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A NOTE ON BLOCK REPRESENTATIONS OF THE GROUP INVERSE OF LAPLACIAN MATRICES*

CHANGJIANG BU†, LIZHU SUN†, JIANG ZHOU†, AND YIMIN WEI‡

Abstract. Let G be a weighted graph with Laplacian matrix L and signless Laplacian matrix Q. In this note, block representations for the group inverse of L and Q are given. The resistance distance in a graph can be obtained from the block representation of the group inverse of L.

Key words. Group inverse, Laplacian matrix, Signless Laplacian matrix, Resistance distance.

AMS subject classifications. 15A09, 05C50, 05C12.

1. Introduction. For an $n \times n$ matrix A, the *group inverse* of A is the unique $n \times n$ matrix X satisfying the matrix equations $AXA = A$, $XAX = X$ and $AX = XA$. It is well known that the group inverse of A exists if and only if $\mathrm{rank}(A) = \mathrm{rank}(A^2)$ (see [28,29]). If the group inverse of A exists, it is unique, which is denoted by $A^{\#}$. The matrix A is *group invertible* if $A^{\#}$ exists. For a matrix B, let B^+ denote the Moore-Penrose inverse of B. Some representations for the group inverse of block matrices (operators) are given in [2–8,11,12,14,16,17,28]. More details for the theory of generalized inverse can be found in [9].

Let G be a undirected weighted graph without loops or multiple edges, and each edge of G has been labeled by a positive real number, which is called the *weight* of the edge. The *adjacency matrix* A of G is the matrix whose (i, j)-entry equals 0 if there is no edge joining vertices i and j and equals the weight of the edge joining vertices i and j otherwise. Let D be the diagonal matrix whose i-th diagonal entry equals the sum of the weights of the edges incident to the vertex i in G. The matrices $D - A$ and $D + A$ are called the *Laplacian matrix* and *signless Laplacian matrix* of G, respectively. It is known that $D - A$ and $D + A$ are positive semidefinite.

Let L be the Laplacian matrix of a weighted graph G. Since L is symmetric,

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$L^{\#}$ exists and $L^{\#} = L^{+}$ (see [9]). In [20], Kirkland et al. gave a representation for the group inverse of irreducible Laplacian matrices in terms of bottleneck matrix and all-ones matrix. In [18], Ho and van Dooren used an SVD (singular value decomposition) approach to calculate the Moore-Penrose (group) inverse of the Laplacian of a bipartite graph. In this note, we give a block representation for the group inverse of (signless) Laplacian matrices. Applying this block representation, we give a formulae for the resistance distance in a graph.

2. Some lemmas. For a group invertible matrix S, let S^{π} denote the projection matrix $I - SS^{\#}$, where I is the identity matrix.

Lemma 2.1. [25] *Let M be a Hermitian positive semidefinite matrix, which is partitioned as $M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$. Then $A^{\pi}B = 0,\ BC^{\pi} = 0$.*

Lemma 2.2. [3] *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A is nonsingular, and $S = D - CA^{-1}B$ is group invertible. Then $M^{\#}$ exists if and only if $R = A^2 + BS^{\pi}C$ is nonsingular. If $M^{\#}$ exists, then*

$$M^{\#} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where

$$\begin{aligned}
X &= AR^{-1}(A + BS^{\#}C)R^{-1}A, \\
Y &= AR^{-1}(A + BS^{\#}C)R^{-1}BS^{\pi} - AR^{-1}BS^{\#}, \\
Z &= S^{\pi}CR^{-1}(A + BS^{\#}C)R^{-1}A - S^{\#}CR^{-1}A, \\
W &= S^{\pi}CR^{-1}(A + BS^{\#}C)R^{-1}BS^{\pi} - S^{\#}CR^{-1}BS^{\pi} - S^{\pi}CR^{-1}BS^{\#} + S^{\#}.
\end{aligned}$$

Klein and Randić introduced the concept of resistance distance in [21]. A graph G can be viewed as an electrical network N by replacing each edge of G with a resistor. For two vertices i and j in G, the *resistance distance* between them is defined to be the effective resistance between them in the electrical network N (see [21]). The resistance distance is a distance function in graphs, it has important applications in chemical graph theory. Some results on resistance distance can be found in [13,21,24,26,27].

