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PROPERTIES OF A MATRIX GROUP ASSOCIATED TO A
\(\{K, S + 1\}\)-POTENT MATRIX

LEILA LEBTAHI† AND NÉSTOR THOME‡

Abstract. In a previous paper, the authors introduced and characterized a new kind of matrices called \(\{K, s + 1\}\)-potent. In this paper, an associated group to a \(\{K, s + 1\}\)-potent matrix is explicitly constructed and its properties are studied. Moreover, it is shown that the group is a semidirect product of \(\mathbb{Z}_2\) acting on \(\mathbb{Z}_{(s+1)^2-1}\). For some values of \(s\), more specifications on the group are derived. In addition, some illustrative examples are given.

Key words. Involutory matrix, \(\{K, s + 1\}\)-potent matrix, Group.

AMS subject classifications. 15A30, 15A24.

1. Introduction. Let \(K \in \mathbb{C}^{n \times n}\) be an involutory matrix, that is \(K^2 = I_n\), where \(I_n\) denotes the \(n \times n\) identity. In [5], the authors introduced and characterized a new kind of matrices called \(\{K, s + 1\}\)-potent matrices where \(K\) is involutory. We recall that for an involutory matrix \(K \in \mathbb{C}^{n \times n}\) and \(s \in \{0, 1, 2, 3, \ldots\}\), a matrix \(A \in \mathbb{C}^{n \times n}\) is called \(\{K, s + 1\}\)-potent if

\[(1.1) \quad KA^{s+1}K = A.\]

These matrices generalize all the following classes of matrices: \(k\)-potent matrices, idempotent matrices, periodic matrices, involutory matrices, centrosymmetric matrices, mirror symmetric matrices, circulant matrices, etc. Several applications of these matrices can be found in the literature [11, 13]. The class of \(\{K, s + 1\}\)-potent matrices was linked to other kind of matrices (as \(\{s + 1\}\)-generalized projectors, \(\{K\}\)-Hermitian matrices, normal matrices, etc.) in [6]. Throughout this paper, we consider \(K \in \mathbb{C}^{n \times n}\) to be an involutory matrix.

Some results on a similar class of \(2 \times 2\) matrices and \(n \times n\) invertible matrices have been presented in [2]. On the other hand, matrices commuting with a permutation and \(\{K\}\)-centrosymmetric matrices (that correspond to \(s = 0\)) have received increasing

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interest in the last twenty years. Some of their properties can be found in [1, 8, 9, 13]. Furthermore, matrices with k-involutory symmetries have been studied in [11, 12]. Moreover, some spectral properties related to similar classes of matrices are given in [3, 5, 10].

Related to the group theory, we recall that if $G$ is a finite group with identity element $e$ and $a \in G$ then $a^m = e$ implies that the order of $a$ divides $m$ for any natural power $m$.

Motivated by the fact that the definition of $\{K, s + 1\}$-potent matrices involves products of the two matrices $A$ and $K$, we wonder if there are any other relationships between products where these matrices appear. As a particular case, when $s$ is the smallest positive integer such that $A^{s+1} = A$, it is clear that $\{A, A^2, A^3, \ldots, A^s\}$ is a cyclic group (and, therefore, commutative and normal) of order $s$. This leads to our main aim, which is to extend these results to $\{K, s + 1\}$-potent matrices.

This paper is organized as follows. First, properties of a $\{K, s + 1\}$-potent matrix $A$ are studied in Section 2 involving products and powers of $A$ and $K$. These properties are necessary to construct, in Section 3, a finite group $G$ from a given $\{K, s + 1\}$-potent matrix. As a consequence, this group is a semidirect product of $\mathbb{Z}_2$ acting on $\mathbb{Z}_{(s+1)^2 - 1}$ where $\mathbb{Z}_r$ is the group of integers modulo $r$. Moreover, the group $G$ is calculated in some simple cases. The case $A^k = A$ for some $k < (s + 1)^2$ is also analyzed in Section 4. Finally, in Section 5, some illustrative examples are given.

