2012

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Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.1562

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A NOTE ON BLOCK REPRESENTATIONS OF THE GROUP INVERSE OF LAPLACIAN MATRICES∗

CHANGJIANG BU†, LIZHU SUN†, JIANG ZHOU†, AND YIMIN WEI‡

Abstract. Let G be a weighted graph with Laplacian matrix L and signless Laplacian matrix Q. In this note, block representations for the group inverse of L and Q are given. The resistance distance in a graph can be obtained from the block representation of the group inverse of L.

Key words. Group inverse, Laplacian matrix, Signless Laplacian matrix, Resistance distance.

AMS subject classifications. 15A09, 05C50, 05C12.

1. Introduction. For an n × n matrix A, the group inverse of A is the unique n × n matrix X satisfying the matrix equations AXA = A, XAX = X and AX = XA. It is well known that the group inverse of A exists if and only if rank(A) = rank(A^2) (see [28,29]). If the group inverse of A exists, it is unique, which is denoted by A#. The matrix A is group invertible if A# exists. For a matrix B, let B+ denote the Moore-Penrose inverse of B. Some representations for the group inverse of block matrices (operators) are given in [2–8,11,12,14,16,17,28]. More details for the theory of generalized inverse can be found in [9].

Let G be a undirected weighted graph without loops or multiple edges, and each edge of G has been labeled by a positive real number, which is called the weight of the edge. The adjacency matrix A of G is the matrix whose (i, j)-entry equals 0 if there is no edge joining vertices i and j and equals the weight of the edge joining vertices i and j otherwise. Let D be the diagonal matrix whose i-th diagonal entry equals the sum of the weights of the edges incident to the vertex i in G. The matrices D − A and D + A are called the Laplacian matrix and signless Laplacian matrix of G, respectively. It is known that D − A and D + A are positive semidefinite.

Let L be the Laplacian matrix of a weighted graph G. Since L is symmetric,
L# exists and L# = L+ (see [9]). In [20], Kirkland et al. gave a representation for the group inverse of irreducible Laplacian matrices in terms of bottleneck matrix and all-ones matrix. In [18], Ho and van Dooren used an SVD (singular value decomposition) approach to calculate the Moore-Penrose (group) inverse of the Laplacian of a bipartite graph. In this note, we give a block representation for the group inverse of (signless) Laplacian matrices. Applying this block representation, we give a formula for the resistance distance in a graph.

2. Some lemmas. For a group invertible matrix $S$, let $S^\pi$ denote the projection matrix $I - SS^\#$, where $I$ is the identity matrix.

**Lemma 2.1.** [25] Let $M$ be a Hermitian positive semidefinite matrix, which is partitioned as $M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$. Then $A^\pi B = 0$, $BC^\pi = 0$.

**Lemma 2.2.** [3] Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A$ is nonsingular, and $S = D - CA^{-1}B$ is group invertible. Then $M^\#$ exists if and only if $R = A^2 + BS^\pi C$ is nonsingular. If $M^\#$ exists, then

$$M^\# = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where

$$X = AR^{-1}(A + BS^\pi C)R^{-1}A,$$

$$Y = AR^{-1}(A + BS^\pi C)R^{-1}BS^\pi - AR^{-1}BS^\#,$$

$$Z = S^\pi CR^{-1}(A + BS^\pi C)R^{-1}A - S^\pi CR^{-1}A,$$

$$W = S^\pi CR^{-1}(A + BS^\pi C)R^{-1}BS^\pi - S^\pi CR^{-1}BS^\pi + S^\pi.$$

Klein and Randić introduced the concept of resistance distance in [21]. A graph $G$ can be viewed as an electrical network $N$ by replacing each edge of $G$ with a resistor. For two vertices $i$ and $j$ in $G$, the resistance distance between them is defined to be the effective resistance between them in the electrical network $N$ (see [21]). The resistance distance is a distance function in graphs, it has important applications in chemical graph theory. Some results on resistance distance can be found in [13,21,24,26,27].