For a matrix M, let M_{ij} denote the (i,j)-entry of M. Let G be a connected weighted graph with Laplacian matrix L. Let Ω_{ij} denote the resistance distance between vertices i and j in G. It is know that $\Omega_{ij} = L^{+}_{ii} + L^{+}_{jj} - L^{+}_{ij} - L^{+}_{ji}$ (see [1]). Note that L is symmetric, $L^{\#} = L^{+}$. Hence, we have the following lemma.

Lemma 2.3. *Let G be a connected weighted graph with vertex set $\{1, 2, \ldots, n\}$ and Laplacian matrix L. Then $\Omega_{ij} = L^{\#}_{ii} + L^{\#}_{jj} - L^{\#}_{ij} - L^{\#}_{ji}$.*

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3. Main results. Some expressions for the Moore-Penrose inverse of a 2×2 block matrix are given in [19,22]. But the expressions in [19,22] are very complicated. We first give a new expression for the group inverse of Laplacian matrices as follow.

THEOREM 3.1. *Let G be a weighted graph with Laplacian matrix L. If L is partitioned as $L = \begin{pmatrix} L_1 & L_2 \\ L_2^\top & L_3 \end{pmatrix}$ (L_1 is square), then*

$$L^\# = \begin{pmatrix} X & Y \\ Y^\top & Z \end{pmatrix},$$

where

$$\begin{aligned}
X &= L_1 R^\# K R^\# L_1,\\
Y &= L_1 R^\# K R^\# L_2 S^\pi - L_1 R^\# L_2 S^\#,\\
Z &= S^\pi L_2^\top R^\# K R^\# L_2 S^\pi - S^\# L_2^\top R^\# L_2 S^\pi - S^\pi L_2^\top R^\# L_2 S^\# + S^\#,\\
R &= L_1^2 + L_2 S^\pi L_2^\top,\\
K &= L_1 + L_2 S^\# L_2^\top,\\
S &= L_3 - L_2^\top L_1^\# L_2.
\end{aligned}$$

Proof. Since $L_1,\ L_3$ are real symmetric, there exist orthogonal matrices $P_1,\ P_2$ such that

$$L_1 = P_1 \begin{pmatrix} \Delta_1 & 0 \\ 0 & 0 \end{pmatrix} P_1^\top,\ L_3 = P_2 \begin{pmatrix} \Delta_2 & 0 \\ 0 & 0 \end{pmatrix} P_2^\top,$$

where $\Delta_1,\ \Delta_2$ are nonsingular diagonal matrices, the zero blocks can be vacuous. Then we have

$$L_1^\# = P_1 \begin{pmatrix} \Delta_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} P_1^\top,\ L_3^\# = P_2 \begin{pmatrix} \Delta_2^{-1} & 0 \\ 0 & 0 \end{pmatrix} P_2^\top.$$

Suppose that $L_2 = P_1 \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} P_2^\top$. By Lemma 2.1, we have $L_1^\pi L_2 = 0$, $L_2 L_3^\pi = 0$. Hence, $M_2 = 0,\ M_3 = 0,\ M_4 = 0$. Then

$$L^\# = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} \Delta_1 & 0 & M_1 & 0 \\ 0 & 0 & 0 & 0 \\ M_1^\top & 0 & \Delta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^\# \begin{pmatrix} P_1^\top & 0 \\ 0 & P_2^\top \end{pmatrix} = U \begin{pmatrix} M^\# & 0 \\ 0 & 0 \end{pmatrix} U^{-1},$$