2. Basic properties of $\{K, s + 1\}$-potent matrices. It is clear that for each $n \in \{1, 2, 3, \ldots\}$, there exists at least one matrix $A \in \mathbb{C}^{n \times n}$ such that $A$ is $\{K, s + 1\}$-potent for each involutory matrix $K$ and for each $s \in \{1, 2, 3, \ldots\}$. It is also easy to see that such a matrix is not unique [5].

Throughout this section, we consider $s \in \{1, 2, 3, \ldots\}$. It is well-known [5] that a matrix $A \in \mathbb{C}^{n \times n}$ is $\{K, s + 1\}$-potent if and only if any of the following conditions are (trivially) equivalent: $KAK = A^{s+1}$, $KA = A^{s+1}K$, and $AK = KA^{s+1}$.

We now establish properties regarding $\{K, s + 1\}$-potent matrices.

**Lemma 2.1.** If $A \in \mathbb{C}^{n \times n}$ is a $\{K, s + 1\}$-potent matrix then the following properties hold

(a) $KA^{s+2} = A^{s+2}K$ and $KA^{s+2}K = A^{s+2}$.
(b) $A^{s+2} = (KA)^2 = (AK)^2$.
(c) $A^{(s+1)^2} = A$.
(d) $(A^{(s+1)^2 - 1})^k = A^{(s+1)^2 - 1}$ for every $k \in \{1, 2, 3, \ldots\}$.
(e) $(A^{s+2})^{s+1} = A^{s+2}$.
(f) $(KA)^{2s+1} = KA$ and $(AK)^{2s+1} = AK$. 
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\( (g) \) \( KA^jK = A^{(s+1)}j \) and \( A^jK = KA^{(s+1)}j \) for every \( j \in \{1, 2, \ldots, (s + 1)^2 - 1\} \).
\( (h) \) \( (KA^j)^2 = A^{(s+2)}j \) for all \( j \in \{1, 2, \ldots, (s + 1)^2 - 1\} \).
\( (i) \) \( A^j A^{(s+1)^2 - 1} = A^{(s+1)^2 - 1} A^j \) and \( (KA^j) A^{(s+1)^2 - 1} = A^{(s+1)^2 - 1} (KA^j) = KA^j \), for all \( j \in \{1, 2, \ldots, (s + 1)^2 - 1\} \).
\( (j) \) For each \( j \in \{1, 2, \ldots, (s + 1)^2 - 1\} \), one has \( (KA^j) K A^k = A^{(s+1)^2 - 1} \), where \( k \) is the unique element of \( \{1, 2, \ldots, (s + 1)^2 - 1\} \) such that \( k \equiv -j (s+1) \) [mod \((s + 1)^2 - 1\)].
\( (k) \) \( K (KA^j)^{s+1} K = \begin{cases} KA^{j+s+1} & \text{if } s \text{ is even} \\ (A^{j+s+1})^{s+1} & \text{if } s \text{ is odd} \end{cases} \) for all \( j \in \{1, 2, \ldots, (s + 1)^2 - 1\} \).

Proof. (a) One has \( KA^{s+2} = KA^{s+1} A = AK A = AA^{s+1} K = A^{s+2} K \). The second equality can be deduced post-multiplying both sides by \( K \).

(b) From (a) and the definition, we have \( A^{s+2} = KA^{s+2} K = KAA^{s+1} K = KAK = (KA)^2 \). The other equality in (b) can be similarly deduced.

(c) By definition we have \( A^{(s+1)^2} = A^{(s+1)^2 + 1} = (KA) K^{s+1} = KA^{s+1} K = A \).

(d) Using Property (c) we get
\[
(A^{(s+1)^2 - 1})^2 = A^{(s+1)^2} A^{(s+1)^2 - 2} = AA^{(s+1)^2 - 2} = A^{(s+1)^2 - 1},
\]
and now Property (d) can be easily shown by induction.

