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CHANGE OF THE *CONGRUENCE CANONICAL FORM OF 2-BY-2 MATRICES UNDER PERTURBATIONS

VYACHESLAV FUTORNY†, LENA KLIMENKO‡, AND VLADIMIR V. SERGEICHUK§

Abstract. It is constructed the Hasse diagram for the closure ordering on the sets of *congruence classes of $2 \times 2$ matrices. In other words, it is constructed the directed graph whose vertices are $2 \times 2$ canonical complex matrices for *congruence and there is a directed path from $A$ to $B$ if and only if $A$ can be transformed by an arbitrarily small perturbation to a matrix that is *congruent to $B$.

Key words. Closure graph, *Congruence canonical form, Perturbations.

AMS subject classifications. 15A21, 15A63, 47A07.

1. Introduction. We study how arbitrarily small perturbations of a $2 \times 2$ complex matrix can change its *canonical form for *congruence (matrices $A$ and $B$ are *congruent if $S^*AS = B$ for a nonsingular $S$). We construct the closure graph $G_2$, which is defined for any natural $n$ as follows.

Definition 1.1. The closure graph $G_n$ for *congruence classes of $n \times n$ complex matrices is the directed graph, in which each vertex $v$ represents in a one-to-one manner a *congruence class $C_v$ of $n \times n$ matrices, and there is a directed path from a vertex $v$ to a vertex $w$ if and only if one (and hence each) matrix from $C_v$ can be transformed to a matrix form $C_w$ by an arbitrarily small perturbation.

The graph $G_n$ is the Hasse diagram of the *congruence classes of $n \times n$ matrices with the following partial order: $a \succeq b$ means that $a$ is contained in the closure of $b$. Thus, the graph $G_n$ shows how the *congruence classes relate to each other in the affine space of $n \times n$ matrices.

Since each $n \times n$ matrix is uniquely represented in the form $P + iQ$ in which $P$ and $Q$ are Hermitian matrices, $G_n$ is also the closure graph for *congruence classes.

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of Hermitian matrix pencils $P + \lambda Q$.

Note that the closure graph $G_2$ for *congruence, which we construct in Theorem 2.2, is more complicated than the closure graphs for congruence classes of 2-by-2 and 3-by-3 matrices, which were constructed by the authors in [4], since an arrow between *congruence classes in $G_2$ may depend on the parameters of their matrices.

Unlike perturbations of matrices under congruence and *congruence, perturbations of matrices under similarity and of matrix pencils have been much studied. For a given matrix $A$, den Boer and Thijssse [3] and, independently, Markus and Parilis [17] described the set of all Jordan canonical matrices such that for each $J$ from this set there exists a matrix that is arbitrarily close to $A$ and is similar to $J$. Their description was extended to Kronecker’s canonical forms of pencils by Pokrzywa [18]. Edelman, Elmroth, and Kågström [7] developed a comprehensive theory of closure relations for similarity classes of matrices, for equivalence classes of matrix pencils, and for their bundles. The software StratiGraph [8] constructs their closure graphs. The closure graph for $2 \times 3$ matrix pencils was constructed and studied by Elmroth and Kågström [9].

The term “*congruence orbit” is often used instead of “*congruence class” (see De Terán and Dopico [2]). The problem that we consider can be called “the stratification of orbits of matrices under *congruence” by analogy with the stratification of orbits of matrices under similarity and of matrix pencils [7, 8, 15]. An informal introduction to perturbations of matrices determined up to similarity, congruence, or *congruence is given by Klimenko and Sergeichuk [16].

All matrices that we consider are complex matrices.

2. The closure graph for *congruence classes of 2-by-2 matrices. Define the $n$-by-$n$ matrices:

$$J_n(\lambda) := \begin{bmatrix} \lambda & 1 & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & \lambda \end{bmatrix}, \quad \Delta_n := \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}.$$

We use the following canonical form for *congruence.

**Proposition 2.1** ([10, Theorem 4.5.21]). *Each square complex matrix is *congruent to a direct sum, uniquely determined up to permutation of summands, of matrices of the form*

$$
\begin{pmatrix}
0 & I_m \\
J_m(\lambda) & 0 \\
\end{pmatrix}
\quad (0 \neq \lambda \in \mathbb{C}, \ |\lambda| < 1), \quad \mu \Delta_n \quad (\mu \in \mathbb{C}, \ |\mu| = 1), \quad J_k(0).
$$
This canonical form obtained in [11] was based on [21, Theorem 3] and was generalized to other fields in [14]. A direct proof that this form is canonical is given in [12, 13].

