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Muhuo Liu
liumuhuo@163.com

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THE (SIGNLESS) LAPLACIAN SPECTRAL RADII OF C-CYCLIC GRAPHS WITH N VERTICES AND K PENDANT VERTICES

MUHUO LIU

Abstract. A connected graph is called a c-cyclic graph if it contains n vertices and \( n + c - 1 \) edges. Let \( C(n, k, c) \) denote the class of connected c-cyclic graphs with n vertices and k pendant vertices. Recently, the unique extremal graph, which has greatest (respectively, signless) Laplacian spectral radius, in \( C(n, k, c) \) has been determined for \( 0 \leq c \leq 3, k \geq 1 \) and \( n \geq 2c + k + 1 \). In this paper, the unique graph with greatest (respectively, signless) Laplacian spectral radius in \( C(n, k, c) \) is determined for \( c \geq 0, k \geq 1 \) and \( n \geq 2c + k + 1 \).

Key words. (Signless) Laplacian spectral radius, c-Cyclic graph, Pendant vertex.

AMS subject classifications. 05C50, 05C75, 05C05.

1. Introduction. Throughout the paper, \( G = (V, E) \) is a connected undirected simple graph with \( V = \{v_1, v_2, \ldots, v_n\} \). Let \( N(v) \) be the neighbor set of vertex \( v \), and let \( d(v) \) be the degree of \( v \). When \( d(v) = 1 \), we call \( v \) a pendant vertex of \( G \). In the following, we enumerate the degrees of \( G \) in non-increasing order, i.e., \( d_1 \geq d_2 \geq \cdots \geq d_n \), where \( d_1 = d(v) \).

If \( G \) contains \( n \) vertices and \( n + c - 1 \) edges, then \( G \) is called a c-cyclic graph. In particular, \( G \) is called a tree, unicyclic graph, bicyclic graph or a tricyclic graph if \( c = 0, 1, 2 \) or 3, respectively. In the coming discussion, \( n \) and \( k \) are two positive integers, and \( c \) is a nonnegative integer. Let \( C(n, k, c) \) denote the class of connected c-cyclic graphs with \( n \) vertices and \( k \) pendant vertices.

Let \( P_n \) and \( C_n \) be a path and a cycle on \( n \) vertices, respectively. Generally, \( C_3 \) is called a triangle and \( C_4 \) is called a quadrilateral. Suppose \( u \) is a vertex of a graph \( G \). Suppose \( P_s = w_1w_2\cdots w_s \) and \( C_q = v_1v_2\cdots v_qv_1 \), where \( w_i \notin V(G) \) for \( 1 \leq i \leq s \) and \( v_j \notin V(G) \) for \( 1 \leq j \leq q \). If we obtain \( G' \) by identifying the vertex \( u \) with \( w_1 \), then we say that \( G' \) is obtained from \( G \) by attaching the path \( P_s \) to \( u \) of \( G \). Similarly,
if we obtain $G'$ by identifying the vertex $u$ with $v_1$, then we say that $G'$ is obtained from $G$ by attaching the cycle $C_q$ to $u$ of $G$.

Paths $P_1, \ldots, P_k$ are said to have almost equal lengths if $|l_i - l_j| \leq 1$ for $1 \leq i \leq j \leq k$. Denote by $F_n(k, C_4^{(1)}, C_3^{(c-1)})$ the unique connected $c$-cyclic graph on $n$ vertices obtained by attaching $t$ quadrilaterals, $c-t$ triangles, and $k$ paths of almost equal lengths, respectively, to a common vertex. Let $F_n(k, C_3^{(1)})$ be the $c$-cyclic graph on $n$ vertices obtained by attaching $k$ paths of almost equal lengths and $c$ triangles, respectively, to a common vertex, and let $F_n(k, C_4^{(c)})$ define the $c$-cyclic graph on $n$ vertices obtained by attaching $k$ paths of almost equal lengths and $c$ quadrilaterals, respectively, to a common vertex. It follows that $F_n(k, C_3^{(1)}) = F_n(k, C_4^{(0)}, C_3^{(c)})$ and $F_n(k, C_4^{(c)}) = F_n(k, C_4^{(0)}, C_3^{(0)})$.

Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of $G$, respectively. The Laplacian matrix of $G$ is $L(G) = D(G) - A(G)$ and the signless Laplacian matrix of $G$ is $Q(G) = D(G) + A(G)$. Denote by $\lambda(G)$ and $\mu(G)$, respectively, the Laplacian spectral radius and signless Laplacian spectral radius of $G$. Thus, $\lambda(G)$ and $\mu(G)$ are equal to the largest eigenvalues of $L(G)$ and $Q(G)$, respectively. It is well-known that $Q(G)$ is positive semidefinite and nonnegative, and, when $G$ is connected, it is irreducible [9]. Thus, when $G$ is connected, by the famous Perron-Frobenius Theorem of non-negative matrices (see e.g. [5]), it follows that $\mu(H) < \mu(G)$ holds for any proper subgraph $H$ of $G$.

We call $G$ an extremal graph in $\mathbb{C}(n, k, c)$ of first (respectively, second) type if $G$ has greatest signless Laplacian (respectively, Laplacian) spectral radius in $\mathbb{C}(n, k, c)$.

Recently, the extremal graphs, which have greatest (signless) Laplacian spectral radii, in $\mathbb{C}(n, k, c)$ has been studied [1, 3, 4, 8, 10, 11, 14, 15]. From these recent results, we can conclude that: $F_n(k, C_3^{(0)})$ is the unique extremal tree of $\mathbb{C}(n, k, 0)$ [3, 14]; $F_n(k, C_3^{(1)})$ is the unique extremal unicyclic graph in $\mathbb{C}(n, k, 1)$ of first type [10, 15] when $n \geq k + 3$ and $F_n(k, C_4^{(1)})$ is the unique extremal unicyclic graph in $\mathbb{C}(n, k, 1)$ of second type [3, 10] when $n \geq k + 4$; $F_n(k, C_3^{(2)})$ is the unique extremal bicyclic graph in $\mathbb{C}(n, k, 2)$ of first type [1, 11] when $n \geq k + 5$ and $F_n(k, C_4^{(2)})$ is the unique extremal bicyclic graph in $\mathbb{C}(n, k, 2)$ of second type [3, 10] when $n \geq k + 7$; $F_n(k, C_3^{(3)})$ is the unique extremal tricyclic graph in $\mathbb{C}(n, k, 3)$ of first type [3, 11] when $n \geq k + 7$ and $F_n(k, C_4^{(3)})$ is the unique extremal tricyclic graph in $\mathbb{C}(n, k, 3)$ of second type [4] when $n \geq k + 10$.

In this paper, the unique extremal graph in $\mathbb{C}(n, k, c)$ of first (respectively, second) type is identified for $c \geq 0$, $k \geq 1$ and $n \geq 2c + k + 1$. Thus, the main results of [1, 3, 4, 8, 10, 11] immediately follow from our results. Our main results can be stated...
as follows.

**Theorem 1.1.** If $k \geq 1$, $c \geq 0$ and $n \geq 2c + k + 1$, then $F_n(k, C_3^{(c)})$ is the unique extremal graph in $\mathbb{C}(n, k, c)$ of first type.

**Theorem 1.2.** Suppose $k \geq 1$, $c \geq 0$ and $n \geq 2c + k + 1$.

(i) If $n \geq 3c + k + 1$, then $F_n(k, C_4^{(c)})$ is the unique extremal graph in $\mathbb{C}(n, k, c)$ of second type.

(ii) If $n = 2c + k + 1 + t$ and $0 \leq t \leq c - 1$, then $F_n(k, C_4^{(t)}, C_3^{(c-t)})$ is the unique extremal graph in $\mathbb{C}(n, k, c)$ of second type.

2. Some preliminaries. The graph $W_G(uv)$ is obtained from $G$ by subdividing the edge $uv$, i.e., adding a new vertex $w$ and edges $wu$, $uv$ in $G - uv$, where $uv \in E(G)$. An internal path, say $P = v_1 \ldots v_{s+1}$ ($s \geq 1$), is a path joining $v_1$ and $v_{s+1}$ (which need not be distinct) such that the degrees of $v_1$ and $v_{s+1}$ are greater than 2, while all other vertices $v_2, \ldots, v_s$ are of degree 2.

**Lemma 2.1.** [10] Let $uv$ be an edge in an internal path of a connected graph $G$. Then, $\mu(G) > \mu(W_G(uv))$.

Suppose $v$ is a vertex of $G$ with at least two vertices. Let $G_{t,l}$ ($l \geq t \geq 2$) be the graph obtained from $G$ by attaching two new paths $P_t = v_1v_2\ldots v_t$ and $P_l = u_1u_2\ldots u_l$, respectively, to $v$ of $G$. Let $G_{t-1,l+1} = G_{t,l} - v_{t-1}v_t + uv_1$.

