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A GENERALIZATION OF MATRIX INVERSION WITH APPLICATION TO LINEAR DIFFERENTIAL-ALGEBRAIC SYSTEMS*

YURI M. NECHEPURENKO[†] AND MILOUD SADKANE[‡]

Abstract. A new generalized inversion for square matrices based on projections is introduced. It includes as special cases known generalized inverses such as the Moore-Penrose and the Drazin inverses. When associated with a regular matrix pencil, it can be expressed by a contour integral formula and can be used, in particular, to write down an explicit representation of the solutions of linear differential algebraic systems. The representation can further be simplified when a well chosen expansion is used for the exponential function. An illustration is given with the expansion in Laguerre functions.

Key words. Matrix pseudo-inversions, Matrix pencils, spectral projections, Linear differential-algebraic systems, Explicit representations of the solutions, Matrix exponential, Laguerre functions.

AMS subject classifications. 15A09, 15A22, 34A30.

1. Introduction. There are several generalizations of matrix inversion, each being useful in a specialized context, but their common point is to extend the notion of invertibility to matrices which are not necessarily square or full rank [1, 4, 17]. Among the most popular ones are the Moore-Penrose pseudo-inverse [10, 12, 13] and the Drazin inverse [6].

The Moore-Penrose pseudo-inverse occurs, for example, in least-squares and quadratic minimization problems (see, e.g. [3]). For a matrix A , it is defined as the unique solution A^\dagger of the system of four matrix equations:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

Here and throughout this paper, the notation X^* stands for the conjugate transpose of X .

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The Drazin inverse occurs, for example, in perturbation theory of eigenvalues and eigenvectors and in iterative solution of singular systems (see, e.g. [15, 14]). For a square matrix A , it is defined to be the unique matrix A^D such that

$$A^D A A^D = A^D, \quad A A^D = A^D A, \quad \text{and} \quad A^D A^{\nu+1} = A^\nu,$$

where $\nu = \text{ind}(A)$ is the index of A , that is, the size of the largest Jordan block of A corresponding to its zero eigenvalue.

The purpose of this paper is to present and study a new generalization of the inverse of a square matrix which includes the Moore-Penrose and the Drazin inverses. In Section 2, we define this new generalization and discuss its main properties and, in Section 3, we explain how to use it for regular matrix pencils. As an application, we show in Section 4 that the proposed generalization can be used to write down the solutions of linear differential-algebraic systems with constant coefficients in a way analogous to [9, Chapter 2], where the Drazin inverse has been used. In Section 5, we show that expanding the exponential function in terms of Laguerre functions simplifies the generalized inverse and leads to more practical representations of the solutions.

2. Generalization of the matrix inversion. The following theorem allows us to define the matrix inversion in a broader sense.

THEOREM 2.1. *Let B be a square matrix of order n and let Q_r and Q_l be two projections in \mathbb{C}^n of the same rank. If*

$$a) Q_l B = B Q_r \quad \text{and} \quad b) \ker B \subset \text{Im}(I - Q_r), \quad (2.1)$$

then the system

$$a) B^+ B = Q_r, \quad b) B B^+ = Q_l, \quad c) B^+ B B^+ = B^+ \quad (2.2)$$

is uniquely solvable for the matrix B^+ .

Proof. The projections Q_r and Q_l can be represented as

$$Q_r = X \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} X^{-1}, \quad Q_l = Y \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Y^{-1}, \quad (2.3)$$

where I is the identity matrix of order $m = \text{rank } Q_r = \text{rank } Q_l$ and X, Y are some nonsingular matrices of order n . Equality (2.1a) is satisfied if and only if

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Y^{-1} B X = Y^{-1} B X \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

or equivalently, if and only if

$$B = Y \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} X^{-1}, \quad (2.4)$$

where B_{11} and B_{22} are some square matrices of orders m and $n - m$, respectively, and then the inclusion (2.1b) is satisfied if and only if B_{11} is nonsingular. Therefore, the condition (2.1) ensures the existence of the matrix

$$B^+ = X \begin{bmatrix} B_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Y^{-1} \quad (2.5)$$

and it is easy to check that the matrices in (2.3), (2.4) and (2.5) satisfy (2.2).