The vertices of $G_n$ can be identified with the $n \times n$ canonical matrices for *congruence since each *congruence class contains exactly one canonical matrix.

For each $A \in \mathbb{C}^{n \times n}$ and a small matrix $X \in \mathbb{C}^{n \times n}$,

$$(I + X)^*A(I + X) = A + X^*A + AX + \underbrace{X^*AX}_{\text{very small}}$$

and so the *congruence class of $A$ in a small neighborhood of $A$ can be obtained by a very small deformation of the real affine matrix space $\{A + X^*A + AX | X \in \mathbb{C}^{n \times n}\}$. (By the local Lipschitz property [20], if $A$ and $B$ are close to each other and $B = S^*AS$ with a nonsingular $S$, then $S$ can be taken near $I_n$.) The real vector space

$$T(A) := \{X^*A + AX | X \in \mathbb{C}^{n \times n}\}$$

is the tangent space to the *congruence class of $A$ at the point $A$. The numbers

$$(2.2) \quad \dim \mathbb{R} T(A), \quad \text{codim} \mathbb{R} T(A) := 2n^2 - \dim \mathbb{R} T(A)$$

are called the dimension and, respectively, codimension over $\mathbb{R}$ of the *congruence class of $A$.

The following theorem proved in Section 3 is the main result of the paper.

**Theorem 2.2.** The closure graph for *congruence classes of $2 \times 2$ matrices is

$$(2.3)$$

$$\begin{bmatrix} \mu & 0 \\ 0 & \nu \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ \sigma & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \tau \\ \tau & \tau^2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \tau \end{bmatrix}, \quad \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}, \quad \begin{bmatrix} \lambda \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
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in which \( \lambda, \mu, \nu, \sigma, \tau \in \mathbb{C}, \mathbb{R}_+ \) denotes the set of nonnegative real numbers, and \( \text{Im}(c) \) denotes the imaginary part of \( c \in \mathbb{C} \). Each *congruence class is given by its canonical matrix, which is a direct sum of blocks of the form (2.1). The graph is infinite: Each vertex except for \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) represents an infinite set of vertices indexed by the parameters of the corresponding canonical matrix. The *congruence classes of canonical matrices that are located at the same horizontal level in (2.3) have the same dimension over \( \mathbb{R} \), which is indicated to the right.

The arrow \( \begin{bmatrix} \lambda & 0 \\ 0 & \nu \end{bmatrix} \to \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \) exists if and only if \( \lambda = \mu a + \nu b \) for some nonnegative \( a, b \in \mathbb{R} \). The arrow \( \begin{bmatrix} \lambda & 0 \\ 0 & \nu \end{bmatrix} \to \begin{bmatrix} 0 & \tau \\ \tau & \tau \end{bmatrix} \) exists if and only if the imaginary part of \( \lambda \tau \) is nonnegative. The arrow \( \begin{bmatrix} \lambda & 0 \\ 0 & \nu \end{bmatrix} \to \begin{bmatrix} \lambda & 0 \\ 0 & -\nu \end{bmatrix} \) exists if and only if \( \tau = ±\lambda \). The arrows \( \begin{bmatrix} \lambda & 0 \\ 0 & \nu \end{bmatrix} \to \begin{bmatrix} \lambda & 0 \\ 0 & \pm \lambda \end{bmatrix} \) exist if and only if the value of \( \lambda \) is the same in both matrices. The other arrows exist for all values of parameters of their matrices.

**Remark 2.3.** Let \( M \) be a 2 \( \times \) 2 canonical matrix for *congruence.

- Let \( N \) be another 2 \( \times \) 2 canonical matrix for *congruence. Each neighborhood of \( M \) contains a matrix whose *congruence canonical form is \( N \) if and only if there is a directed path from \( M \) to \( N \) in (2.3) (if \( M = N \), then there is the “lazy” path of length 0 from \( M \) to \( N \)).

- The closure of the *congruence class of \( M \) is equal to the union of the *congruence classes of all canonical matrices \( N \) such that there is a directed path from \( N \) to \( M \) (if \( M = N \) then the “lazy” path exists).

**Remark 2.4.** It is not surprising that \( \text{diag}(\lambda, ±\lambda) \) and \( \text{diag}(\mu, \nu) \) \( (|\lambda| = |\mu| = |\nu| = 1 \text{ and } \mu ≠ ±\nu) \) have different behavior under perturbation: many properties of a nonsingular matrix \( A \) with respect to *congruence are determined by its *cosquare \( (A^*)^{-1}A \) (see [13, 14, 19]), the *cosquare of \( \text{diag}(\lambda, ±\lambda) \) has a multiple eigenvalue, and the *cosquare of \( \text{diag}(\mu, \nu) \) has two distinct eigenvalues.