(e) From (c) we get \( (A^{s+2})^{s+1} = (A^{s+1} A)^{s+1} = (s+1)^{s+1} A^{s+1} = A^{(s+1)^2} A^{s+1} = AA^{s+1} K = A^{s+2} \).

(f) From (b) and (c) the equalities \( (KA)^{2s+1} = KA(KA)^{2s} = KA(A^{s+2})^s = KA^{s+2} = KA \) hold, and in a similar way it can be shown the equality \( (AK)^{2s+1} = AK \).

(g) We proceed by recurrence. In fact, by definition we have
\[
(2.1) \quad KAK = A^{s+1}.
\]
Then Equality \((2.1)\) yields \( KA^2 K = KAA K = A^{s+1} K A^{s+1} = A^{2(s+1)} \). Following a similar reasoning it can be proven that \( KA^j K = A^{j(s+1)} \) for all \( j \in \{1, 2, \ldots, s\} \). Now, by using the definition and \( A^{(s+1)^2} = A \) we get the property for \( j = s + 1 \) as follows: \( KA^{s+1} K = A = A^{(s+1)^2} = A^{(s+1)(s+1)} \). From now on, following a similar reasoning as before it can be proven that \( KA^j K = A^{j(s+1)} \) for all \( j \in \{1, 2, \ldots, (s + 1)^2 - 1\} \), and the other equality in (g) is easily obtained from \( K^2 = I_n \).

(h) Using (g) one has
\[
(KA^j)^2 = (A^{j(s+1)})^2 = A^{j(s+1)} (KA^{j(s+1)} K) = A^{j(s+1)} A^j = A^{j(s+2)}.
\]
A set of potent matrix, Lemma 2.1 allows us to construct a group containing a cyclic subgroup.

KA is a group with respect to the matrix product satisfying the following properties:

\( \text{if } k \neq j(s+1)[\text{mod } ((s+1)^2-1)] \).

\( KA^n KA^{k} = A^{j(s+1)+k} \). The right hand side is \( A^{(s+1)^2-1} \) if \( k = -j(s+1)[\text{mod } ((s+1)^2-1)] \).

Case 1. \( s \) is even. Using Properties (g) and (c), we get

\[
KKA^{s+1}K = A^j A^{(s+1)+j} A_{(s+1)} K = A^{j(s+1)+j} K = A^{(s+1)+j} K A^{(s+1)+j}
\]

\[
= K A^{(s+1)\left[(s+1)+j\right]} K A^{(s+1)\left[(s+1)+j\right]}
\]

\[
= K (A^{(s+1)^2})^{j} A^{\left((s+1)^2\right)} K A^{(s+1)^2}
\]

Case 2. \( s \) is odd. Using Property (g), we get

\[
KKA^{s+1}K = A^j A^{(s+1)+j} A_{(s+1)} K = A^{j(s+1)+j} K A^{j(s+1)+j}
\]

\[
= K (A^{(s+1)^2})^{j} A^{\left((s+1)^2\right)} K A^{j(s+1)^2}
\]

3. Construction of a matrix group. Firstly, we note that, from a \( \{K, s+1\} \)-potent matrix, Lemma 2.1 allows us to construct a group containing a cyclic subgroup of \( \{K, s+1\} \)-potent matrices. Throughout this section we assume that \( s \geq 1 \).

THEOREM 3.1. Let \( A \in \mathbb{C}^{n \times n} \) be a \( \{K, s+1\} \)-potent matrix. If \( A^i \neq A^j \) for all distinct \( i, j \in \{1, 2, \ldots, (s+1)^2-1\} \) then the set

\[ G = \{ A, A^2, A^3, \ldots, A^{(s+1)^2-1}, KA, KA^2, KA^3, \ldots, KA^{(s+1)^2-1} \} \]

is a group with respect to the matrix product satisfying the following properties:

(a) \( A \) is an element of order \( (s+1)^2-1 \), and then the set

\[
S_A = \{ A, A^2, A^3, \ldots, A^{(s+1)^2-1} \}
\]

is a cyclic subgroup of \( G \).