For a matrix $M$, let $M_{ij}$ denote the $(i,j)$-entry of $M$. Let $G$ be a connected weighted graph with Laplacian matrix $L$. Let $\Omega_{ij}$ denote the resistance distance between vertices $i$ and $j$ in $G$. It is known that $\Omega_{ij} = L_{ii}^+ + L_{jj}^+ - L_{ij}^+ - L_{ji}^+$ (see [1]). Note that $L$ is symmetric, $L^# = L^+$. Hence, we have the following lemma.

**Lemma 2.3.** Let $G$ be a connected weighted graph with vertex set $\{1, 2, \ldots, n\}$ and Laplacian matrix $L$. Then $\Omega_{ij} = L_{ii}^# + L_{jj}^# - L_{ij}^# - L_{ji}^#$.  

Electronic Journal of Linear Algebra  ISSN 1081-3810  A publication of the International Linear Algebra Society  Volume 23, pp. 866-876, October 2012
3. Main results. Some expressions for the Moore-Penrose inverse of a $2 \times 2$ block matrix are given in [19,22]. But the expressions in [19,22] are very complicated. We first give a new expression for the group inverse of Laplacian matrices as follow.

**Theorem 3.1.** Let $G$ be a weighted graph with Laplacian matrix $L$. If $L$ is partitioned as $L = \begin{pmatrix} L_1 & L_2 \\ L_2^\top & L_3 \end{pmatrix}$ ($L_1$ is square), then

$$L^\# = \begin{pmatrix} X & Y \\ Y^\top & Z \end{pmatrix},$$

where

$$X = L_1 R^\# K R^\# L_1,$$
$$Y = L_1 R^\# K R^\# L_2 S^\pi - L_1 R^\# L_2 S^\#,$$
$$Z = S^\pi L_2^\top R^\# K R^\# L_2 S^\pi - S^\# L_2^\top R^\# L_2 S^\pi - S^\pi L_2^\top R^\# L_2 S^\# + S^\#,$$
$$R = L_1^+ + L_2 S^\# L_2^\top,$$
$$K = L_1 + L_2 S^\# L_2^\top,$$
$$S = L_3 - L_2^\top L_1^+ L_2.$$

**Proof.** Since $L_1$, $L_3$ are real symmetric, there exist orthogonal matrices $P_1$, $P_2$ such that

$$L_1 = P_1 \begin{pmatrix} \Delta_1 & 0 \\ 0 & 0 \end{pmatrix} P_1^\top, \quad L_3 = P_2 \begin{pmatrix} \Delta_2 & 0 \\ 0 & 0 \end{pmatrix} P_2^\top,$$

where $\Delta_1$, $\Delta_2$ are nonsingular diagonal matrices, the zero blocks can be vacuous. Then we have

$$L_1^\# = P_1 \begin{pmatrix} \Delta_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} P_1^\top, \quad L_3^\# = P_2 \begin{pmatrix} \Delta_2^{-1} & 0 \\ 0 & 0 \end{pmatrix} P_2^\top.$$

Suppose that $L_2 = P_1 \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} P_2^\top$. By Lemma 2.1, we have $L_1^\top L_2 = 0$, $L_2 L_2^\top = 0$. Hence, $M_2 = 0$, $M_3 = 0$, $M_4 = 0$. Then

$$L^\# = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} \Delta_1 & 0 & M_1 & 0 \\ 0 & 0 & 0 & 0 \\ M_1^\top & 0 & \Delta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^\# \begin{pmatrix} P_1^\top & 0 \\ 0 & P_2^\top \end{pmatrix} = U \begin{pmatrix} M^\# & 0 \\ 0 & 0 \end{pmatrix} U^{-1},$$
where \( M = \begin{pmatrix} \Delta_1 & M_1 \\ M_1^T & \Delta_2 \end{pmatrix}, U = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} \). Recall that \( \Delta_1 \) is a nonsingular diagonal matrix. Since \( \Delta_2 - M_1^T \Delta_1^{-1} M_1 \), the Schur complement of \( M \), is real symmetric, it is group invertible. By Lemma 2.2, we have