### 3. Proof of Theorem 2.2

The following lemma is a weak form of [6, Example 2.1] (which is a special case of [6, Theorem 2.2] about \( n \times n \) matrices).

**Lemma 3.1.** Let \( A \) be any 2 \( \times \) 2 matrix. Then all matrices \( A + X \) that are sufficiently close to \( A \) can be simultaneously reduced by some transformation

\[
S(X)^*(A + X)S(X),
\]

\( S(X) \) is nonsingular and continuous on a neighborhood of zero,
to one of the following forms:

\[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} \oplus \begin{bmatrix}
* & * \\
* & *
\end{bmatrix}, \quad
\begin{bmatrix}
\lambda & 0 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
\varepsilon & 0 \\
0 & \delta
\end{bmatrix} (|\lambda| = 1), \quad
\begin{bmatrix}
\lambda & 0 \\
0 & \mu
\end{bmatrix} + \begin{bmatrix}
\varepsilon & 0 \\
0 & \delta
\end{bmatrix} (\lambda \pm \mu), \quad
\begin{bmatrix}
0 & 1 \\
\lambda & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
* & *
\end{bmatrix} (|\lambda| < 1), \quad
\begin{bmatrix}
0 & \lambda \\
\lambda & 0
\end{bmatrix} + \begin{bmatrix}
* & 0 \\
0 & *
\end{bmatrix} (|\lambda| = 1).
\]

Each of these matrices has the form $A_{\text{can}} + D$, in which $A_{\text{can}}$ is a direct sum of blocks of the form (2.1), the *'s in $D$ are complex numbers, all $\varepsilon, \delta, \delta_\mu$ are either real numbers if $\lambda, \mu \in \mathbb{R}$ or pure imaginary numbers if $\lambda, \mu \in \mathbb{C}$. (Clearly, $D$ tends to zero as $X$ tends to zero.) For each $A_{\text{can}} + D$, twice the number of its stars plus the number of its entries of the form $\varepsilon, \delta, \delta_\mu$ is equal to the codimension over $\mathbb{R}$ (defined in (2.2)) of the *congruence class of $A_{\text{can}}$.

Note that the codimensions of congruence and *congruence classes were calculated in [1] [5] and [2] [6], respectively.

By [22] Part III, Theorem 1.7, the boundary of each *congruence class is a union of *congruence classes of strictly lower dimension, which ensures the following lemma.

**Lemma 3.2.** If $M \to N$ is an arrow in the closure graph $G_2$, then the *congruence class $C_M$ of $M$ is contained in the closure of the *congruence class $C_N$ of $N$, and so the dimension of $C_M$ is lower than the dimension of $C_N$.

For each vertex $M$ in $G_2$, the dimension $d_M$ over $\mathbb{R}$ of the *congruence class of $M$ is indicated in (2.6). It was calculated as follows: By [22], $d_M = 8 - c_M$ in which $c_M$ is the codimension of the *congruence class of $M$; $c_M$ was taken from Lemma 3.1.

The proof of Theorem 2.2 is divided into two steps.

**Step 1:** Let us prove that each arrow in (2.3) is correct. To make sure that an arrow $M \to N$ is correct, we need to prove that the canonical matrix $M$ can be transformed by an arbitrarily small perturbation to a matrix whose *congruence canonical form is $N$. Consider each of the arrows of (2.3).

- The arrows $\begin{bmatrix}
0 & 0 \\
0 & \nu
\end{bmatrix} \to \begin{bmatrix}
0 & 0 \\
0 & \nu
\end{bmatrix}$, $\begin{bmatrix}
\lambda & 0 \\
0 & 0
\end{bmatrix} \to \begin{bmatrix}
\lambda & 0 \\
0 & 0
\end{bmatrix}$, and $\begin{bmatrix}
0 & \tau \\
0 & 0
\end{bmatrix} \to \begin{bmatrix}
\tau & 0 \\
0 & \tau
\end{bmatrix}$ are correct.

Let $A := \begin{bmatrix}
\lambda & 0 \\
0 & \nu
\end{bmatrix}$, or $\begin{bmatrix}
\lambda & 0 \\
0 & \nu
\end{bmatrix}$. Then $A$ is *congruent to $\varepsilon A$, in which $\varepsilon$ is any positive real number, and each neighborhood of $\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}$ contains $\varepsilon A$ with a sufficiently small $\varepsilon$.