(b) \( KA^n \) and \( KA^{(s+1)^2-1} \) are elements of order 2 of \( G \).

(c) \( (KA^n)A(KA^n) = A^{n+1} \).

(d) The set \( S_A \) is a normal subgroup of \( G \) and all its elements are \( \{K, s+1\} \)-potent matrices.

(e) The order of \( G \) is:

- \( (s+1)^2-1 \) if \( KA = A^j \) for some \( j \in \{1, 2, \ldots, (s+1)^2-1\} \) and, in this case, the group \( G \) is commutative.

- \( 2((s+1)^2-2) \) if \( KA \neq A^j \) for all \( j \in \{1, 2, \ldots, (s+1)^2-1\} \) and, in this case, the group \( G \) is noncommutative.

(f) For every \( j \in \{1, 2, \ldots, (s+1)^2-1\} \), the element \( KA^j \) of the set \( G \setminus S_A \) (when it is nonempty) is \( \{K, s+1\} \)-potent if and only if \( s \) is even and one of the following conditions \( \{4|s, \frac{s}{2}+1|\} \) or \( \{4|s, s+2| \} \) holds.
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**Proof.** From the properties given in Lemma 2.1 it can be checked that \( A^{(s+1)^2-1} \)

is the identity element of the group \( G \).

(a) \( G \) contains clearly a cyclic subgroup generated by the element \( A \) of order 

\((s + 1)^2 - 1\).

(b) Using Property (h) of Lemma 2.1 one has \( (KA^s)^2 = A^{(s+2)} = A^{(s+1)^2-1} \).

Similarly, \( (KA^{(s+1)^2-1})^2 = A^{(s+1)^2-1} \).

(c) Using the definition we get \( (KA^s)A(KA^s) = KA^{s+1}KA^s = AA^s = A^{s+1} \).

(d) The set \( S_A \) is a subgroup of \( G \) of index 2. Then it is normal. As a direct consequence of Property (f) of Lemma 2.1 we obtain the second part of Property (d) since \( A^2 \) are \( \{K, s + 1\} \)-potent matrices for all \( j \in \{1, 2, \ldots, (s + 1)^2 - 1\} \).

(e) Case 1. There exists \( j \in \{1, 2, \ldots, (s + 1)^2 - 1\} \) such that \( KA = A^j \). Then 

\( KA^i = A^{j+i} \), that is, it is also one of the \( (s + 1)^2 - 1 \) powers of \( A \) indicated in (3.1) for all \( i \in \{1, 2, \ldots, (s + 1)^2 - 1\} \). Thus, the order of \( G \) is \( (s + 1)^2 - 1 \) and then \( G \) is commutative.

Case 2. For every \( j \in \{1, 2, \ldots, (s + 1)^2 - 1\} \) one has \( KA \neq A^j \). It is clear that the order of \( G \) is \( 2((s + 1)^2 - 1) \) because if \( i, j \in \{1, 2, \ldots, (s + 1)^2 - 1\} \) such that \( KA^i = KA^j \) with \( i \neq j \) exist, then \( A^i = K^2A^i = K^2A^j = A^j \), which is impossible. Assume that \( G \) is commutative. Then \( (KA)(KA^{s+1}) = (KA^{s+1})(KA) \), that implies \( A^{2s+1} = A \). We obtain a contradiction.

(f) Assume that \( G \setminus S_A \neq \emptyset \) and \( KA^j \in G \setminus S_A \). Then the fact that \( KA^j \) is 

\( \{K, s + 1\} \)-potent implies that \( s \) is even, by Property (k) of Lemma 2.1 This same property assures that, in this case, \( (KA^j)^{s+1}K = KA^{j(s+4)+j} \). Then,

\( KA^j \) is \( \{K, s + 1\} \) - potent \( \iff \) \( KA^{j(s+4)+j} = KA^j \iff A^j \tilde{A}^{(s+4)+j} = A^j \)

\( \iff A^j \tilde{A}^{(s+4)+j}A^{(s+1)^2-1-j} = A^{(s+1)^2-1} \iff A^j \tilde{A}^{(s+4)} = A^{(s+1)^2-1} \)

and so, this is equivalent to the statement \( (s+1)^2-1 = s(s+2) \) divides \( j\tilde{A}^{(s+4)} \). Now, it can be checked that \( s(s+2)|j\tilde{A}^{(s+4)} \) is equivalent to \( \{4|s, \tilde{A}+1|j\} \) or \( \{4/j, s, s+2|j\} \).