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where $M = \begin{pmatrix} \Delta_1 & M_1 \\ M_1^\top & \Delta_2 \end{pmatrix}$, $U = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$. Recall that Δ_1 is a nonsingular diagonal matrix. Since $\Delta_2 - M_1^\top \Delta_1^{-1} M_1$, the Schur complement of M, is real symmetric, it is group invertible. By Lemma 2.2, we have

$$M^\# = \begin{pmatrix} \widetilde{X} & \widetilde{Y} \\ \widetilde{Y}^\top & \widetilde{W} \end{pmatrix},$$

where

$$\begin{aligned}
\widetilde{X} &= \Delta_1 \widetilde{R}^{-1} \widetilde{K} \widetilde{R}^{-1} \Delta_1, \\
\widetilde{Y} &= \Delta_1 \widetilde{R}^{-1} \widetilde{K} \widetilde{R}^{-1} M_1 \widetilde{S}^\pi - \Delta_1 \widetilde{R}^{-1} M_1 \widetilde{S}^\#, \\
\widetilde{W} &= \widetilde{S}^\pi M_1^\top \widetilde{R}^{-1} \widetilde{K} \widetilde{R}^{-1} M_1 \widetilde{S}^\pi - \widetilde{S}^\# M_1^\top \widetilde{R}^{-1} M_1 \widetilde{S}^\pi - \widetilde{S}^\pi M_1^\top \widetilde{R}^{-1} M_1 \widetilde{S}^\# + \widetilde{S}^\#, \\
\widetilde{R} &= \Delta_1^2 + M_1 \widetilde{S}^\pi M_1^\top, \\
\widetilde{K} &= \Delta_1 + M_1 S^\# M_1^\top, \\
\widetilde{S} &= \Delta_2 - M_1^\top \Delta_1^{-1} M_1.
\end{aligned}$$

By $L^\# = U \begin{pmatrix} M^\# & 0 \\ 0 & 0 \end{pmatrix} U^{-1}$, we can obtain the representation of $L^\#$. ∎

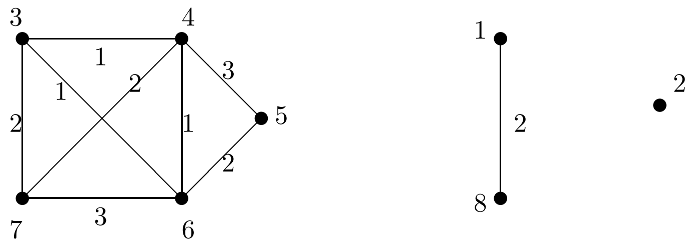

Figure 1: Weighted graph G.

Let F_1, F_2 be two subsets of the set $\{1, 2, \ldots, n\}$. The complement of F_1 and F_2 in $\{1, 2, \ldots, n\}$ are denoted by $\overline{F_1}$ and $\overline{F_2}$, respectively. For a matrix L of order n, let $L[F_1|F_2]$ denote the submatrix of L determined by the rows whose index is in F_1 and the columns whose index is in F_2. Here we give an example for Theorem 3.1.

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Considering the weighted graph G shown in Figure 1. The Laplacian matrix of G is

$$L = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -1 & 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 7 & -3 & -1 & -2 & 0 \\ 0 & 0 & 0 & -3 & 5 & -2 & 0 & 0 \\ 0 & 0 & -1 & -1 & -2 & 7 & -3 & 0 \\ 0 & 0 & -2 & -2 & 0 & -3 & 7 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Let $L_1 = L[\{1\}|\{1\}],\ L_2 = L[\{1\}|\overline{\{1\}}],\ L_3 = L[\overline{\{1\}}|\overline{\{1\}}]$. Then

$$S = L_3 - L_2^\top L_1^\# L_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -1 & 0 & -1 & -2 & 0 \\ 0 & -1 & 7 & -3 & -1 & -2 & 0 \\ 0 & 0 & -3 & 5 & -2 & 0 & 0 \\ 0 & -1 & -1 & -2 & 7 & -3 & 0 \\ 0 & -2 & -2 & 0 & -3 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S^\# = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2052/11725 & -548/11725 & -139/1675 & -438/11725 & -93/11725 & 0 \\ 0 & -548/11725 & 1152/11725 & 11/1675 & -363/11725 & -318/11725 & 0 \\ 0 & -139/1675 & 11/1675 & 261/1675 & -34/1675 & -99/1675 & 0 \\ 0 & -438/11725 & -363/11725 & -34/1675 & 1122/11725 & -83/11725 & 0 \\ 0 & -93/11725 & -318/11725 & -99/1675 & -83/11725 & 1187/11725 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S^\pi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\ 0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\ 0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\ 0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$K = L_1 + L_2 S^\# L_2^\top = 2,\ R = L_1^2 + L_2 S^\pi L_2^\top = 8,\ R^\# = 1/8.$$