To show uniqueness of the solution (2.5), it is sufficient to note that the system

$$\Delta B = 0, \quad B\Delta = 0, \quad (B^+ + \Delta)B(B^+ + \Delta) = B^+ + \Delta$$

has the only solution $\Delta = 0$. \square

Due to (2.2), the matrix B^+ possesses the following properties

$$Q_r B^+ = B^+ Q_l = B^+. \quad (2.6)$$

If $\ker B = \text{Im}(I - Q_r)$, then $B(I - Q_r) = 0$. Thus, in this case in addition to (2.6), we also have from (2.1a) that

$$Q_l B = B Q_r = B,$$

and therefore, $BB^+B = B$.

We will say that a square matrix B is (Q_r, Q_l) -pseudo-invertible if it satisfies the conditions (2.1) of Theorem 2.1. The corresponding matrix B^+ will be referred to as the (Q_r, Q_l) -pseudo-inverse of B . This matrix is uniquely defined by choosing the projections Q_r and Q_l .

By an appropriate choice of the projections Q_r and Q_l one can recover known generalized inverses. For example:

If Q_r and Q_l are the orthogonal projections onto respectively $\text{Im } B^*$ and $\text{Im } B$, then B^+ coincides with B^\dagger , the Moore-Penrose pseudo-inverse of B .

If $Q_r = Q_l = Q$, where Q is the spectral projection onto the invariant subspace associated with the nonzero eigenvalues of B , then $B^+ = B^D$, the Drazin inverse of B .

3. Spectral pseudo-inversion. Let E and A be square matrices of order n such that the pencil

$$\lambda E - A \tag{3.1}$$

is regular (see, e.g. [7, Chapter 10], [8, Chapter 2], [15, Chapter 6]). Denote by $\Lambda(A, E)$ the set of its finite eigenvalues. Then the spectral projections \mathcal{P}_r and \mathcal{P}_l onto the right and left deflating subspaces corresponding to $\Lambda(A, E)$ are given by

$$\mathcal{P}_r = \frac{1}{2\pi i} \oint_{\Gamma} (\lambda E - A)^{-1} E \, d\lambda \tag{3.2}$$

and

$$\mathcal{P}_l = \frac{1}{2\pi i} \oint_{\Gamma} E (\lambda E - A)^{-1} \, d\lambda, \tag{3.3}$$

where Γ is a closed contour surrounding $\Lambda(A, E)$. Here and throughout this paper, closed contour means a positively oriented Jordan curve.

Note that \mathcal{P}_r and \mathcal{P}_l have the same rank, which is equal to the sum of algebraic multiplicities of the finite eigenvalues of (3.1). Moreover, \mathcal{P}_r and \mathcal{P}_l possess the properties

$$a) \mathcal{P}_l E = E \mathcal{P}_r, \quad b) \mathcal{P}_l A = A \mathcal{P}_r, \tag{3.4}$$

and

$$\ker E \subset \text{Im} (I - \mathcal{P}_r). \tag{3.5}$$

The identity (3.4a) and the inclusion (3.5) follow directly from (3.2) and (3.3), for the identity (3.4b), see e.g. [8, p. 50].

From (3.4a) and (3.5) we see that the conditions of Theorem 2.1 are satisfied for $B = E$, $\mathcal{Q}_r = \mathcal{P}_r$ and $\mathcal{Q}_l = \mathcal{P}_l$. Therefore, E is $(\mathcal{P}_r, \mathcal{P}_l)$ -pseudo-invertible. The corresponding matrix E^+ will be referred to as the *spectral pseudo-inverse* of E .