\[
M^\# = \begin{pmatrix} \tilde{X} \\ \tilde{Y}^T \end{pmatrix} \begin{pmatrix} \tilde{Y} \\ \tilde{W} \end{pmatrix},
\]

where

\[
\begin{align*}
\tilde{X} &= \Delta_1 \tilde{R}^{-1} \tilde{K} \tilde{R}^{-1} \Delta_1, \\
\tilde{Y} &= \Delta_1 \tilde{R}^{-1} \tilde{K} \tilde{R}^{-1} M_1 \tilde{S}^\pi - \Delta_1 \tilde{R}^{-1} M_1 \tilde{S}^\#, \\
\tilde{W} &= \tilde{S}^\pi M_1^T \tilde{R}^{-1} \tilde{K} \tilde{R}^{-1} M_1 \tilde{S}^\pi - \tilde{S}^\# M_1^T \tilde{R}^{-1} M_1 \tilde{S}^\pi - \tilde{S}^\pi M_1^T \tilde{R}^{-1} M_1 \tilde{S}^\# + \tilde{S}^\#, \\
\tilde{R} &= \Delta_2^2 + M_1 \tilde{S}^\# M_1^T, \\
\tilde{K} &= \Delta_1 + M_1 S^\# M_1^T, \\
\tilde{S} &= \Delta_2 - M_1^T \Delta_1^{-1} M_1.
\end{align*}
\]

By \( L^\# = U \begin{pmatrix} M^\# & 0 \\ 0 & 0 \end{pmatrix} U^{-1} \), we can obtain the representation of \( L^\# \). \( \square \)

Figure 1: Weighted graph \( G \).

Let \( F_1, F_2 \) be two subsets of the set \( \{1, 2, \ldots, n\} \). The complement of \( F_1 \) and \( F_2 \) in \( \{1, 2, \ldots, n\} \) are denoted by \( \overline{F_1} \) and \( \overline{F_2} \), respectively. For a matrix \( L \) of order \( n \), let \( L[F_1|F_2] \) denote the submatrix of \( L \) determined by the rows whose index is in \( F_1 \) and the columns whose index is in \( F_2 \). Here we give an example for Theorem 3.1.
By Theorem 3.1, we get

\[ L = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & -1 & 0 & -1 & -2 & 0 & 0 \\
0 & 0 & -1 & 7 & -3 & -1 & -2 & 0 \\
0 & 0 & 0 & -3 & 5 & -2 & 0 & 0 \\
0 & 0 & -1 & -1 & -2 & 7 & -3 & 0 \\
0 & 0 & -2 & -2 & 0 & -3 & 7 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}. \]

Let \( L_1 = L[[1]]^{11} \), \( L_2 = L[[1]]^{11} \), \( L_3 = L[[1]]^{11} \). Then

\[ S = L_3 - L_2^\top L_1 L_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & -1 & 0 & -1 & -2 & 0 \\
0 & -1 & 7 & -3 & -1 & -2 & 0 \\
0 & 0 & -3 & 5 & -2 & 0 & 0 \\
0 & -1 & -1 & -2 & 7 & -3 & 0 \\
0 & -2 & -2 & 0 & -3 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \]

\[ S^\# = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
2052/11725 & -548/11725 & -139/1675 & -438/11725 & -93/11725 & 0 \\
0 & -548/11725 & 1152/11725 & 11/1675 & -363/11725 & -318/11725 & 0 \\
0 & -139/1675 & 11/1675 & 261/1675 & -34/1675 & -99/1675 & 0 \\
0 & -438/11725 & -363/11725 & -34/1675 & 1122/11725 & -83/11725 & 0 \\
0 & -93/11725 & -318/11725 & -99/1675 & -83/11725 & 1187/11725 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \]

\[ S^\pi = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\
0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\
0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \]

\[ K = L_1 + L_2 S^\# L_2^\top = 2, \ R = L_2^2 + L_2 S^\pi L_2^\top = 8, \ R^\# = 1/8. \]