In the first case, there are 2s such \( j \) and in the second one, there are \( s \) such \( j \).

**Corollary 3.2.** The group \( G \) is a semidirect product of \( \mathbb{Z}_2 \) acting on \( \mathbb{Z}_{(s+1)^2-1} \) when \( G \) has order \( 2((s + 1)^2 - 1) \).

**Proof.** We consider a semidirect product of \( \mathbb{Z}_2 \) acting on \( \mathbb{Z}_r \). Then its presentation is in the form \( \langle a, b | a^2 = e, b^r = e, aba = b^m \rangle \) where \( m, r \) are coprime. Here \( r = (s + 1)^2 - 1, a = KA^s, b = A, m = s + 1 \). [4]

Now, we need the following definition.
Definition 3.3 ([L]).

(i) The dihedral group, denoted by \( D_m \), is generated by an element \( a \) of order \( m \) and another element \( b \) of order 2 such that \( b^{-1}ab = a^{-1} \). It has order \( 2m \).

(ii) The quasi-dihedral group, denoted by \( Q_{2m} \) for \( m \geq 4 \), is generated by an element \( a \) of order \( 2^{m-1} \) and another element \( b \) of order 2 such that \( b^{-1}ab = a^{2^{m-2}-1} \). It has order \( 2^m \).

We now classify the group \( G \) defined in Theorem 3.1. It is clear that \( G \) depends on \( A, K, \) and \( s \). That is why, for the following result, we will denote \( G \) by \( G_s \) and so we stress on the parameter \( s \). We shall see a relation between the group \( G_s \) and dihedral and quasi-dihedral groups in the following result.

Proposition 3.4. Assume that \( G_s \) has order \( 2((s+1)^2 - 1) \). One has

(a) \( G_s \) is a dihedral group if and only if \( s = 1 \). In this case, \( G_1 \simeq D_3 \).

(b) Let \( s > 1 \). Then \( G_s \) is a quasi-dihedral group if and only if \( s = 2 \). In this case, \( G_2 \simeq Q_{16} \).

Proof. (a) If \( s = 1 \), the proof for case (a) can be deduced from the fact that \( G_s \) is generated by the element \( A \) of order 3, the element \( KA \) of order 2, and furthermore, the relation \((KA)A(KA) = A^2 \) holds, where \( A^2 \) is the inverse of \( A \).

Let \( s > 1 \). In this case, it is known that the group \( G_s \) has an element \( A \) of order \( (s+1)^2 - 1 \). On the other hand, we check that for any \( \alpha \in \{1, 2, \ldots, (s+1)^2 - 1\} \) the element \( KA^\alpha \) has order 2 if and only if \( \alpha \) is a multiple of \( s \). Indeed,

\[
A^{(s+1)^2 - 1} = (KA^\alpha)^2 \iff A^{s(s+2)} = KA^\alpha K A^\alpha = A^{\alpha(s+1)} A^\alpha = A^{\alpha(s+2)}
\]

\[
\iff s(s+2) \text{ divides } \alpha(s+2) \iff s \text{ divides } \alpha,
\]

and hence all the elements of order 2 of \( G_s \) have the form \( KA^{st} \). However, for any element of the form \( KA^{st} \), where

\[
t \in \left\{ \frac{1}{s}, \frac{2}{s}, \ldots, \frac{(s+1)^2 - 1}{s} \right\} \cap \mathbb{N} = \{1, 2, 3, \ldots, s + 2\},
\]

the equality \((KA^{st})A(KA^{st}) = A^{(s+1)^2 - 2} \) is not satisfied since

\[
(KA^{st})A(KA^{st}) = (KA^{st+1}K)A^{st} = A^{((s+1)^2 - 1)t + (s+1)} = (A^{(s+1)^2 - 1})^t A^{s+1} = A^{s+1}.
\]

Finally \( s > 1 \) implies that \( s + 1 < (s+1)^2 - 2 \), a contradiction.