3.1. Properties of the spectral pseudo-inverse matrix. According to Theorem 2.1, the spectral pseudo-inverse matrix E^+ is the unique solution of the system

$$a) E^+ E = \mathcal{P}_r, \quad b) E E^+ = \mathcal{P}_l, \quad c) E^+ E E^+ = E^+, \tag{3.6}$$

and according to (2.6), it satisfies the equalities

$$\mathcal{P}_r E^+ = E^+ \mathcal{P}_l = E^+. \tag{3.7}$$

The spectral pseudo-inverse matrix has a very simple explicit representation given by the following theorem which may be considered as its alternative definition.

THEOREM 3.1.

$$E^+ = \frac{1}{2\pi i} \oint_{\Gamma} (\lambda E - A)^{-1} d\lambda, \quad (3.8)$$

where Γ is a closed contour surrounding $\Lambda(A, E)$.

Proof. Due to uniqueness of the solution of (3.6), it is sufficient to show that the matrix E^+ given in (3.8) satisfies this system. The first two equalities in (3.6) follow directly from (3.2), (3.3) and (3.8). To show the third one, let Γ' be a closed contour surrounding Γ . Then

$$\begin{aligned} E^+ E E^+ &= \frac{1}{(2i\pi)^2} \oint_{\Gamma'} \oint_{\Gamma} (\lambda E - A)^{-1} E (\lambda' E - A)^{-1} d\lambda d\lambda' \\ &= \frac{1}{(2i\pi)^2} \oint_{\Gamma'} \oint_{\Gamma} \frac{(\lambda E - A)^{-1} - (\lambda' E - A)^{-1}}{\lambda' - \lambda} d\lambda d\lambda'. \end{aligned}$$

The result follows by noting that for $\lambda \in \Gamma$ and $\lambda' \in \Gamma'$

$$\oint_{\Gamma} \frac{d\lambda}{\lambda' - \lambda} = 0 \quad \text{and} \quad \oint_{\Gamma'} \frac{d\lambda'}{\lambda' - \lambda} = 2i\pi. \quad \square$$

The next theorem shows that when A is nonsingular, the spectral properties of the matrix $E^+ A$ corresponding to its nonzero eigenvalues are the same as those of the matrix pencil (3.1) corresponding to all its finite eigenvalues.

THEOREM 3.2. *If A is nonsingular, then the nonzero eigenvalues of the matrix $E^+ A$ and the corresponding invariant subspaces are the same as the finite eigenvalues of the pencil (3.1) and the corresponding right deflating subspaces. In particular, the right spectral projection \mathcal{P}_r corresponding to $\Lambda(A, E)$ is the spectral projection of $E^+ A$ corresponding to its nonzero eigenvalues.*

Proof. Fix a subset \mathcal{E} of nonzero eigenvalues of $E^+ A$. Let \mathcal{U} be the corresponding invariant subspace and U be a full rank rectangular matrix such that $\text{Im } U = \mathcal{U}$. Then $\mathcal{E} = \Lambda(C)$, where $C = (U^* U)^{-1} U^* E^+ A U$, and $E^+ A U = U C$. Multiplying the last equality on the left by $I - \mathcal{P}_r$ and using (3.7) and non-singularity of C we deduce that $(I - \mathcal{P}_r)U = 0$ and therefore $U = \mathcal{P}_r U$. Now, multiplying the same equality on the left by E and using (3.6b) and (3.4b), we obtain $AU = EUC$. Therefore, \mathcal{E} is a subset of finite eigenvalue of (3.1) and \mathcal{U} is the corresponding right deflating subspace.

Conversely, fix a subset \mathcal{E} of finite eigenvalues of (3.1). Let \mathcal{U} be the corresponding right deflating subspace and U be a full rank rectangular matrix such that $\text{Im } U = \mathcal{U}$. Then $U = \mathcal{P}_r U$ and there exists some square matrix C such that $\mathcal{E} = \Lambda(C)$ and

$AU = EUC$. Multiplying the last equality on the left by E^+ and using (3.6a), we deduce that $E^+AU = UC$. Under the assumption that A is nonsingular the matrix C is nonsingular as well. Therefore, \mathcal{E} is a subset of nonzero eigenvalues of E^+A and \mathcal{U} is the corresponding invariant subspace.