By Theorem 3.1, we get $L^\# = \begin{pmatrix} X & Y \\ Y^\top & Z \end{pmatrix}$, where

$$X = 1/8, \quad Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1/8 \end{pmatrix},$$

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$$Z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2052/11725 & -548/11725 & -139/1675 & -438/11725 & -93/11725 & 0 \\ 0 & -548/11725 & 1152/11725 & 11/1675 & -363/11725 & -318/11725 & 0 \\ 0 & -139/1675 & 11/1675 & 261/1675 & -34/1675 & -99/1675 & 0 \\ 0 & -438/11725 & -363/11725 & -34/1675 & 1122/11725 & -83/11725 & 0 \\ 0 & -93/11725 & -318/11725 & -99/1675 & -83/11725 & 1187/11725 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/8 \end{pmatrix}.$$

If we let $L_1 = L[\{1,2\}|\{1,2\}],\ L_2 = L[\{1,2\}|\overline{\{1,2\}}],\ L_3 = L[\overline{\{1,2\}}|\overline{\{1,2\}}]$, then

$$S = L_3 - L_2^\top L_1^\# L_2 = \begin{pmatrix} 4 & -1 & 0 & -1 & -2 & 0 \\ -1 & 7 & -3 & -1 & -2 & 0 \\ 0 & -3 & 5 & -2 & 0 & 0 \\ -1 & -1 & -2 & 7 & -3 & 0 \\ -2 & -2 & 0 & -3 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S^\# = \begin{pmatrix} 2052/11725 & -548/11725 & -139/1675 & -438/11725 & -93/11725 & 0 \\ -548/11725 & 1152/11725 & 11/1675 & -363/11725 & -318/11725 & 0 \\ -139/1675 & 11/1675 & 261/1675 & -34/1675 & -99/1675 & 0 \\ -438/11725 & -363/11725 & -34/1675 & 1122/11725 & -83/11725 & 0 \\ -93/11725 & -318/11725 & -99/1675 & -83/11725 & 1187/11725 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S^\pi = \begin{pmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R = L_1^2 + L_2 S^\pi L_2^\top = \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix},\ R^\# = \begin{pmatrix} 1/8 & 0 \\ 0 & 0 \end{pmatrix},\ K = L_1 + L_2 S^\# L_2^\top = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

By Theorem 3.1, we get $L^\# = \begin{pmatrix} X & Y \\ Y^\top & Z \end{pmatrix}$, where

$$X = \begin{pmatrix} 1/8 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

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$$Z = \begin{pmatrix} 2052/11725 & -548/11725 & -139/1675 & -438/11725 & -93/11725 & 0 \\ -548/11725 & 1152/11725 & 11/1675 & -363/11725 & -318/11725 & 0 \\ -139/1675 & 11/1675 & 261/1675 & -34/1675 & -99/1675 & 0 \\ -438/11725 & -363/11725 & -34/1675 & 1122/11725 & -83/11725 & 0 \\ -93/11725 & -318/11725 & -99/1675 & -83/11725 & 1187/11725 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/8 \end{pmatrix}.$$

Let L be the Laplacian matrix of a weighted graph, and L is partitioned as $L = \begin{pmatrix} L_1 & L_2 \\ L_2^\top & L_3 \end{pmatrix}$, where L_1 is square. The group inverse of generalized Schur complement $S = L_3 - L_2^\top L_1^\# L_2$ plays the key role in the block representation of $L^\#$ (cf. Theorem 3.1). The Laplacian matrix is an M-matrix. It is known that the Schur complement of an M-matrix is an M-matrix (see [15]). Hence, S is an M-matrix. Clearly, we have $Le = 0$, where e denotes an all-ones column vector with suitable dimension. By $Le = 0$, we get $L_1 e + L_2 e = 0, L_2^\top e + L_3 e = 0$. Then we have

$$Se = L_3 e - L_2^\top L_1^\# L_2 e = -L_2^\top e + L_2^\top L_1^\# L_1 e = -L_2^\top L_1^\pi e = -(L_1^\pi L_2)^\top e.$$

Lemma 2.1 implies that $Se = 0$. Clearly, S is symmetric. Since S is an M-matrix and $Se = 0$, S is the Laplacian matrix of a weighted graph. Hence, we can obtain a block representation for $S^\#$ from Theorem 3.1. We give a algorithm for $L^\#$ as follows.