**Lemma 2.2.** [10] Let $G$ be a connected graph with at least two vertices. If $l \geq t \geq 2$, then $\mu(G_{t,l}) > \mu(G_{t-1,l+1})$.

Let $m(v)$ denote the average of the degrees of the vertices adjacent to $v$, i.e., $m(v) = \sum_{u \in N(v)} \frac{d(u)}{d(v)}$. Next we shall introduce some bounds for $\lambda(G)$ and $\mu(G)$, which will play prominent roles in the proof of our main results.

**Lemma 2.3.** [12, 13] If $G$ is a connected graph with $n$ vertices, then $\mu(G) \geq \lambda(G) \geq d_1 + 1$, where the first equality holds if and only if $G$ is bipartite, and the second equality holds if and only if $d_1 = n - 1$.

**Lemma 2.4.** [8, 13] If $G$ is connected, then

$$\mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)},$$

where the equality holds if and only if $G$ is regular or a star or a path with four vertices.

**Lemma 2.5.** [7, 13] If $G$ is connected, then

$$\mu(G) \leq \max \left\{ \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)} : uv \in E(G) \right\} ,$$
where the equality holds if and only if $G$ is regular or a bipartite semiregular graph.

**Lemma 2.6.** [2] Let $v$ be a vertex of a connected graph $G$. Suppose that $v_1, \ldots, v_s$ are pendant vertices of $G$ which are adjacent to $v$. Let $G'$ be the graph obtained from $G$ by adding any $b \left( 1 \leq b \leq \frac{s(k-1)}{2} \right)$ edges among $v_1, \ldots, v_s$. Then, $\lambda(G) = \lambda(G')$.

3. **The proofs of Theorems 1.1 and 1.2.** The following simple necessary condition turns out to be surprisingly useful in the proof of our main results.

**Lemma 3.1.** Suppose $G$ is a graph of $C(n, k, c)$, where $c \geq 2$ and $k \geq 1$. If either $\lambda(G) \geq k + 2c + 1$ or $\mu(G) \geq k + 2c + 1$, then $G$ is obtained by attaching $k$ paths and $c$ cycles, respectively, to a common vertex.

**Proof.** Suppose the degree sequence of $G$ is $(d_1, d_2, \ldots, d_n)$. Since $G \in C(n, k, c)$, we have $2(n + c - 1) = \sum_{i=1}^{n} d_i$.

If $d_1 + d_2 \geq k + 2c + 3$, then $2(n + c - 1) = \sum_{i=1}^{n} d_i \geq k + 2c + 3 + 2(n - 2 - k) + k = 2n + 2c - 1$, a contradiction. By Lemmas 2.3–2.4 and the facts that $c \geq 2$ and $k \geq 1$, if $d_1 + d_2 \leq k + 2c + 1$, then

$$\lambda(G) \leq \mu(G) < 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} \leq 2 + \sqrt{(d_1 + d_2 - 2)^2} = d_1 + d_2 \leq k + 2c + 1,$$

a contradiction.

Thus, $\lambda(G) \geq k + 2c + 1$ or $\mu(G) \geq k + 2c + 1$ implies that $d_1 + d_2 = k + 2c + 2$. Note that $2(n + c - 1) = \sum_{i=1}^{n} d_i$ and $G$ contains exactly $k$ pendant vertices. So, $d_3 = \cdots = d_{n-k} = 2$ and $d_{n-k+1} = d_{n-k+2} = \cdots = d_n = 1$. Since $d_1 = k + 2c + 2 - d_2 \leq k + 2c$, we divide the proof into the following three cases.

**Case 1.** $d_1 \leq k + 2c - 2$.

By Lemmas 2.3–2.4,

$$\lambda(G) \leq \mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} = 2 + \sqrt{(k + 2c)d_1} < k + 2c + 1,$$

a contradiction.

**Case 2.** $d_1 = k + 2c - 1$.

Then, $d_2 = 3$, which implies that $d(w) \in \{k + 2c - 1, 3, 2, 1\}$ holds for any $w \in V(G)$. By $c \geq 2$ and $k \geq 1$, $G$ is neither a regular nor a bipartite semiregular graph. According to Lemmas 2.3 and 2.5, we have

$$\lambda(G) \leq \mu(G) < \max \left\{ \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)}, w, v \in E(G) \right\}. \quad (3.1)$$
Suppose that \( f(u_0v_0) = \max \left\{ \frac{d(u)d(v) + m(u) + d(v) + m(v)}{d(u) + d(v)}, uv \in E(G) \right\} \) occurs at the edge \( u_0v_0 \), where \( d(u_0) \geq d(v_0) \). Then, \( d(u_0) \in \{ k + 2c - 1, 3, 2 \} \), as \( G \) is connected and \( c \geq 2 \).