As it follows from the two paragraphs above, $\text{Im } \mathcal{P}_r$ is the invariant subspace of E^+A corresponding to the complete set of its nonzero eigenvalues. Equalities (3.7) and (3.4b) imply that the projection \mathcal{P}_r commutes with E^+A . Thus, \mathcal{P}_r is the spectral projection of E^+A corresponding to its nonzero eigenvalues. \square

When A is nonsingular, Theorem 3.2 shows that the spectral projection \mathcal{P}_r can also be expressed as

$$\mathcal{P}_r = \frac{1}{2\pi i} \oint_{\Gamma} (\lambda I - E^+A)^{-1} d\lambda,$$

where Γ is a closed contour which includes the finite eigenvalues of (3.1) and excludes zero.

Let the matrix E be singular and

$$X(\lambda E - A)Y = \begin{bmatrix} \lambda I - J & 0 \\ 0 & \lambda N - I \end{bmatrix} \quad (3.9)$$

be the Weierstrass canonical form of (3.1) (see, e.g. [9, p. 16] or [15, p. 280]), where the matrices X and Y are nonsingular, J corresponds to the finite eigenvalues and N is nilpotent and corresponds to the infinite eigenvalue. The formula (3.8) together with (3.9) gives another representation for E^+ :

$$E^+ = Y \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} X.$$

Thus, we have the following representation for the matrix E^+A :

$$E^+A = Y \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} Y^{-1}$$

which can be used for an alternative proof of Theorem 3.2.

3.2. Difference between E^+ and E^D . We clarify the main differences between the spectral pseudo-inverse E^+ and the Drazin inverse E^D matrices. Denote by $\nu = \text{ind}(E)$ the index of the matrix E and by $\mu = \text{ind}(A, E)$ the index of (3.1), namely, $\mu = 0$ if E is nonsingular and μ is the quantity such that $N^{\mu-1} \neq 0$ and $N^\mu = 0$ otherwise.

LEMMA 3.1. *If E is nonsingular, then $\nu = \mu = 0$, $\mathcal{P}_r = \mathcal{P}_l = I$ and $E^+ = E^D = E^{-1}$. If $AE = EA$, then $\nu = \mu$, $\mathcal{P}_r = \mathcal{P}_l$ and $E^+ = E^D$.*

Proof. If E is nonsingular, then the pencil (3.1) has only finite eigenvalues and all mentioned properties hold.

When E is singular, we have with the notation of (3.9)

$$\mathcal{P}_r = Y \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Y^{-1}, \quad \mathcal{P}_l = X^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} X.$$

If $AE = EA$, then $\mathcal{P}_r = \mathcal{P}_l$, and therefore,

$$X^{-1} = Y \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix},$$

where P_1 and P_2 are nonsingular. Thus,

$$E = Y \begin{bmatrix} P_1 & 0 \\ 0 & P_2 N \end{bmatrix} Y^{-1}, \quad A = Y \begin{bmatrix} P_1 J & 0 \\ 0 & P_2 \end{bmatrix} Y^{-1}.$$

Since A and E commute, we have $P_1 J = J P_1$ and $P_2 N = N P_2$. As a consequence, $(P_2 N)^{\mu-1} = P_2^{\mu-1} N^{\mu-1} \neq 0$ and $(P_2 N)^\mu = P_2^\mu N^\mu = 0$. Thus, we have $\nu = \mu$, and

$$E^+ = Y \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} X = Y \begin{bmatrix} P_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} Y^{-1} = E^D. \quad \square$$

The following examples show that when $AE \neq EA$, then we may have $\nu \neq \mu$, $\mathcal{P}_r \neq \mathcal{P}_l$ and $E^+ \neq E^D$.