(b) It is necessary to take into account the fact that the properties of \( G_s \) for \( s = 2 \) coincide with those of \( Q_{2m} \) for \( m = 4 \). In fact, if \( s = 2 \) then \( G_s \) has an element \( A \) of order 8 = \( 2^{4-1} \), another element \( KA^2 \) of order 2 and, moreover, by Property (c)
of Theorem 3.1 we have \((KA^2)A(KA^2) = A^3 = A^{2^k-1}\) and therefore \(G_2 \cong Q_{2^n}\). Conversely, if \(G_s \cong Q_{2^m}\) then we have \(2^{m-1} = (s+1)^2 - 1 = s(s+2)\). Thus \(s\) is a power of 2 and \(s = 2, m = 4\). 

### 4. What about the group when \(A^k = A\) for \(k < (s+1)^2\)?

We can ask: what would happen if a power of \(A\) less than \((s+1)^2\) and such that \(A^k = A\) existed?

If \(A \in \mathbb{C}^{n \times n}\) is a \(\{K, s+1\}\)-potent matrix then Property \((b)\) of Lemma 2.1 allows us to construct, by Theorem 3.1, the group \(G\) considering the subgroup \(S_A\) of order \((s+1)^2 - 1\) when all the powers of \(A\) are different. But, it may occur that there exists an integer \(k\) such that \(A^k = A\) with \(2 \leq k < (s+1)^2\). In this case, it is also possible to consider the group \(G_{s,k} = \{A, A^2, \ldots, A^{k-1}, KA, KA^2, \ldots, KA^{k-1}\}\) associated to the matrix \(A\). Therefore, the subset \(S_A^k = \{A, A^2, \ldots, A^{k-1}\}\) is a (cyclic) subgroup of the group \(S_A\) and then \(k - 1\) has to divide \((s+1)^2 - 1 = s(s+2)\).

How many groups \(G_{s,k}\) can we construct in this way? One only: the group corresponding to the smallest power \(k\) such that \(A^k = A\) (otherwise, we obtain exactly the same group \(G_{s,k}\)). Consequently, the only possibilities for the order of the group are: \((s+1)^2 - 1, 2((s+1)^2 - 1), k - 1\) or \(2(k - 1)\) (if such \(k\) exists).

For some values of \(s\) and \(k\), more specifications on the group are given in the following result. In order to analyze these special cases we recall the following definition.

**Definition 4.1.** \([4]\) The quaternion group, denoted by \(Q\), is generated by three elements \(a, b, c\) of order 4 such that \(a^2 = b^2 = c^2\) and \(bab^{-1} = a^{-1}\). It has order 8.

**Proposition 4.2.** Assuming that \(G_{s,k}\) has order \(2(k - 1)\), the following statements hold.

\(a\) Let \(s = 1\). Then \(k = 2\). In this case, \(G_{1,2} \cong \mathbb{Z}_2\).

\(b\) Let \(s = 2\). Then one of the following statements hold:

- \(i\) \(k = 2\). In this case, \(G_{2,2} \cong \mathbb{Z}_2\).
- \(ii\) \(k = 3\). In this case, \(G_{2,3} \cong D_2\).
- \(iii\) \(k = 5\). In this case, \(G_{2,3} \cong D_4\) or \(G_{2,5} \cong Q\).

\(c\) Let \(s > 2\). Then

- \(i\) \(G_{s,s+1} \cong \mathbb{Z}_{2s}\) when \(s\) is prime.
- \(ii\) \(G_{s,s+3} \cong D_{s+2}\).