By Theorem 3.1, we get \( L^\# = \begin{pmatrix} X \\ Y \end{pmatrix} \), where

\[ X = 1/8, \ Y = (0 \ 0 \ 0 \ 0 \ 0 \ -1/8), \]
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\[
Z = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2052/11725 & -548/11725 & -139/1675 & -438/11725 & -93/11725 \\
0 & -548/11725 & 1152/11725 & 11/1675 & -363/11725 & -318/11725 \\
0 & -139/1675 & 11/1675 & 261/1675 & -34/1675 & -99/1675 \\
0 & -438/11725 & -363/11725 & -34/1675 & 1122/11725 & -83/11725 \\
0 & -93/11725 & -318/11725 & -99/1675 & -83/11725 & 1187/11725 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/8
\end{pmatrix}
\]

If we let \( L_1 = L[[1,2]],[1,2]] \), \( L_2 = L[[1,2]],[1,2]] \), \( L_3 = L[[1,2]],[1,2]] \), then

\[
S = L_3 - L_2^2 L_1^2 L_2 = \begin{pmatrix}
4 & -1 & 0 & -1 & -2 & 0 \\
-1 & 7 & -3 & -1 & -2 & 0 \\
0 & -3 & 5 & -2 & 0 & 0 \\
-1 & -1 & -2 & 7 & -3 & 0 \\
-2 & -2 & 0 & -3 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
S^\# = \begin{pmatrix}
2052/11725 & -548/11725 & -139/1675 & -438/11725 & -93/11725 & 0 \\
-548/11725 & 1152/11725 & 11/1675 & -363/11725 & -318/11725 & 0 \\
-139/1675 & 11/1675 & 261/1675 & -34/1675 & -99/1675 & 0 \\
-438/11725 & -363/11725 & -34/1675 & 1122/11725 & -83/11725 & 0 \\
-93/11725 & -318/11725 & -99/1675 & -83/11725 & 1187/11725 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
S^\pi = \begin{pmatrix}
1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\
1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\
1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\
1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
R = L_1^2 + L_2 S^\pi L_2^\top = \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}, \quad R^\# = \begin{pmatrix} 1/8 & 0 \\ 0 & 0 \end{pmatrix}, \quad K = L_1 + L_2 S^\# L_2^\top = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.
\]

By Theorem 3.1, we get \( L^\# = \begin{pmatrix} X & Y \\ Y^\top & Z \end{pmatrix} \), where

\[
X = \begin{pmatrix} 1/8 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]
Let $L$ be the Laplacian matrix of a weighted graph, and $L$ is partitioned as $L = \begin{pmatrix} L_1 & L_2 \\ L_2^\top & L_3 \end{pmatrix}$, where $L_1$ is square. The group inverse of generalized Schur complement $S = L_3 - L_2^\top L_1^# L_2^\top L_1$ plays the key role in the block representation of $L^\#$ (cf. Theorem 3.1). The Laplacian matrix is an M-matrix. It is known that the Schur complement of an M-matrix is an M-matrix (see [15]). Hence, $S$ is an M-matrix. Clearly, we have $Le = 0$, where $e$ denotes an all-ones column vector with suitable dimension. By $Le = 0$, we get $L_1e + L_2e = 0$, $L_2^\top e + L_3e = 0$. Then we have

$$Se = L_3e - L_2^\top L_1^# L_2e = -L_2^\top e + L_2^\top L_1^# L_1e = -L_1^\top L_3 e = -(L_1^\top L_2^\top)^\top e.$$ 

Lemma 2.1 implies that $Se = 0$. Clearly, $S$ is symmetric. Since $S$ is an M-matrix and $Se = 0$, $S$ is the Laplacian matrix of a weighted graph. Hence, we can obtain a block representation for $S^\#$ from Theorem 3.1. We give an algorithm for $L^\#$ as follows.