1. Let

$$\lambda E - A = \lambda \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda - 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then $\nu = 2$, $\mu = 1$ (note that $\nu > \mu$), and

$$\mathcal{P}_r = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \neq \mathcal{P}_l = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad E^+ = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \neq E^D = 0.$$

2. Let

$$\lambda E - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & \lambda \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then $\nu = 1$, $\mu = 2$ (note that $\nu < \mu$), $\mathcal{P}_r = \mathcal{P}_l = 0$, $E^+ = 0 \neq E^D = E$.

Moreover, if A and E are Hermitian and A is positive or negative definite, then $\nu = \mu$, $\mathcal{P}_l = \mathcal{P}_r^*$ and a representation of the form (3.9) exists with $X = Y^*$, Hermitian

J and zero matrix N . But if the projections are not orthogonal, i.e. Y couldn't be chosen unitary, then we have

$$E^+ = Y \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Y^*$$

with $Y^* \neq Y^{-1}$, and therefore, $E^+ \neq E^D$.

4. Explicit formulas for solutions of linear differential algebraic systems. Consider the linear differential algebraic system

$$a) E \frac{dx}{dt} = Ax + f, \quad b) x(0) = x^0, \quad (4.1)$$

where $f(t)$ is a sufficiently smooth n -component vector-function and the matrix pencil $\lambda E - A$ is regular. Recall that a solution of (4.1a) always exists and the solution is unique if the initial condition (4.1b) is consistent. More details can be found in [4, Chapter 9] or [9, Chapter 2]. The system (4.1) arises in a wide variety of applications. For example in the interconnect analysis of VLIC [5] and the linear stability analysis of hydrodynamic flows [2]. Without loss of generality we will assume that the matrix A is nonsingular since otherwise the change of variable $x = \tilde{x}e^{\rho t}$, where ρ is chosen such that $A - \rho E$ is nonsingular, leads to a system for \tilde{x} of the same form as (4.1) but with a nonsingular matrix $A - \rho E$ instead of A .

Due to the non-singularity of A we can rewrite (4.1a) as

$$x = A^{-1}E \frac{dx}{dt} - A^{-1}f$$

with the existence of x guaranteed. Substituting $\mu - 1$ times we obtain the following formula:

$$x = (A^{-1}E)^\mu \frac{d^\mu x}{dt^\mu} - \sum_{k=0}^{\mu-1} (A^{-1}E)^k A^{-1} \frac{d^k f}{dt^k}.$$

Multiplying this formula on the left by $I - \mathcal{P}_r$ and noting that $(I - \mathcal{P}_r)(A^{-1}E)^\mu = 0$, we obtain

$$(I - \mathcal{P}_r) \left(x + \sum_{k=0}^{\mu-1} (A^{-1}E)^k A^{-1} \frac{d^k f}{dt^k} \right) = 0. \quad (4.2)$$

This gives the following necessary condition for the well-posedness of (4.1):

$$(I - \mathcal{P}_r) \left(x^0 + \sum_{k=0}^{\mu-1} (A^{-1}E)^k A^{-1} \frac{d^k f}{dt^k}(0) \right) = 0. \quad (4.3)$$

As the following theorem shows, this condition is also sufficient.

THEOREM 4.1. *Under the condition (4.3), the solution of (4.1) exists, is unique and can be represented in the following form:*

$$\begin{aligned}
 x(t) &= e^{tE^+A}E^+Ex^0 + \int_0^t e^{(t-s)E^+A}E^+f(s)ds \\
 &- (I - E^+E) \sum_{k=0}^{\mu-1} (A^{-1}E)^k A^{-1} \frac{d^k f}{dt^k}(t). \tag{4.4}
 \end{aligned}$$

