \geq \kappa(G)$.

A minor of a graph $G$ is obtained by performing a series of edge deletions, deletions of isolated vertices, and edge contractions. If $\beta(H) \leq \beta(G)$ for any minor $H$ of $G$, then the graph parameter $\beta$ is said to be minor monotone. Colin de Verdière [7] in English introduced a graph parameter $\mu(G)$ that was the first of several parameters that bound the maximum nullity from below and are minor monotone. The parameter $\nu(G)$ [6] is minor monotone and is the maximum nullity among matrices $A \in S_n$ satisfying:

- $G(A) = G$.
- $A$ is positive semidefinite.
- $A$ satisfies the Strong Arnold Hypothesis.

ˇSpacapan established the following result concerning the vertex connectivity of a Cartesian product. We use this result to determine the vertex connectivity of the Cartesian product of a complete graph and a path, thereby providing a lower bound for the maximum positive semidefinite nullity.

**Theorem 3.2.** [22] Let $G$ and $H$ be graphs of order at least two. Then

$$\kappa(G \square H) = \min\{\kappa(G)|H|, \kappa(H)|G|, \delta(G) + \delta(H)\}.$$ 

**Proposition 3.3.** For $s \geq 2$ and $t \geq 2$, $M^+_+(K_s \square P_t) = Z^+_+(K_s \square P_t) = s$ and $\text{mr}^+_+(K_s \square P_t) = s(t - 1)$.

*Proof.* By Theorem 3.2 $\kappa(K_s \square P_t) = \min\{\kappa(K_s)|P_t|, \kappa(P_t)|K_s|, \delta(K_s) + \delta(P_t)\} = \min\{(s - 1)t, s, s\} = s$. Hence, $s = \kappa(K_s \square P_t) \leq M^+_+(K_s \square P_t) \leq Z^+_+(K_s \square P_t) \leq Z(K_s \square P_t) = s$ by Theorem 3.1 and [1]. Hence, all inequalities are equalities. \[\square\]

For the complete multipartite graph, the vertex connectivity and the positive semidefinite zero forcing number are equal.

**Proposition 3.4.** For $n_1 \geq n_2 \geq \cdots \geq n_k > 0$,

$$M^+_+(K_{n_1, n_2, \ldots, n_k}) = Z^+_+(K_{n_1, n_2, \ldots, n_k}) = n_2 + n_3 + \cdots + n_k$$

and $\text{mr}^+_+(K_{n_1, n_2, \ldots, n_k}) = n_1$.

If a graph has very low or very high maximum positive semidefinite nullity or positive semidefinite zero forcing number, then the two parameters are equal. In
positive semidefinite maximum nullity and zero forcing number

Observation 3.7. $M_+ (G) = Z_+(G)$ if $M_+ (G) \leq 2$ or $Z_+(G) \geq |G| - 2$. The technique of using $M_+ (G) = Z_+(G)$ is used to prove the next proposition.

**Proposition 3.5.** $M_+ (G) = Z_+(G) = 2$ and $\max_+(G) = |G| - 2$ for $G$ a polygonal path and $G$ a unicyclic graph. Let $T$ be a tree with $|T| = n$, $n \geq 4$, and $T \neq K_{1,n-1}$. Then $M_+ (\overline{T}) = Z_+(\overline{T}) = n - 3$ and $\max_+(\overline{T}) = 3$.

**Corollary 3.6.** For $s \geq 2$, $M_+ (P_s \square P_2) = Z_+(P_s \square P_2) = 2$ and $\max_+(P_s \square P_2) = 2s - 2$.

**Proof.** Since $P_s \square P_2$ is a polygonal path, the result follows from Proposition 3.5.

If we can find a minor $H$ of $G$ with $\nu(H) = Z(G)$, then the maximum nullity, maximum positive semidefinite nullity, zero forcing number, and positive semidefinite zero forcing number of $G$ are all equal, as stated in the next observation. We use this technique to prove Proposition 3.9, Proposition 3.10, and Corollary 3.13.

**Observation 3.7.** If $G$ has a minor $H$ with $\nu(H) = Z(G)$, then $\nu(H) = \nu(G) = M_+(G) = M(G) = Z_+(G) = Z(G)$.