**Step 1.** Let $L = \begin{pmatrix} L_1 & L_2 \\ L_2^\top & L_3 \end{pmatrix}$, where $L_1$ is square. Apply Theorem 3.1 to represent $L^\#$.

**Step 2.** Let $S = L_3 - L_2^\top L_1^# L_2 = \begin{pmatrix} S_1 & S_2 \\ S_2^\top & S_3 \end{pmatrix}$, where $S_1$ is square. Go to step 1 to calculate $S^\#$.

The group inverse of matrices has numerous applications in singular differential equations, Markov chains and iterative methods etc (see [9,10,23]). Here we give a new application for the group inverses of $2 \times 2$ block matrices.

**Theorem 3.2.** Let $G$ be a weighted graph with Laplacian matrix $L$. Let $i$ and $j$ be two vertices of $G$, and $i$ and $j$ belong to the same component of $G$. Then the resistance distance between $i$ and $j$ is $\Omega_{ij} = \epsilon X e^\top$, where

$$\epsilon = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad X = L_1 R^# K R^# L_1, \quad R = L_1^2 + L_2 S^\top L_2^\top, \quad K = L_1 + L_2 S^# L_2^\top, \quad S = L_3 - L_2^\top L_1^# L_2, \quad L_1 = L([i,j]([i,j])), \quad L_2 = L([i,j]([i,j])), \quad L_3 = L([i,j]([i,j])).$$
Lemma 2.3 implies that $\Omega_{ij}$ where $L, G, G$ and 6 in the weighted graph $K = L$. Let $\Omega = \begin{pmatrix} X & Y \\ Y^\top & Z \end{pmatrix}$, where

\[
X = L_1 R^# K R^# L_1, \quad Y = L_1 R^# K R^# L_2 S^\# - L_1 R^# L_2 S^#,
Z = S^\# L_1^2 R^# K R^# L_2 S^\# - S^\# L_1^2 R^# L_2 S^\# + S^#,
R = L_2^2 + L_2 S^\# L_2^\top, \quad K = L_1 + L_2 S^# L_2^\top, \quad S = L_3 - L_2^\top L_1\, L_2.
\]

Lemma 2.3 implies that $\Omega_{ij} = (1 \quad -1) X (1 \quad -1)^\top$. □

Now we use Theorem 3.2 to calculate the resistance distance between vertices 4 and 6 in the weighted graph $G$ shown in Figure 1. Let $L$ be the Laplacian matrix of $G$. Let $L_1 = L[\{4, 6\}|\{4, 6\}], L_2 = L[\{4, 6\}|\{4, 6\}], L_3 = L[\{4, 6\}|\{4, 6\}]$. Then

\[
S = L_3 - L_2^\top L_1\, L_2 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 11/3 & -5/6 & -17/6 & 0 \\ 0 & 0 & -5/6 & 137/48 & -97/48 & 0 \\ 0 & 0 & -17/6 & -97/48 & 233/48 & 0 \\ -2 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},
\]

\[
S^# = \begin{pmatrix} 1/8 & 0 & 0 & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 188/1407 & -142/1407 & -46/1407 & 0 \\ 0 & 0 & -142/1407 & 227/1407 & -85/1407 & 0 \\ 0 & 0 & -46/1407 & -85/1407 & 131/1407 & 0 \\ -1/8 & 0 & 0 & 0 & 0 & 1/8 \end{pmatrix},
\]

\[
S^\# = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix},
\]

\[
K = L_1 + L_2 S^# L_2^\top = \begin{pmatrix} 3516/469 & -372/469 \\ -372/469 & 3420/469 \end{pmatrix}, \quad R = L_2^2 + L_2 S^\# L_2^\top = \begin{pmatrix} 62 & -2 \\ -2 & 62 \end{pmatrix}.
\]
By Theorem 3.2, the resistance distance between vertices 4 and 6 is

\[
\Omega_{46} = (1 - 1) \begin{pmatrix} 1152/11725 & -363/11725 \\ -363/11725 & 1122/11725 \end{pmatrix} (1 - 1) = 120/469.
\]

For a connected graph \( G \), let \( d(u, v) \) denote the distance between two vertices \( u, v \) in \( G \).