Proof. Multiplying on the left the first equality in (4.1) by \mathcal{P}_r and the second one by E^+ we obtain the differential equation

$$\tilde{x}(0) = \mathcal{P}_r x^0, \quad \frac{d\tilde{x}}{dt} = E^+ A \tilde{x} + E^+ f \tag{4.5}$$

for $\tilde{x} = \mathcal{P}_r x$ whose unique solution is given by

$$\mathcal{P}_r x(t) = e^{tE^+A} \mathcal{P}_r x^0 + \int_0^t e^{(t-s)E^+A} E^+ f(s) ds. \tag{4.6}$$

The identity (4.2) implies that

$$(I - \mathcal{P}_r)x(t) = -(I - \mathcal{P}_r) \sum_{k=0}^{\mu-1} (A^{-1}E)^k A^{-1} \frac{d^k f}{dt^k}(t). \tag{4.7}$$

The representation (4.4) follows from (4.6), (4.7) and (3.6a). The uniqueness of the solution follows from the uniqueness of the solution (4.6) of (4.5). \square

When the matrices A and E commute, the following explicit representation for the solution of (4.1) can be obtained through the Drazin inverse matrix E^D [9, Theorem 2.29], [4, Theorem 9.2.3]:

$$\begin{aligned}
 x(t) &= e^{tE^D A} E^D E x^0 + \int_0^t e^{(t-s)E^D A} E^D f(s) ds \\
 &- (I - E^D E) \sum_{k=0}^{\mu-1} (A^{-1}E)^k A^{-1} \frac{d^k f}{dt^k}(t). \tag{4.8}
 \end{aligned}$$

This representation can also be obtained from (4.4) by replacing E^+ by E^D , since in the considered case $E^D = E^+$ (see Lemma 3.1). When $AE \neq EA$, the representation (4.8) does not hold. To obtain an explicit representation of the solution with the Drazin inverse matrix, one needs to reduce the initial system (4.1) to an equivalent one with commuting matrices. This can always be done by dividing (4.1) on the left by any nonsingular matrix of the form $A - \rho E$ [9], but the representation thus obtained is complicated. The spectral pseudo-inverse allows us to obtain (4.4) regardless of the commutativity of A and E .

5. More practical representations. In this section, we adopt the assumptions of Section 4 and show how to avoid E^+ in the explicit representation (4.4). This helps to construct efficient algorithms for solving problems of type (4.1).

LEMMA 5.1. *The solution (4.4) can be written in the following form:*

$$x(t) = W(t)z^0 + \int_0^t W(t-s)(A^{-1}E)^{\mu-1}A^{-1}\frac{d^\mu f}{dt^\mu}(s)ds - \sum_{k=0}^{\mu-1}(A^{-1}E)^k A^{-1}\frac{d^k f}{dt^k}(t), \quad (5.1)$$

where

$$z^0 = x^0 + \sum_{k=0}^{\mu-1}(A^{-1}E)^k A^{-1}\frac{d^k f}{dt^k}(0)$$

and

$$W(t) = e^{tE^+A}\mathcal{P}_r.$$

Proof. The change of variables

$$z = x + \sum_{k=0}^{\mu-1}(A^{-1}E)^k A^{-1}\frac{d^k f}{dt^k} \quad (5.2)$$

reduces (4.1) to the following differential algebraic system

$$E\frac{dz}{dt} = Az + E(A^{-1}E)^{\mu-1}A^{-1}\frac{d^\mu f}{dt^\mu}, \quad z(0) = z^0. \quad (5.3)$$

From (4.2) we see that $z(t) \in \text{Im } \mathcal{P}_r$. Therefore, multiplying the second equality in (5.3) on the left by E^+ gives

$$\frac{dz}{dt} = E^+Az + \mathcal{P}_r(A^{-1}E)^{\mu-1}A^{-1}\frac{d^\mu f}{dt^\mu}, \quad z(0) = z^0. \quad (5.4)$$

If the condition (4.3) is satisfied, then $z^0 \in \text{Im } \mathcal{P}_r$. Thus, (5.1) directly follows from (5.2) and (5.4). \square

Without loss of generality we will assume further that all finite eigenvalues of the matrix pencil $\lambda E - A$ have negative real parts:

$$\rho_0 = \max \text{Re } \Lambda(A, E) < 0. \quad (5.5)$$

This assumption is no restriction since otherwise putting $x = \tilde{x}e^{\rho t}$, where $\rho > \rho_0$, we obtain a system for \tilde{x} of the same form as (4.1) but where the corresponding matrix pencil possesses the property (5.5).