**Proof.** (a) Let \(s = 1\). If \(k\) is an integer such that \(2 \leq k \leq 3\) and \(A^k = A\), it must be \(k = 2\) because \(k - 1\) divides 3. Thus, \(KAK = A^2 = A\) and then the group \(G_{1,2} = \{KA, (KA)^2 = A\}\) is generated by the element \(KA\) of order 2. Hence, \(G_{1,2} \cong \mathbb{Z}_2\). 

(b) Let $s = 2$. If $k$ is an integer such that $2 \leq k \leq 8$ and $A^k = A$, it must be $k = 2$, $k = 3$ or $k = 5$ because $k - 1$ divides 8. We now analyze these three cases:

- $k = 2$: the same reasoning as in (a).
- $k = 3$: in this case, $G_{2,3} = \{A, A^2, KA, KA^2\}$ where $A$ and $KA^2$ have order 2 and $KA^2 A K A^2 = A$. Hence, $G_{2,3} \simeq D_2$.
- $k = 5$: in this case, $G_{2,5} = \{A, A^2, A^3, A^4, KA, KA^2, KA^3, KA^4\}$ is a non-commutative group (for instance, $A(KA^2) \neq (KA^2)A$). Then, Proposition 6.3 in [4] assures that $G_{2,5} \simeq D_4$ or $G_{2,5} \simeq Q$.

(c) Let $s > 2$. If $k$ is an integer such that $2 \leq k \leq s(s + 2)$ and $A^k = A$, we get $k = s + 1$ or $k = s + 3$ as particular values of $k$ because $k - 1$ divides $s(s + 2)$. Now, we analyze these two cases:

- $k = s + 1$: in this case, $G_{s,s+1} = \{A, A^2, \ldots, A^s, KA, KA^2, \ldots, KA^s\}$ is a commutative group of order $2s$. Then, Corollary 6.2 in [4] assures that $G_{s,s+1} \simeq \mathbb{Z}_{2s}$.
- $k = s + 3$: in this case, $G_{s,s+3} = \{A, A^2, \ldots, A^{s+2}, KA, KA^2, \ldots, KA^{s+2}\}$ is a non-commutative group (for example, $A(KA^{s+1}) \neq (KA^{s+1})A$). $A$ is of order $s + 2$, $KA^s$ of order 2 and $KA^s AKA^s = A^{s+1}$. Hence, $G_{s,s+3} \simeq D_{s+2}$.

**Remark 4.3.** If $G_{s,k}$ has order $2(k - 1)$ then $G_{s,k}$ is a semidirect product of $\mathbb{Z}_2$ acting on $\mathbb{Z}_{k-1}$, when $k - 1, s + 1$ are coprime. Its proof is similar to the proof of Corollary 6.2 where $a = KA^{k-1}, b = A, r = k - 1, m = s + 1$. The Property of Lemma 2.1 allows us to show that $KA^{k-1}$ has order 2. In fact, $a^2 = (KA^{k-1})^2 = A^{(k-1)(s+2)} = A^{k-1} = e$. Moreover, $aba = (KA^{k-1})A(KA^{k-1}) = KA^k KA^{k-1} = A^{s+1} A^{k-1} = A^{s+1} = b^m$.

**Remark 4.4.** We observe that $G_{2,5}$ is isomorphic to $D_4$ or $Q$ because there are (up to isomorphism) exactly two distinct non-commutative groups of order 8. We will see, in Example 2, that these two possibilities can be realized.

**Remark 4.5.** We observe that if $s > 2$ and $k = s + 1$, then $A$ satisfies $KA = AK$ since $A^{s+1} = A$. Such a matrix is said to be $\{K\}$-centrosymmetric [9]. In this case, the group associated to such a matrix $A$ has order 2s, commutative and then isomorphic to $\mathbb{Z}_{2s}$ if $s$ is prime.

We close this section with the following remark.