Step 1. Let $L = \begin{pmatrix} L_1 & L_2 \\ L_2^\top & L_3 \end{pmatrix}$, where L_1 is square. Apply Theorem 3.1 to represent $L^\#$.

Step 2. Let $S = L_3 - L_2^\top L_1^\# L_2 = \begin{pmatrix} S_1 & S_2 \\ S_2^\top & S_3 \end{pmatrix}$, where S_1 is square. Go to step 1 to calculate $S^\#$.

The group inverse of matrices has numerous applications in singular differential equations, Markov chains and iterative methods etc (see [9,10,23]). Here we give a new application for the group inverses of 2×2 block matrices.

Theorem 3.2. *Let G be a weighted graph with Laplacian matrix L. Let i and j be two vertices of G, and i and j belong to the same component of G. Then the resistance distance between i and j is $\Omega_{ij} = \epsilon X \epsilon^\top$, where*

$$\epsilon = \begin{pmatrix} 1 & -1 \end{pmatrix},\ X = L_1 R^\# K R^\# L_1,\ R = L_1^2 + L_2 S^\pi L_2^\top,\ K = L_1 + L_2 S^\# L_2^\top,$$
$$S = L_3 - L_2^\top L_1^\# L_2,\ L_1 = L[\{i,j\}|\{i,j\}],\ L_2 = L[\{i,j\}|\overline{\{i,j\}}],\ L_3 = L(\overline{\{i,j\}}|\overline{\{i,j\}}).$$

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Proof. There exists a permutation matrix P such that $L = P\begin{pmatrix} L_1 & L_2 \\ L_2^\top & L_3 \end{pmatrix} P^\top$, where $L_1 = L[\{i,j\}|\{i,j\}]$. By Theorem 3.1, we have

$$L^\# = \begin{pmatrix} X & Y \\ Y^\top & Z \end{pmatrix},$$

where

$$\begin{aligned} &X = L_1 R^\# K R^\# L_1,\ Y = L_1 R^\# K R^\# L_2 S^\pi - L_1 R^\# L_2 S^\#, \\ &Z = S^\pi L_2^\top R^\# K R^\# L_2 S^\pi - S^\# L_2^\top R^\# L_2 S^\pi - S^\pi L_2^\top R^\# L_2 S^\# + S^\#, \\ &R = L_1^2 + L_2 S^\pi L_2^\top,\ K = L_1 + L_2 S^\# L_2^\top,\ S = L_3 - L_2^\top L_1^\# L_2. \end{aligned}$$

Lemma 2.3 implies that $\Omega_{ij} = \begin{pmatrix} 1 & -1 \end{pmatrix} X \begin{pmatrix} 1 & -1 \end{pmatrix}^\top$. $\square$

Now we use Theorem 3.2 to calculate the resistance distance between vertices 4 and 6 in the weighted graph G shown in Figure 1. Let L be the Laplacian matrix of G. Let $L_1 = L[\{4,6\}|\{4,6\}],\ L_2 = L[\{4,6\}|\overline{\{4,6\}}],\ L_3 = L(\overline{\{4,6\}}|\overline{\{4,6\}})$. Then

$$S = L_3 - L_2^\top L_1^\# L_2 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 11/3 & -5/6 & -17/6 & 0 \\ 0 & 0 & -5/6 & 137/48 & -97/48 & 0 \\ 0 & 0 & -17/6 & -97/48 & 233/48 & 0 \\ -2 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$S^\# = \begin{pmatrix} 1/8 & 0 & 0 & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 188/1407 & -142/1407 & -46/1407 & 0 \\ 0 & 0 & -142/1407 & 227/1407 & -85/1407 & 0 \\ 0 & 0 & -46/1407 & -85/1407 & 131/1407 & 0 \\ -1/8 & 0 & 0 & 0 & 0 & 1/8 \end{pmatrix},$$

$$S^\pi = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix},$$

$$K = L_1 + L_2 S^\# L_2^\top = \begin{pmatrix} 3516/469 & -372/469 \\ -372/469 & 3420/469 \end{pmatrix},\ R = L_1^2 + L_2 S^\pi L_2^\top = \begin{pmatrix} 62 & -2 \\ -2 & 62 \end{pmatrix},$$