Subcase 2.1. \( d(u_0) = k + 2c - 1 \).

Then, \( d(u_0) = d_1 \geq 4 > d(v_0) \).

If \( d(v_0) = 3 \), then \( d(u_0) = d_1 \) and \( d(v_0) = d_2 \). By inequality (3.1),

\[
\mu(G) < f(u_0v_0) \leq \frac{d_1^2 + d_2 + 2(d_1 - 1) + d_2^2 + d_1 + 2(d_2 - 1)}{d_1 + d_2} = k + 2c - 1 + \frac{14}{k + 2c} \leq k + 2c + 1,
\]
a contradiction.

If \( d(v_0) = 2 \), then by inequality (3.1),

\[
\mu(G) < f(u_0v_0) \leq \frac{d_1^2 + 3 + 2(d_1 - 1) + 4 + d_1 + 3}{d_1 + 2} = k + 2c + \frac{6}{k + 2c + 1} \leq k + 2c + 1,
\]
a contradiction.

If \( d(v_0) = 1 \), then by inequality (3.1),

\[
\mu(G) < f(u_0v_0) \leq \frac{d_1^2 + 1 + 3 + 2(d_1 - 2) + 1 + d_1}{d_1 + 1} = k + 2c + 1 - \frac{1}{k + 2c} < k + 2c + 1,
\]
a contradiction.

Subcase 2.2. \( d(u_0) = 3 \).

Since \( d_1 \geq 4 \) and \( d(u_0) = 3 = d_2 > d_3 \), we have \( 1 \leq d(v_0) \leq 2 \).

If \( d(v_0) = 2 \), then by inequality (3.1),

\[
\mu(G) < f(u_0v_0) \leq \frac{d_1^2 + d_1 + 2(d_2 - 1) + 4 + d_1 + d_2}{d_2 + 2} = \frac{2(k + 2c) + 18}{5} < k + 2c + 1,
\]
a contradiction.

If \( d(v_0) = 1 \), then by inequality (3.1),

\[
\mu(G) < f(u_0v_0) \leq \frac{d_1^2 + d_1 + 2 + 1 + d_2}{d_2 + 1} = \frac{k + 2c + 15}{4} < k + 2c + 1,
\]
a contradiction.
Subcase 2.3. \( d(u_0) = 2 \).

Then, \( 1 \leq d(v_0) \leq 2 \).

If \( d(v_0) = 2 \), then by inequality (3.1),
\[
\lambda(G) \leq \mu(G) < f(u_0v_0) \leq \frac{2(4 + d_1 + 2)}{2 + 2} = \frac{k + 2c + 5}{2} \leq k + 2c + 1,
\]
a contradiction.

If \( d(v_0) = 1 \), then by inequality (3.1),
\[
\lambda(G) \leq \mu(G) < f(u_0v_0) \leq \frac{4 + d_1 + 1 + 1 + 2}{2 + 1} = \frac{k + 2c + 7}{3} < k + 2c + 1,
\]
a contradiction.

Case 3. \( d_1 = k + 2c \).

Then, \( d_2 = 2 \), and hence \( d_2 = \cdots = d_{n-k} = 2 \) and \( d_{n-k+1} = \cdots = d_n = 1 \), which implies that \( G \) is obtained by attaching \( k \) paths and \( c \) cycles to a common vertex. \( \square \)

Proof of Theorem [11] When \( 0 \leq c \leq 1 \), the result had been proved in [8, 10, 14, 15]. So, we may suppose that \( c \geq 2 \) and \( G \) is an extremal graph in \( C(n, k, c) \) of first type in the sequel.

Since \( F_n(k, C_3^{(c)}) \in C(n, k, c) \) and \( F_n(k, C_3^{(c)}) \) is non-bipartite, by the choice of \( G \) and Lemma 2.3, \( \mu(G) \geq \mu(F_n(k, C_3^{(c)})) > k + 2c + 1 \). Thus, by Lemma 3.1, \( G \) is obtained by attaching \( k \) paths and \( c \) cycles, respectively, to a common vertex, say \( u_0 \).