**Remark 4.6.** (The case $s = 0$). It corresponds to $\{K\}$-centrosymmetric matrices. It is sometimes possible to construct a similar group as before. Observe that the condition $A^{(s+1)t} = A$ does not give any information for $s = 0$. So, if we assume that $A^t = A$ for some positive integer $t$ (where $A^t \neq A^m$ for all $l, m < t, l \neq m$),
then $G = \{A, A^2, \ldots, A^{t-1}, KA, KA^2, \ldots, KA^{t-1}\}$ is a group with the same features mentioned before.

However, when the assumption is not fulfilled, since the powers of $A$ cannot be the identity element, the group does not exist. An example will clarify this situation. For the matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

it is impossible to construct such a group because

$$A^{m+1} = \begin{bmatrix} 2^m & 0 & 2^m \\ 0 & 1 & 0 \\ 2^m & 0 & 2^m \end{bmatrix} \quad \text{for every } m \geq 1.$$

5. Examples. Now we present some more examples illustrating the results we have obtained.

**Example 5.1.** Let

$$A = \begin{bmatrix} 0 & 0 & -i \\ i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. $$

$A$ is $\{K, 2\}$-potent as the authors showed in [39]. $G = \{A, A^2, A^3, KA, KA^2, KA^3\}$ is a group of order 6 because $A^3 = I_3$ and, in this case, $G \simeq D_3$ by Proposition 3.4.

**Example 5.2.**

(1) Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. $$

$A$ is $\{K, 3\}$-potent and $A^2 = -I_2$. This example shows a matrix of the class considered in Section 4. For this matrix, $A^5 = A$ holds and so it is possible to construct a group with similar features as in Proposition 3.2. Since $s = 2$, $k = 5$, $G_{2,5}$ has order 4 or 8. In this example, the group is $G_{2,5} = \{\pm I_2, \pm A, \pm K, \pm KA\}$ and, in this case, $A$ is an element of order 4, $KA^2 = -K$ is an element of order 2, and $(KA^2)A(KA^2) = A^3$. So $G_{2,5}$ is isomorphic to $D_4$.

(2) Let $A$ be a $\{K, 3\}$-potent matrix given by

$$A = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1+i & 1-i \\ 0 & 0 & 1-i & 1+i \\ 1+i & 1-i & 0 & 0 \\ 1-i & 1+i & 0 & 0 \end{bmatrix} \quad \text{and} \quad K = \frac{1}{8} \begin{bmatrix} 9 & -1 & 3 & -3 \\ -1 & 9 & -3 & 3 \\ -3 & 3 & -1 & 9 \\ 3 & -3 & 9 & -1 \end{bmatrix}. $$
$A^4 = I_4$ and the associated matrix group (of order 8) is

$$G_{2,5} = \{A, A^2, A^3, I_4, KA, KA^2, KA^3, K\}.$$ 

$G_{2,5}$ is generated by the three elements $A^2, KA, KA^3$ of order 4 such that $(A^2)^2 = (KA)^2 = (KA^3)^2$ and $(KA)(A^2)(KA^3) = A^2$. So $G_{2,5}$ is isomorphic to $Q$.

**Example 5.3.** Let

$$A = \begin{bmatrix} \sqrt{5} & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & \sqrt{5} \\ 0 & -1 \end{bmatrix}.$$ 

$A$ is $\{K, 4\}$-potent and $A^5 = I_2$. The associated group (of order 10) is

$$G = \{A, A^2, A^3, A^4, I_2, KA, KA^2, KA^3, KA^4, K\}.$$ 

$G$ is generated by $A$ of order 5, $KA^2$ of order 2, and $(KA^2)A(KA^2) = A^4$. Then $G$ is isomorphic to $D_5$.

**Example 5.4.** Let $A$ be a $\{K, 5\}$-potent matrix given by

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}.$$ 

The associated group (of order 6) is $G = \{A, A^2, I_2, KA, KA^2, K\}$ since $A^3 = I_2$. As $G$ is generated by $A$ of order 3, $KA^2$ of order 2 and $(KA^2)A(KA^2) = A^2$, then $G$ is isomorphic to $D_3$.

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REFERENCES

Properties of a matrix group associated to a \( \{K, s + 1\} \)-potent matrix