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$$R^{\#} = \begin{pmatrix} 31/1920 & 1/1920 \\ 1/1920 & 31/1920 \end{pmatrix},\ X = L_1 R^{\#} K R^{\#} L_1 = \begin{pmatrix} 1152/11725 & -363/11725 \\ -363/11725 & 1122/11725 \end{pmatrix}.$$

By Theorem 3.2, the resistance distance between vertices 4 and 6 is

$$\Omega_{46} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1152/11725 & -363/11725 \\ -363/11725 & 1122/11725 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 120/469.$$

For a connected graph G, let $d(u, v)$ denote the distance between two vertices u, v in G.

Theorem 3.3. *Let G be a connected bipartite graph with signless Laplacian matrix Q. Let u and v be two vertices of G. Then*

$$\Omega_{uv} = \begin{cases} Q^{\#}_{uu} + Q^{\#}_{vv} + Q^{\#}_{uv} + Q^{\#}_{vu} & \text{if } d(u,v) \text{ is odd,} \\ Q^{\#}_{uu} + Q^{\#}_{vv} - Q^{\#}_{uv} - Q^{\#}_{vu} & \text{if } d(u,v) \text{ is even.} \end{cases}$$

Proof. Since G is a bipartite graph, its adjacency matrix can be written as $A = \begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix}$, where two zero sub-blocks of A correspond to the two color classes of G. Suppose that $Q = \begin{pmatrix} D_1 & B \\ B^\top & D_2 \end{pmatrix}$ is the signless Laplacian matrix of G. Then $L = \begin{pmatrix} D_1 & -B \\ -B^\top & D_2 \end{pmatrix}$ is the Laplacian matrix of G. Clearly, we have

$$Q = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} D_1 & -B \\ -B^\top & D_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$
$$Q^{\#} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} D_1 & -B \\ -B^\top & D_2 \end{pmatrix}^{\#} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

If $d(u, v)$ is odd, then u, v belong to different color classes of G. Lemma 2.3 implies that $\Omega_{uv} = Q^{\#}_{uu} + Q^{\#}_{vv} + Q^{\#}_{uv} + Q^{\#}_{vu}$. If $d(u, v)$ is even, then u, v belong to the same color class of G. Lemma 2.3 implies that $\Omega_{uv} = Q^{\#}_{uu} + Q^{\#}_{vv} - Q^{\#}_{uv} - Q^{\#}_{vu}$. ☐

Let G be a weighted graph with signless Laplacian matrix Q, and Q is partitioned as $Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_2^\top & Q_3 \end{pmatrix}$, where Q_1 is square. It is know that Q is positive semidefinite. By Lemma 2.1, we have $Q_1^\pi Q_2 = 0,\ Q_2 Q_3^\pi = 0$. It is not difficult to get the representation for $Q^{\#}$.

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THEOREM 3.4. *Let G be a weighted graph with signless Laplacian matrix Q. If Q is partitioned as $Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_2^\top & Q_3 \end{pmatrix}$ (Q_1 is square), then*

$$Q^\# = \begin{pmatrix} X & Y \\ Y^\top & Z \end{pmatrix},$$

where

$$\begin{aligned} &X = Q_1 R^\# K R^\# Q_1,\ Y = Q_1 R^\# K R^\# Q_2 S^\pi - Q_1 R^\# Q_2 S^\#, \\ &Z = S^\pi Q_2^\top R^\# K R^\# Q_2 S^\pi - S^\# Q_2^\top R^\# Q_2 S^\pi - S^\pi Q_2^\top R^\# Q_2 S^\# + S^\#, \\ &R = Q_1^2 + Q_2 S^\pi Q_2^\top,\ K = Q_1 + Q_2 S^\# Q_2^\top,\ S = Q_3 - Q_2^\top Q_1^\# Q_2. \end{aligned}$$

Proof. The proof is similar to the proof of Theorem 3.1. ☐

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