Hein van der Holst [16] showed that $\xi(Q_3) = 4$. Let $X_3$ and $X_4$ be the vector representations in Definition 2.2. One can verify by computation with software that $X_3^T X_3 \in S_+(Q_3)$ satisfies the Strong Arnold Hypothesis. This result is useful because $Q_3$ is a minor of $C_3 \square P_2$ for $s \geq 4$ and of $C_4 \square P_2$, and the standard zero forcing number of these graphs is equal to $\nu(Q_3)$. Note that $X_3^T X_4 \in S_+(Q_4)$ does not satisfy the Strong Arnold Hypothesis.

**Proposition 3.8.** Let $X_3$ be the vector representation in Definition 2.2. Then the matrix $X_3^T X_3 \in S_+(Q_3)$ satisfies the Strong Arnold Hypothesis, and so $\nu(Q_3) = 4$.

**Proposition 3.9.** For $s \geq 4$, $M_+(C_s \square P_2) = Z_+(C_s \square P_2) = 2s - 4$ and for $t \geq 2$, $M_+(C_t \square P_1) = Z_+(C_t \square P_1) = 4$ and $\max_+(C_t \square P_1) = 4t - 4$.

**Proof.** Observe that $Q_3$ is a minor of $C_s \square P_2$ for $s \geq 4$ and of $C_4 \square P_2$. We know that $\nu(Q_3) = 4$ by Proposition 3.8 so $\nu(Q_3) = 4 = Z(C_s \square P_2) = Z(C_t \square P_1)$ by [11]. By Observation 3.7, $M_+(C_s \square P_2) = Z_+(C_s \square P_2) = M_+(C_t \square P_1) = Z_+(C_t \square P_1) = 4$.

The $s$th half-graph $H_s$ is the graph constructed from the disjoint union of $K_s$ and $\overline{K}_s$ by adding all edges $\{u_i, v_j\}$ such that $i + j \leq s + 1$, where $u_1, \ldots, u_s$ are the vertices of $K_s$ and $v_1, \ldots, v_s$ are the vertices of $\overline{K}_s$. Although the next proposition uses minors, it is worth noting that the matrix $A_s$ of rank $s$ constructed in [8] whose graph is $H_s$ is positive semidefinite. A wheel graph $W_n$ is constructed from $C_{n-1}$ by
adding one vertex that is adjacent to each of the vertices of \( C_{n-1} \).

**Proposition 3.10.** For \( n \geq 4 \), \( M_+ (W_n) = Z_+ (W_n) = 3 \) and \( \text{mr}_+ (W_n) = n - 3 \) and \( M_+ (H_s) = Z_+ (H_s) = s \) and \( \text{mr}_+ (H_s) = s \).

**Proof.** Observe that \( K_4 \) is a minor of \( W_n \). Moreover, \( \nu (K_4) = 3 = Z(W_n) \) by [14]. By Observation 3.7, \( M_+ (W_n) = Z_+ (W_n) = 3 \).

Let \( R = \{ u_1, \ldots , u_s, v_1 \} \subseteq V(H_s) \). Then \( H_s [R] = K_{s+1} \) is a minor of \( H_s \). Moreover, \( \nu (K_{s+1}) = s = Z(H_s) \) by [8]. By Observation 3.7, \( M_+ (H_s) = Z_+ (H_s) = s \).

Although the maximum positive semidefinite nullity and positive semidefinite zero forcing number have not been determined in general for the Cartesian product of two complete graphs and the Cartesian product of a cycle and a complete graph, we use vector representations to establish these values for the Cartesian product of a complete graph on three vertices with itself and the Cartesian product of a four cycle with a complete graph on three vertices. Since the maximum nullity, maximum positive semidefinite nullity, zero forcing number, and positive semidefinite zero forcing number are all equal for these particular graphs, we conjecture that the same is true in general for the Cartesian product of two complete graphs and the Cartesian product of a cycle and a complete graph. We number the vertices of \( K_3 \square K_3 \) and \( C_4 \square K_3 \) as in Figure 3.1.

![Fig. 3.1. The Cartesian products \( K_3 \square K_3 \) and \( C_4 \square K_3 \).](image)

**Proposition 3.11.** \( M_+ (K_3 \square K_3) = Z_+ (K_3 \square K_3) = 5 \) and \( \text{mr}_+ (K_3 \square K_3) = 4 \).