Suppose that \( G \) contains a cycle, say \( C \), of length at least four. Let \( u, v \) and \( w \) be three vertices of \( C \) such that \( uv \in E(C) \), \( vw \in E(C) \) and \( u_0 \not\in \{u, v, w\} \). Suppose \( x \) is a pendant vertex of \( G \). Let \( G_1 = G + wx - vw - xw \), \( G_2 = G_1 - v \) and let \( G_3 = G_1 + xv \). Then, \( G_1 \in C(n, k, c) \). Since \( c \geq 2 \), \( uv \) lies on an internal path of \( G_2 \) and \( G = W_{G_2}(uw) \). By Lemma 2.4, \( \mu(G) < \mu(G_2) \). Furthermore, since \( G_1 \subset G_3 \), we have \( \mu(G_1) < \mu(G_3) \). Thus, \( \mu(G) < \mu(G_2) = \mu(G_1) < \mu(G_3) \), contrary to the choice of \( G \). So, every cycle of \( G \) is a triangle, and hence \( G \) is obtained by attaching \( k \) paths and \( c \) cycles, respectively, to a common vertex, say \( u_0 \).

If there exists two paths, the length of which differ at least two, without loss of generality, we suppose that \( l_1 - l_2 \geq 2 \). Suppose \( P_{l_1} = u_0w_1w_2 \cdots w_{l_1-1} \) and \( P_{l_2} = u_0z_2z_3 \cdots z_{l_2-1} \). Let \( G_5 = G - w_{l_1-2}w_{l_1-1} + z_{l_2-1}z_{l_2-1} \). Then, \( G_5 \in C(n, k, c) \). By Lemma 2.2, we have \( \mu(G) < \mu(G_5) \), which contradicts the choice of \( G \). Thus, the \( k \) paths have almost equal lengths and hence the result follows. \( \square \)

Suppose \( P_1 = u_1u_2 \cdots u_i \) is a path. If \( d(u_2) = d(u_3) = \cdots = d(u_{i-1}) = 2 \) and \( d(u_i) = 1 \), then \( P_1 \) is called a pendant path. Denote by \( g(G) \) the girth, i.e., the length
of a shortest cycle of $G$. Let $F_n^s(k, C_4^{(c-s)}, C_3^{(s)})$ be the connected $(c-s)$-cyclic graph obtained from $F_n(k, C_4^{(c-s)}, C_3^{(s)})$ by deleting $s$ edges, the degrees of whose end vertices are two, in the $s$ triangles of $F_n(k, C_4^{(c-s)}, C_3^{(s)})$. In other words, $F_n^s(k, C_4^{(c-s)}, C_3^{(s)})$ is obtained from $F_{n-2s}(k, C_4^{(c-s)})$ by attaching $2s$ pendant edges to the vertex of degree $k + 2(c - s)$ of $F_{n-2s}(k, C_4^{(c-s)})$.

**Lemma 3.2.** Suppose $G$ is a connected $c$-cyclic graph on $n$ vertices obtained by attaching $k$ paths, $s$ triangles, and $c - s$ cycles of order at least four, respectively, to a common vertex, where $1 \leq s \leq c$ and $k \geq 1$. Then,

$$\lambda(G) \leq \lambda(F_n^s(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})),$$

where the first equality holds if and only if $G = F_n^s(k, C_4^{(c-s)}, C_3^{(s)})$.

**Proof.** Let $G_1$ be the connected $(c-s)$-cyclic graph obtained from $G$ by deleting $s$ edges, the degrees of whose end vertices are two, in the $s$ triangles of $G$. Then, $g(G_1) \geq 4$. Suppose $u_0$ has the maximum degree of $G$. Then, $d(u_0) = k + 2c$. By Lemma 2.3.

$$\lambda(F_n^s(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)}))$$

We consider the following two cases.

**Case 1. The length of every cycle of $G_1$ is four.**

Then, $G_1$ is a connected $(c-s)$-cyclic graph obtained by attaching $k + 2s$ paths (among which at least $2s$ paths have lengths 1), and $c - s$ quadrilaterals, respectively, to $u_0$. Suppose that $P_1, P_2, \ldots, P_k$ are $k$ pendant paths of $G_1$ with the first $k$ largest lengths among all the pendant paths of $G_1$. If there exists two pendant paths of $\{P_1, P_2, \ldots, P_k\}$, the length of which differ at least two, without loss of generality, we may suppose that $l_1 - l_2 \geq 2$. Suppose $P_1 = u_0w_1w_2 \cdots w_{l_1-1}$ and $P_2 = u_0z_1z_2 \cdots z_{l_2-1}$. Let $G_2 = G_1 - w_1w_2 \cdots w_{l_1-1} + z_1z_2 \cdots z_{l_2-1}$. By Lemma 2.3.