Denote by F_k^α the Laguerre function

$$F_k^\alpha(t) = \sqrt{2\alpha}e^{-\alpha t}L_k(2\alpha t)$$

where $\alpha > 0$ and $L_k(s)$ is the normalized Laguerre polynomial of degree k . Note that

$$\int_0^\infty F_k^\alpha(t)F_l^\alpha(t)dt = \begin{cases} 0, & k \neq l, \\ 1, & k = l. \end{cases}$$

LEMMA 5.2. For any $\alpha > 0$ the following decomposition holds:

$$W(t) = \sum_{k=0}^\infty M_k^\alpha \mathcal{P}_r F_k^\alpha(t), \quad t \geq 0, \quad (5.6)$$

where

$$M_k^\alpha = \sqrt{2\alpha} \{(A - \alpha E)^{-1}(A + \alpha E)\}^k (\alpha E - A)^{-1} E. \quad (5.7)$$

Proof. For all $s \geq 0$ and β such that $\operatorname{Re}\beta > -1/2$ the following decomposition holds [16, 11]:

$$e^{-\beta s} = \sum_{k=0}^\infty \frac{\beta^k}{(1 + \beta)^{k+1}} L_k(s).$$

Fix $\alpha > 0$. The change of variables $\beta = \tilde{\beta}/(2\alpha)$, $s = 2\alpha t$ and $\gamma = \tilde{\beta} + \alpha$ gives

$$e^{-\gamma t} = \sqrt{2\alpha} \sum_{k=0}^\infty \frac{(\gamma - \alpha)^k}{(\gamma + \alpha)^{k+1}} F_k^\alpha(t). \quad (5.8)$$

The decomposition holds for all γ such that $\operatorname{Re}\gamma > 0$ and $t \geq 0$. It follows then from (5.8), (5.5) and Theorem 3.2 that

$$e^{tE^+ A} \mathcal{P}_r = \sqrt{2\alpha} \sum_{k=0}^\infty \{(E^+ A - \alpha I)^{-1}(E^+ A + \alpha I)\}^k (\alpha I - E^+ A)^{-1} \mathcal{P}_r F_k^\alpha(t).$$

To complete the proof of (5.6), (5.7) it suffices to show that

$$\begin{aligned} & \{(E^+ A - \alpha I)^{-1}(E^+ A + \alpha I)\}^k (\alpha I - E^+ A)^{-1} \mathcal{P}_r \\ &= \{(A - \alpha E)^{-1}(A + \alpha E)\}^k (\alpha E - A)^{-1} E \mathcal{P}_r. \end{aligned} \quad (5.9)$$

From (3.7), (3.6b) and (3.4b) we have

$$(A - \alpha E)\mathcal{P}_r = E\mathcal{P}_r(E^+A - \alpha I). \quad (5.10)$$

Since the matrices $A - \alpha E$ and $E^+A - \alpha I$ are nonsingular and the matrices E^+A and \mathcal{P}_r commute, (5.10) implies that

$$(A - \alpha E)^{-1}E\mathcal{P}_r = \mathcal{P}_r(E^+A - \alpha I)^{-1} = (E^+A - \alpha I)^{-1}\mathcal{P}_r. \quad (5.11)$$