**Proof.** Let

\[
X = \begin{bmatrix}
1 & 1/2 & -1/2 & 2 & 0 & 0 & -2 & 0 & 0 \\
0 & 1 & 0 & -2 & 1 & 1 & 0 & 1/2 & 0 \\
0 & 0 & -2 & 0 & 0 & -1 & 1 & 1/2 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}.
\]
Then the set of column vectors of $X$ is a vector representation of $K_3 \square K_3$. \[\Box\]

**Proposition 3.12.** $M_+(C_4 \square K_3) = Z_+(C_4 \square K_3) = 6 = \nu(C_4 \square K_3)$ and $\text{mr}_+(C_4 \square K_3) = 6$.

**Proof.** Let

$$X = \begin{pmatrix} 1 & 1/2 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1/2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1/2 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1/2 & 1 & . \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$  

Then the set of column vectors of $X$ is a vector representation of $C_4 \square K_3$. One can verify by computation with software that $X^TX \in S_+ (C_4 \square K_3)$ satisfies the Strong Arnold Hypothesis, so $\nu(C_4 \square K_3) = 6$. \[\Box\]

**Corollary 3.13.** For $s \geq 4$,

$$M_+(C_s \square K_3) = Z_+(C_s \square K_3) = 6$$

and $\text{mr}_+(C_s \square K_3) = 3s - 6$.

**Proof.** Observe that $C_4 \square K_3$ is a minor of $C_s \square K_3$ for $s \geq 4$. Thus, $\nu(C_4 \square K_3) = 6 = Z(C_s \square K_3)$ by Proposition 3.12 and \[1\]. By Observation 3.7, $M_+(C_s \square K_3) = Z_+(C_s \square K_3) = 6$. \[\Box\]

Members of the AIM Minimum Rank - Special Graphs Work Group \[1\] used the technique in the next observation (using cliques) to determine the minimum positive semidefinite rank of the supertriangle $T_n$, the strong product of two paths $P_s \boxtimes P_t$, and the corona $K_t \circ K_s$. The $(m, k)$-pineapple (with $m \geq 3$, $k \geq 2$) is $P_{m,k} = K_m \cup K_{1,k}$ where $K_m \cap K_{1,k}$ is the degree $k$ vertex of $K_{1,k}$. The $s$-helm $W(1, s)$ is constructed from an $s$-sun by adding a star vertex adjacent to each vertex on the $s$-cycle.

**Observation 3.14.** If $G = \cup_{i=1}^h G_i$, then $\text{mr}_+^R(G) \leq \sum_{i=1}^h \text{mr}_+^R(G_i)$ and $\text{mr}_-^S(G) \leq \sum_{i=1}^h \text{mr}_-^S(G_i)$.

**Proposition 3.15.** For $k \geq 2$, $m \geq 3$, $M_+(K_m \cup K_{1,k}) = Z_+(K_m \cup K_{1,k}) = m-1$ and $\text{mr}_+(K_m \cup K_{1,k}) = k + 1$, $M_+(C_t \circ K_s) = Z_+(C_t \circ K_s) = st - t + 2$ and $\text{mr}_+(C_t \circ K_s) = 2t - 2$, and $M_+(W(1, s)) = Z_+(W(1, s)) = 3$ and $\text{mr}_+(W(1, s)) = 2s - 2$.

The line graph $L(G)$ of a graph $G = (V, E)$ is the graph with vertex set $E$, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ share an endpoint. For a graph $G = (V, E)$, an orientation $G^T$ assigns to each edge $\{u, v\}$
exactly one of the two arcs \((u, v), (v, u)\). The \textit{incidence matrix} of an orientation \(G^\tau\) is the \(|V| \times |E| \{0, \pm 1\}\)-matrix \(D(G^\tau) = [d_{ve}]\) having rows indexed by the vertices and columns indexed by the oriented edges of \(G\), where 
\[
d_{ve} = \begin{cases} 
0 & \text{if } v \notin e, \\
1 & \text{if } \exists w, e = (w, v), \\
-1 & \text{if } \exists w, e = (v, w). 
\end{cases}
\]

If \(G\) is connected, \(\text{rank}(D(G^\tau)) = |G| - 1 \) [11, Theorem 8.3.1].