**Theorem 3.3.** Let \( G \) be a connected bipartite graph with signless Laplacian matrix \( Q \). Let \( u \) and \( v \) be two vertices of \( G \). Then

\[
\Omega_{uv} = \begin{cases} 
Q_{uu}^# + Q_{vv}^# + Q_{uv}^# + Q_{vu}^# & \text{if } d(u, v) \text{ is odd}, \\
Q_{uu}^# + Q_{vv}^# - Q_{uv}^# - Q_{vu}^# & \text{if } d(u, v) \text{ is even}.
\end{cases}
\]

**Proof.** Since \( G \) is a bipartite graph, its adjacency matrix can be written as \( A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \), where two zero sub-blocks of \( A \) correspond to the two color classes of \( G \). Suppose that \( Q = \begin{pmatrix} D_1 \\ B^T \\ D_2 \end{pmatrix} \) is the signless Laplacian matrix of \( G \). Then \( L = \begin{pmatrix} D_1 & -B \\ -B^T & D_2 \end{pmatrix} \) is the Laplacian matrix of \( G \). Clearly, we have

\[
Q = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} D_1 & -B^T \\ -B & D_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},
\]

\[
Q^# = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} D_1 & -B^T \\ -B & D_2 \end{pmatrix}^\# \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.
\]

If \( d(u, v) \) is odd, then \( u, v \) belong to different color classes of \( G \). Lemma 2.3 implies that \( \Omega_{uv} = Q_{uu}^# + Q_{vv}^# + Q_{uv}^# + Q_{vu}^# \). If \( d(u, v) \) is even, then \( u, v \) belong to the same color class of \( G \). Lemma 2.3 implies that \( \Omega_{uv} = Q_{uu}^# + Q_{vv}^# - Q_{uv}^# - Q_{vu}^# \). \( \square \)

Let \( G \) be a weighted graph with signless Laplacian matrix \( Q \), and \( Q \) is partitioned as \( Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{pmatrix} \), where \( Q_1 \) is square. It is known that \( Q \) is positive semidefinite. By Lemma 2.1, we have \( Q_1^TQ_2 = 0 \), \( Q_2Q_3^T = 0 \). It is not difficult to get the representation for \( Q^# \).
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THEOREM 3.4. Let $G$ be a weighted graph with signless Laplacian matrix $Q$. If $Q$ is partitioned as $Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_2^\top & Q_3 \end{pmatrix}$ ($Q_1$ is square), then

$$Q^\# = \begin{pmatrix} X & Y \\ Y^\top & Z \end{pmatrix},$$

where

$$X = Q_1 R^\# K R^\# Q_1, \quad Y = Q_1 R^\# K R^\# Q_2 S - Q_1 R^\# Q_2 S^\#,\nZ = S^\# Q_1 R^\# K R^\# Q_2 S - Q_1 R^\# K R^\# Q_2 S^\# - S^\# Q_2 R^\# Q_2 S^\# + S^\#,$$

$$R = Q_1^2 + Q_2 S^\# Q_2^\top, \quad K = Q_1 + Q_2 S^\# Q_2^\top, \quad S = Q_3 - Q_2^\top Q_2^\# Q_2.$$

Proof. The proof is similar to the proof of Theorem 3.1. □

Acknowledgment. The authors would like to thank the referee for giving valuable comments and suggestions.

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