Due to (3.6b), (3.7), (5.11) and (3.4b) we have

$$\begin{aligned} (E^+A - \alpha I)^{-1}E^+A\mathcal{P}_r &= (E^+A - \alpha I)^{-1}\mathcal{P}_rE^+A \\ &= (A - \alpha E)^{-1}E\mathcal{P}_rE^+A = (A - \alpha E)^{-1}\mathcal{P}_lA = (A - \alpha E)^{-1}A\mathcal{P}_r. \end{aligned} \quad (5.12)$$

From (5.11) and (5.12) we have

$$(E^+A - \alpha I)^{-1}(E^+A + \alpha I)\mathcal{P}_r = (A - \alpha E)^{-1}(A + \alpha E)\mathcal{P}_r, \quad (5.13)$$

and from (5.11) and (5.13) we have

$$\begin{aligned} &\{(E^+A - \alpha I)^{-1}(E^+A + \alpha I)\}^k (\alpha I - E^+A)^{-1}\mathcal{P}_r \\ &= \{(E^+A - \alpha I)^{-1}(E^+A + \alpha I)\mathcal{P}_r\}^k (\alpha I - E^+A)^{-1}\mathcal{P}_r \\ &= \{(A - \alpha E)^{-1}(A + \alpha E)\mathcal{P}_r\}^k (\alpha E - A)^{-1}E\mathcal{P}_r. \end{aligned}$$

Taking into account now that \mathcal{P}_r commutes with $(A - \alpha E)^{-1}(A + \alpha E)$ and $(\alpha E - A)^{-1}E$ we obtain (5.9). \square

Note that if $\mu = 1$ then the matrix N in (3.9) is zero and, therefore,

$$\mathcal{P}_lE = E\mathcal{P}_r = E.$$

Thus, in this case we can simplify (5.6) by deleting the projection \mathcal{P}_r . In the general case, this is not possible but we always can delete the projection after substituting (5.6) in to (5.1).

THEOREM 5.1. *Under the conditions (4.3) and (5.5), the solution of (4.1) exists, is unique and can be represented as in (5.1) with*

$$W(t) = \sum_{k=0}^{\infty} M_k^\alpha F_k^\alpha(t),$$

where M_k^α is given in (5.7) and α is an arbitrary positive parameter.

Proof. The theorem follows from Lemmas 5.1 and 5.2 and the equalities:

$$E\mathcal{P}_r(A^{-1}E)^{\mu-1}A^{-1} = \mathcal{P}_l(EA^{-1})^\mu = (EA^{-1})^\mu = E(A^{-1}E)^{\mu-1}A^{-1}$$

and $z^0 = \mathcal{P}_r z^0$. \square

As an example, consider a system of the form (4.1) with $f(t) = Bu(t)$ where B is an $n \times p$ matrix with $p \ll n$, and $u(t)$ is a sufficiently smooth control function, and matrices A and E satisfy (5.5). Suppose for simplicity that

$$x^0 = 0, \quad \frac{d^k u}{dt^k}(0) = 0, \quad k = 0, \dots, \mu - 1,$$

which imply that $z^0 = 0$. Then the hypotheses of Theorem 5.1 are satisfied and we have

$$x(t) = \sum_{k=0}^{\infty} S_k \int_0^t F_k^\alpha(t-s) \frac{d^\mu u}{dt^\mu}(s) ds - \sum_{k=0}^{\mu-1} T_k \frac{d^k u}{dt^k}(t), \quad (5.14)$$

where T_k and S_k are $n \times p$ matrices defined by the following recurrence formulas:

$$AT_0 = B, \quad AT_k = ET_{k-1}, \quad k = 1, \dots, \mu - 1,$$

and

$$(A - \alpha E)S_0 = -\sqrt{2\alpha}ET_{\mu-1}, \quad (A - \alpha E)S_k = (A + \alpha E)S_{k-1}, \quad k \geq 1.$$

Truncating the infinite sum in (5.14) yields an approximate solution of (5.14). The matrices T_k and S_k can be computed once and then used for different functions u .

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