**Proposition 3.16.** \(M_+ (L(K_n)) = Z_+ (L(K_n)) = \frac{n(n-1)}{2} - n + 2 \) and \(\text{mr}_R (L(K_n)) = n - 2\).

**Proof.** To show that \(\text{mr} (L(K_n)) \leq n - 2\), the authors in [1] constructed a matrix in \(S(L(K_n))\) of rank at most \(n - 2\). We show that the matrix constructed in [1] is in fact positive semidefinite. Let \(D\) denote the incidence matrix of an orientation of \(K_{n-1}\). Then \(\text{rank}(D) = n - 2\). Let 
\[
M = \begin{bmatrix} B & D \\ D^T & D^T D \end{bmatrix},
\]
where \(J_{n-1}\) is the matrix of ones and \(B = I_{n-1} - \frac{1}{n-1} J_{n-1}\). Recall that the vertices of \(L(K_n)\) are the edges of \(K_n\). The matrix partition corresponds to the edges that are incident with vertex 1 of \(K_n\) and those that are not; thus \(B\) is \((n-1) \times (n-1)\).

Observe that \(M \in S(L(K_n))\). Since \(D^T J_{n-1} = 0\), we have 
\[
D^T (I_{n-1} - \frac{1}{n-1} J_{n-1}) = D^T.
\]

So 
\[
\begin{bmatrix} I & 0 \\ -D^T & I \end{bmatrix} \begin{bmatrix} B & D \\ D^T & D^T D \end{bmatrix} = \begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix}.
\]

The columns of \(B\) and of \(D\) are orthogonal to the all 1s vector, so \(\text{rank}(M) = \text{rank}([B \ D]) \leq n - 2\). Observe that \(B = B^T\). Computation shows that \(B^T B = B\). Since \(D^T (I_{n-1} - \frac{1}{n-1} J_{n-1}) D = D^T, (I_{n-1} - \frac{1}{n-1} J_{n-1}) D = D\), that is \(BD = D\). For any real matrix \(X\), \(X^T X\) is positive semidefinite. Let 
\[
X = \begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix}.
\]

Then 
\[
X^T X = \begin{bmatrix} B^T & 0 \\ D^T & 0 \end{bmatrix} \begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B^T B & B^T D \\ D^T B & D^T D \end{bmatrix} = \begin{bmatrix} B & D \\ D^T & D^T D \end{bmatrix} = M.
\]
Thus, \(M\) is positive semidefinite and \(\text{mr}_R^2 (L(K_n)) \leq n - 2\).
Since \( K_n \) has a Hamiltonian path, \( L(P_n) = P_{n-1} \) is an induced subgraph of \( L(K_n) \). As a consequence of this fact and [1] Proposition 4.6, \( Z(L(K_n)) = \frac{n(n-1)}{2} - n + 2 \). So
\[
\frac{n(n-1)}{2} - n + 2 \leq M^c_+(L(K_n)) \leq Z_+(L(K_n)) \leq Z(L(K_n)) = \frac{n(n-1)}{2} - n + 2.
\]

Hence, all inequalities are equalities.  

It is known that if \( H \) is a subgraph of \( G \) (not necessarily induced), then \( L(H) \) is an induced subgraph of \( L(G) \). Any graph \( G \) of order \( n \) is a subgraph of \( K_n \), so \( L(G) \) is an induced subgraph of \( L(K_n) \). Now, if \( G \) contains \( P_n \) as a subgraph, that is, \( G \) has a Hamiltonian path, then \( L(G) \) contains \( L(P_n) = P_{n-1} \) as an induced subgraph. As in [1], the next two results then follow from Proposition 3.16.

**Corollary 3.17.** If \( |G| = n \), then \( \text{mr}_+(L(G)) \leq n - 2 \).

**Corollary 3.18.** If \( G \) contains a Hamiltonian path and \( |G| = n \geq 2 \), then \( \text{mr}_+(L(G)) = n - 2 \).

The **necklace** \( N_s \), with \( s \) diamonds is a 3-regular graph formed from a 3s-cycle by attaching \( s \) diamond vertices, where each diamond vertex is adjacent to three consecutive cycle vertices and distinct diamond vertices have disjoint neighborhoods.

**Remark 3.19.** [21] Observe that each diamond vertex is a duplicate vertex \( (u \text{ and } v) \) is said to be duplicate vertices if \( N[u] = N[v] \), where \( N[u] \) denotes the closed neighborhood of \( u \) of the middle vertex of the three consecutive cycle vertices to which it is adjacent. The induced subgraph obtained by a sequential deletion of the \( s \) duplicate vertices of \( N_s \) is a 3s-cycle. We know \( \text{mr}_+(C_{3s}) = 3s - 2 \). By [5] Corollary 2.3, \( 3s - 2 = \text{mr}_+(C_{3s}) = \text{mr}_+(N_s) \). Hence, \( \text{M}^c_+(N_s) = s + 2 \). By [9] Proposition 5.11, \( 2 = Z_+(C_{3s}) = Z_+(N_s) = s \), and so \( Z_+(N_s) = s + 2 \).