$\mu(G_1) < \mu(G_2)$. Repeating the above process, we see by Lemma 2.3.

$$\mu(G_1) \leq \mu(F_n^s(k, C_4^{(c-s)}, C_3^{(s)}))$$

namely, $G = F_n^s(k, C_4^{(c-s)}, C_3^{(s)})$.

Since $\lambda(F_n^s(k, C_4^{(c-s)}, C_3^{(s)}))$ is bipartite, $\mu(F_n^s(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n^s(k, C_4^{(c-s)}, C_3^{(s)}))$ by Lemma 2.3. Thus, by Lemma 2.3.

$$\lambda(G) = \lambda(G_1) \leq \mu(F_n^s(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n^s(k, C_4^{(c-s)}, C_3^{(s)})).$$

If $\lambda(G) = \lambda(F_n^s(k, C_4^{(c-s)}, C_3^{(s)}))$, then $\mu(G_1) = \mu(F_n^s(k, C_4^{(c-s)}, C_3^{(s)}))$, and hence $G = F_n(k, C_4^{(c-s)}, C_3^{(s)})$. Conversely, if $G = F_n(k, C_4^{(c-s)}, C_3^{(s)})$, then $\lambda(G) = \lambda(F_n^s(k, C_4^{(c-s)}, C_3^{(s)}))$.

**Case 2.** $G_1$ contains a cycle, say $C$, of length at least five.

Let $u, v$ and $w$ be three vertices of $C$ such that $uv \in E(C)$, $vw \in E(C)$ and $u_0 \not\in \{u, v, w\}$. Let $P$ be a longest pendant path of $G_1$ with initial vertex $u_0$, and
let \( x \) be the pendant vertex of \( P \). Let \( G_2 = G_1 + uw - vw - uw \), \( G_3 = G_2 - v \) and let \( G_4 = G_2 + xw \). Since \( d(u_0) = k + 2c \geq 3 \), \( uw \) lies on an internal path of \( G_3 \) and \( G_1 = W_{G_1}(uw) \). By Lemma 2.1, \( \mu(G_1) < \mu(G_2) \). Furthermore, since \( G_2 \subseteq G_4 \), we have \( \mu(G_2) < \mu(G_4) \). Thus, \( \mu(G_1) < \mu(G_3) = \mu(G_2) < \mu(G_4) \).

Note that \( G_4 \) contains exactly \( k + 2s \) pendant vertices, and there are at least \( 2s \) pendant vertices being adjacent to \( u_0 \) in \( G_4 \). By repeating the above process, we see that there exists some \((c-s)\)-cyclic graph, say \( G_5 \), such that \( \mu(G_4) \leq \mu(G_5) \), where \( G_5 \) is obtained by attaching \( k + 2s \) paths (among which at least \( 2s \) paths have lengths 1) and \( c-s \) quadrilaterals, respectively, to \( u_0 \). By the proof of Case 1, we have \( \mu(G_5) \leq \mu(F_n(k, C_4^{(c-s)}, C_3^{(s)})) \).

Since \( F_n^*(k, C_4^{(c-s)}, C_3^{(s)}) \) is bipartite, Lemma 2.3 implies that \( \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})) \). Now, by Lemma 2.3, we can conclude that

\[
\lambda(G) = \lambda(G_1) \leq \mu(G_1) < \mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})).
\]

This completes the proof of this result.

**Lemma 3.3.** Suppose \( 1 \leq s \leq c, k \geq 1 \) and \( n \geq k + 3c + 2 - s \). Then, \( \lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})) < \lambda(F_n(k, C_4^{(c-s+1)}, C_3^{(s-1)})) \).

**Proof.** Suppose \( u_0 \) has the maximum degree of \( F_n^*(k, C_4^{(c-s)}, C_3^{(s)}) \). Then, \( d(u_0) = k + 2c \). Let \( P \) be a longest pendant path of \( F_n^*(k, C_4^{(c-s)}, C_3^{(s)}) \) with initial vertex \( u_0 \), and let \( y \) be the pendant vertex of \( P \). If \( |V(P)| = 2 \), then \( n \leq 2s + 3(c-s) + 1 + k = k + 3c + 1 + s \), a contradiction. Thus, \( |V(P)| \geq 3 \). Let \( x \) be a pendant vertex, which is adjacent to \( u_0 \).

Let \( G_1 = F_n^*(k, C_4^{(c-s)}, C_3^{(s)}) + xy \). Then, \( g(G_1) \geq 4 \) and \( G_1 \) is obtained by attaching \( k + 2(s-1) \) paths (among which at least \( 2(s-1) \) paths have lengths 1), \( c-s \) quadrilaterals and a cycle (say \( C_q \), where \( q \geq 4 \)), respectively, to \( u_0 \).

If \( q = 4 \), then by Lemma 2.2, \( \mu(G_1) \leq \mu(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)})) \). If \( q \geq 5 \), then since \( d(u_0) = k + 2c \geq 3 \), by Lemma 2.1, there exists some graph, say \( G_2 \), such that \( \mu(G_2) < \mu(G_1) \), where \( G_2 \) is obtained by attaching \( k + 2(s-1) \) paths (among which at least \( 2(s-1) \) paths have lengths 1) and \( c-s+1 \) quadrilaterals, respectively, to \( u_0 \). Now, Lemma 2.2 implies that \( \mu(G_2) \leq \mu(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)})) \). Thus, \( \mu(G_1) \leq \mu(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)})) \).

Note that \( F_n^*(k, C_4^{(c-s)}, C_3^{(s)}) \subseteq G_1 \). Then, \( \mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) < \mu(G_1) \). By Lemma 2.6, we have \( \lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) \) and \( \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)})) \). Since \( F_n^*(k, C_4^{(c-s)}, C_3^{(s)}) \) and \( F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)}) \) are two bipartite graphs, by Lemma 2.3, \( \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) \).
Now, we can conclude that
\[
\lambda(F_n(k, C_4^{(c-s)}), C_3^{(s)})) = \lambda(F_n^*(k, C_4^{(c-s)}), C_3^{(s)})) = \mu(F_n^*(k, C_4^{(c-s)}), C_3^{(s)}))
\]
\[
< \mu(G_1) \leq \mu(F_n^*(k, C_4^{(c-s+1)}), C_3^{(s-1)}))
\]
\[
= \lambda(F_n^*(k, C_4^{(c-s+1)}), C_3^{(s-1)})) = \lambda(F_n(k, C_4^{(c-s+1)}), C_3^{(s-1)}))
\].

Thus, \(\lambda(F_n(k, C_4^{(c-s)}), C_3^{(s)})) < \lambda(F_n(k, C_4^{(c-s+1)}), C_3^{(s-1)}))\).

Proof of Theorem 1.2 (i). When \(0 \leq c \leq 1\), the result had been proved in [3, 8, 11, 14]. So, we may suppose that \(c \geq 2\) and \(G\) is an extremal graph in \(\mathbb{C}(n, k, c)\) of second type in the sequel.

Note that \(F_n(k, C_4^{(c)}) \in \mathbb{C}(n, k, c)\). By Lemma 2.2 and the choice of \(G\), \(\lambda(G) \geq \lambda(F_n(k, C_4^{(c)})) > k + 2c + 1\). Thus, \(G\) is obtained by attaching \(k\) paths and \(c\) cycles, respectively, to a common vertex by Lemma 3.1. We consider the following two cases.

Case 1. \(g(G) \geq 4\).

If every cycle of \(G\) is a quadrilateral, by Lemma 2.2 it follows that \(\mu(G) \leq \mu(F_n(k, C_4^{(c)}))\), where the equality holds if and only if \(G = F_n(k, C_4^{(c)})\). If \(G\) contains at least one cycle of length at least five, since \(c \geq 2\), by Lemma 2.1 there exists some \(c\)-cyclic graph, say \(G_1\), such that \(\mu(G) < \mu(G_1)\), where \(G_1\) is obtained by attaching \(k\) paths and \(c\) quadrilaterals, respectively, to a common vertex. Furthermore, Lemma 2.2 implies that \(\mu(G_1) \leq \mu(F_n(k, C_4^{(c)}))\), and hence, \(\mu(G) < \mu(F_n(k, C_4^{(c)}))\).

So, we can conclude that \(\mu(G) \leq \mu(F_n(k, C_4^{(c)}))\), where the equality holds if and only if \(G = F_n(k, C_4^{(c)})\). Since \(F_n(k, C_4^{(c)})\) is bipartite, by Lemma 2.3 we have \(\lambda(G) \leq \mu(F_n(k, C_4^{(c)})) = \lambda(F_n(k, C_4^{(c)}))\). Now, if \(\lambda(G) = \lambda(F_n(k, C_4^{(c)}))\), then \(\mu(G) = \mu(F_n(k, C_4^{(c)}))\), and hence, \(G = F_n(k, C_4^{(c)})\).