A block of a graph is a maximal nonseparable induced subgraph. A graph is block-clique if every block is a clique. Recall that a graph is chordal if it does not contain an induced cycle on four or more vertices.

**Remark 3.20.** [21] Let \( G \) be a block-clique graph and let \( b(G) \) denote the number of blocks of \( G \). Since block-clique graphs are chordal, \( \text{mr}_+(G) = \text{cc}(G) = b(G) \) by [5] Theorem 3.6. Furthermore, \( \text{OS}(G) = \text{cc}(G) \) by [12] Proposition 3.6 (see [12] for a definition of the OS-number). As \( Z_+(G) + \text{OS}(G) = |G| \) by [3] Theorem 3.6, \( \text{M}^c_+(G) = Z_+(G) = |G| - b(G) \). Note that \( \text{cc}(G) = \text{mr}_+(G) \leq \text{M}^c_+(G) \leq \text{cc}(G) \), and so we have equality throughout. It follows that \( \text{M}_+(G) = Z_+(G) = |G| - b(G) \) and \( \text{mr}_+(G) = b(G) \).
Table 3.1
Summary of positive semidefinite maximum nullity and zero forcing number results.

<table>
<thead>
<tr>
<th>result #</th>
<th>G</th>
<th>order</th>
<th>(Z_+(G))</th>
<th>(M_+(G))</th>
<th>(= 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>(P_n)</td>
<td>(n)</td>
<td>1</td>
<td>1</td>
<td>Y</td>
</tr>
<tr>
<td>1.2</td>
<td>(C_n)</td>
<td>(n)</td>
<td>2</td>
<td>2</td>
<td>Y</td>
</tr>
<tr>
<td>1.3</td>
<td>(K_n)</td>
<td>(n)</td>
<td>(n-1)</td>
<td>(n-1)</td>
<td>Y</td>
</tr>
<tr>
<td>1.4</td>
<td>(T) (tree)</td>
<td>(n)</td>
<td>1</td>
<td>1</td>
<td>N</td>
</tr>
<tr>
<td>2.4</td>
<td>(Q_n) (hypercube)</td>
<td>(2^n)</td>
<td>(2^{n-1})</td>
<td>(2^{n-1})</td>
<td>Y</td>
</tr>
<tr>
<td>3.1</td>
<td>(K_{n_1,n_2,\ldots,n_k})</td>
<td>(n_1 + n_2 + \cdots + n_k)</td>
<td>(n_2 + \cdots + n_k)</td>
<td>(n_2 + \cdots + n_k)</td>
<td>N</td>
</tr>
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<td>3.10</td>
<td>(\mathcal{T}), (T) a tree, (</td>
<td>T</td>
<td>= n, n \geq 4, T \neq K_{1,n-1})</td>
<td>(n)</td>
<td>(n-3)</td>
</tr>
<tr>
<td>3.15</td>
<td>(P_s \Box P_t), (s \geq 2)</td>
<td>(2s)</td>
<td>2</td>
<td>2</td>
<td>Y</td>
</tr>
<tr>
<td></td>
<td>Conj</td>
<td>(P_s \Box P_t), (s \geq t)</td>
<td>(t)</td>
<td>(t)</td>
<td>Y</td>
</tr>
<tr>
<td>3.16</td>
<td>(C_s \Box P_t), (s \geq 4)</td>
<td>(2s)</td>
<td>4</td>
<td>4</td>
<td>Y</td>
</tr>
<tr>
<td>3.17</td>
<td>(C_s \Box P_t), (t \geq 2)</td>
<td>(4t)</td>
<td>4</td>
<td>4</td>
<td>Y</td>
</tr>
<tr>
<td></td>
<td>Conj</td>
<td>(C_s \Box P_t)</td>
<td>(st)</td>
<td>(\min{s,2t})</td>
<td>(\min{s,2t})</td>
</tr>
<tr>
<td>3.3</td>
<td>(W_n, n \geq 4)</td>
<td>(n)</td>
<td>3</td>
<td>3</td>
<td>Y</td>
</tr>
<tr>
<td>3.10</td>
<td>(H_s), (half-graph)</td>
<td>(2s)</td>
<td>(s)</td>
<td>(s)</td>
<td>Y</td>
</tr>
<tr>
<td>3.11</td>
<td>(K_s \Box K_3)</td>
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<td>5</td>
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<td></td>
<td>Conj</td>
<td>(K_s \Box K_t)</td>
<td>(st)</td>
<td>(st - s - t + 2)</td>
<td>(st - s - t + 2)</td>
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<td>(C_s \Box K_3), (s \geq 4)</td>
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<td>Y</td>
</tr>
<tr>
<td></td>
<td>Conj</td>
<td>(C_s \Box K_t), (s \geq 4)</td>
<td>(2t)</td>
<td>(2t)</td>
<td>Y</td>
</tr>
<tr>
<td>3.14</td>
<td>(P_{m,k} = K_m \cup K_{1,k}), (m \geq 2, m \geq 3)</td>
<td>(m+k)</td>
<td>(m-1)</td>
<td>(m-1)</td>
<td>N</td>
</tr>
<tr>
<td></td>
<td>(C_t \circ K_s)</td>
<td>(st + t)</td>
<td>(st - t + 2)</td>
<td>(st - t + 2)</td>
<td>N, (s = 1)</td>
</tr>
<tr>
<td>3.15</td>
<td>(W(1,s)), ((s)-helm)</td>
<td>(2s+1)</td>
<td>3</td>
<td>3</td>
<td>N</td>
</tr>
<tr>
<td>3.16</td>
<td>(L(K_n))</td>
<td>(\frac{n(n-1)}{2})</td>
<td>(\frac{n(n-1)}{2} - n + 2)</td>
<td>(\frac{n(n-1)}{2} - n + 2)</td>
<td>Y</td>
</tr>
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</table>