Case 2. \(g(G) = 3\).

We may suppose that \(G\) contains exactly \(s \geq 1\) triangles. By Lemma 3.2 \(\lambda(G) \leq \lambda(F_n(k, C_4^{(c-s)}), C_3^{(s)}))\). Since \(s \geq 1\), Lemma 3.2 implies that
\[
\lambda(F_n(k, C_4^{(c-s)}), C_3^{(s)})) < \lambda(F_n(k, C_4^{(c-s+1)}), C_3^{(s-1)})) \leq \cdots \leq \lambda(F_n(k, C_4^{(c)}), C_3^{(0)}))
\].

Thus, \(\lambda(G) < \lambda(F_n(k, C_4^{(c)}), C_3^{(s)})) = \lambda(F_n(k, C_4^{(c)}))\).

Proof of Theorem 1.2 (ii). When \(c = 1\), then \(n = k + 3\), and the result clearly follows. So, we may suppose that \(c \geq 2\) and \(G\) is an extremal graph in \(\mathbb{C}(n, k, c)\) of second type in the sequel. Since \(F_n(k, C_4^{(c)}), C_3^{(c-k)}) \in \mathbb{C}(n, k, c)\), by Lemma 2.3 and the choice of \(G\), \(\lambda(G) \geq \lambda(F_n(k, C_4^{(c)}), C_3^{(c-k)})) \geq k + 2c + 1\). Thus, \(G\) is obtained by attaching \(k\) paths and \(c\) cycles, respectively, to a common vertex by Lemma 3.1.

Note that \(n \leq 3c + k\). So, we may suppose that \(G\) contains exactly \(s \geq 1\) triangles.
If \( s \leq c - t - 1 \), then

\[
n \geq 2s + 3(c - s) + k + 1 = 3c + k + 1 - s \\
\geq 3c + k + 1 - (c - t - 1) = k + 2c + t + 2,
\]

a contradiction. Thus, \( s \geq c - t \).

If \( s = c - t \), then by Lemma 3.2, \( \lambda(G) \leq \lambda(F_n(k, C_4^{(c-t)}, C_3^{(c-t)})) \), where the equality holds if and only if \( G = F_n(k, C_4^{(c-t)}, C_3^{(c-t)}) \). If \( s \geq c - t + 1 \), then since \( n = 2c + k + 1 + t \geq 3c + k + 2 - s \), by Lemmas 3.2–3.3 we can conclude that

\[
\lambda(G) \leq \lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})) < \lambda(F_n(k, C_4^{(c-s+1)}, C_3^{(s-1)})) \\
\leq \cdots \leq \lambda(F_n(k, C_4^{(c-t)}, C_3^{(c-t)})) \quad \square
\]

4. Further discussion. By Theorems 1.1 and 1.2, the unique extremal graph in \( \mathbb{C}(n, k, c) \) of first type and the unique extremal graph in \( \mathbb{C}(n, k, c) \) of second type are, respectively, determined for \( c \geq 0, \ k \geq 1 \) and \( n \geq 2c + k + 1 \).

When \( c \geq 0, k \geq 1 \) and \( n \leq 2c + k \), for any \( G \in \mathbb{C}(n, k, c) \), by Lemma 2.3 we have \( \lambda(G) \leq n \), where the equality holds if and only if \( d_1 = n - 1 \). Furthermore, when \( c \geq 3 \), the extremal graphs in \( \mathbb{C}(n, k, c) \) of second type are always not unique. For instance, let \( W_1 \) and \( W_2 \) be the two tricyclic graphs on \( n \) vertices as shown in Fig. 4.1. Then, \( \{W_1, W_2\} \subseteq \mathbb{C}(n, k, 3) \). Since \( d_1(W_1) = n - 1 = d_1(W_2) \), by Lemma 2.3 we have \( \lambda(W_1) = n = \lambda(W_2) \).

When \( c \geq 0, k \geq 1 \) and \( n \leq 2c + k \), it is still an open problem to characterize the extremal graphs in \( \mathbb{C}(n, k, c) \) of first type.

![Fig. 4.1. The tricyclic graphs \( W_1 \) and \( W_2 \).](http://math.technion.ac.il/iic/ela)

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