\(^1\) \(M(G) = Z(G) = M_+(G) = Z_+(G)\)?
Positive Semidefinite Maximum Nullity and Zero Forcing Number 829

<table>
<thead>
<tr>
<th>result</th>
<th>G</th>
<th>order</th>
<th>$Z_+(G)$</th>
<th>$M_+(G)$</th>
<th></th>
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<td>1</td>
<td>$L(T)$, $T$ a tree and $l = #$ pendent vertices of $T$</td>
<td>$</td>
<td>T</td>
<td>- 1$</td>
<td>$l - 1$</td>
</tr>
<tr>
<td>2</td>
<td>$T_n$, (supertriangle)</td>
<td>$\frac{1}{2}n(n + 1)$</td>
<td>$n$</td>
<td>$n$</td>
<td>Y</td>
</tr>
<tr>
<td>3</td>
<td>$P_n \boxplus P_t$</td>
<td>$st$</td>
<td>$s + t - 1$</td>
<td>$s + t - 1$</td>
<td>Y</td>
</tr>
<tr>
<td>4</td>
<td>$K_t \ast K_s$, $t \geq 2$</td>
<td>$st + t$</td>
<td>$st - 1$</td>
<td>$st - 1$</td>
<td>Y</td>
</tr>
<tr>
<td>5</td>
<td>Möbius ladder $ML_{2n}$</td>
<td>$2n$</td>
<td>(\begin{cases} 3 &amp; \text{if } n = 3, \ 4 &amp; \text{if } n = 4, \ 4 &amp; \text{else.} \end{cases})</td>
<td>(\begin{cases} 3 &amp; \text{if } n = 3, \ 4 &amp; \text{if } n = 4, \ 4 &amp; \text{else.} \end{cases})</td>
<td>Y, $n &gt; 4$</td>
</tr>
<tr>
<td>6</td>
<td>block-clique $(1$-chordal)</td>
<td>$</td>
<td>G</td>
<td>$</td>
<td>$</td>
</tr>
<tr>
<td>7</td>
<td>$C_n$, $n \geq 5$</td>
<td>$n$</td>
<td>$n - 3$</td>
<td>$n - 3$</td>
<td>Y</td>
</tr>
<tr>
<td>8</td>
<td>complement of a 2-tree, $n \geq 3$</td>
<td>$n - l$</td>
<td>(\begin{cases}</td>
<td>H</td>
<td>= 3, \</td>
</tr>
<tr>
<td>9</td>
<td>$l = #$ dominating vertices</td>
<td>$n - l$</td>
<td>(\begin{cases}</td>
<td>H</td>
<td>\geq 5, \</td>
</tr>
</tbody>
</table>

\(1\) $M(G) = Z(G) = M_+(G) = Z_+(G)$?

\(2\) Note that $3 = M_+(ML_8) < Z_+(ML_8) = 4$.

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**REFERENCES**