- The arrow $\begin{bmatrix}
0 & 0 \\
0 & \nu
\end{bmatrix} \to \begin{bmatrix}
0 & 0 \\
\mu & 0
\end{bmatrix}$ (with given $\lambda, \mu, \nu \in \mathbb{C}$ such that $|\lambda| = |\mu| = |\nu| = 1$)
exists if and only if \( \lambda \in \mu \mathbb{R}_+ + \nu \mathbb{R}_+ = \{\mu a + \nu b | a, b \in \mathbb{R}, a \geq 0, b \geq 0\} \) (in particular, [\( \begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix} \) \( \rightarrow \) [\( \begin{smallmatrix} \sigma & 0 \\ 0 & 0 \end{smallmatrix} \)] and [\( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \) \( \rightarrow \) [\( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \)] exist).

The arrow [\( \begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix} \) \( \rightarrow \) [\( \begin{smallmatrix} \nu & 0 \\ 0 & 0 \end{smallmatrix} \)] exists if and only if there exists an arbitrarily small perturbation

\[
\begin{bmatrix}
\lambda & 0 \\
0 & 0 \\
\end{bmatrix} + E =
\begin{bmatrix}
\lambda + \varepsilon_{11} & \varepsilon_{12} \\
\varepsilon_{21} & \varepsilon_{22}
\end{bmatrix}
\]

of

\[
\begin{bmatrix}
\lambda & 0 \\
0 & 0 \\
\end{bmatrix},
\]

i.e.,

\[
\bar{x}x + \bar{z}z = \lambda + \varepsilon_{11} \quad \bar{y}y + \bar{t}t = \varepsilon_{12}
\]

For fixed \( \lambda, \mu, \nu \) and an arbitrarily small \( \varepsilon_{11} \), the first equation with unknowns \( x \) and \( z \) has a solution only if \( \lambda \in \mu \mathbb{R}_+ + \nu \mathbb{R}_+ \).

Conversely, let \( \lambda \in \mu \mathbb{R}_+ + \nu \mathbb{R}_+ \). Take \( \varepsilon_{11} = 0 \) and chose \( x \) and \( z \) for which the first equality in (3.2) holds. Then take arbitrarily small \( y, t \) for which \( S \) is nonsingular and get arbitrarily small \( \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22} \) for which the other equalities in (3.2) hold.

- **The arrow** [\( \begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix} \) \( \rightarrow \) [\( \begin{smallmatrix} \omega & 1 \\ 0 & 0 \end{smallmatrix} \)] \( (|\lambda| = 1, |\sigma| < 1) \) **exists** for all \( \lambda \) and \( \sigma \).

The arrow [\( \begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix} \) \( \rightarrow \) [\( \begin{smallmatrix} \omega & 0 \\ 0 & 0 \end{smallmatrix} \)] exists if and only if there exists an arbitrarily small perturbation (3.1) that is *congruent* to [\( \begin{smallmatrix} \omega & 0 \\ 0 & 0 \end{smallmatrix} \)]. This means that there exists a nonsingular \( S = [\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}] \) such that

\[
\begin{bmatrix}
x & y \\
z & t
\end{bmatrix}
\begin{bmatrix}
\omega & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x & y \\
z & t
\end{bmatrix}
= \begin{bmatrix}
\lambda & 0 \\
0 & 0
\end{bmatrix}
+ E,
\]

i.e.,

\[
\bar{x}x + \bar{z}z = \lambda + \varepsilon_{11} \quad \bar{y}y + \bar{t}t = \varepsilon_{12}
\]

Suppose that \( \bar{x}x = u + iv, \sigma = \alpha + \beta i \), and \( \lambda + \varepsilon_{11} = a + bi \), in which \( u, v, \alpha, \beta, a, b \in \mathbb{R} \). Then the first equation in (3.3) takes the form

\[
(a + \beta i)(u + vi) = a + bi,
\]

which gives the system

\[
\begin{align*}
(1 + \alpha)u - \beta v &= a \\
\beta u + (\alpha - 1)v &= b
\end{align*}
\]
with respect to the unknowns $u$ and $v$. Its determinant $\alpha^2 + \beta^2 - 1$ is nonzero since $|\sigma| < 1$. Therefore, the first equation in (3.3) holds for some $x$ and $z$. Taking arbitrarily small $y, t$ for which $S$ is nonsingular, we get arbitrarily small $\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_2$ for which the other equalities in (3.3) hold.

- The arrow $\left[ \begin{array}{c} \lambda \\ 0 \\ \lambda \end{array} \right] \rightarrow \left[ \begin{array}{c} \varphi \\ \tau \end{array} \right]$ (|\lambda| = |\tau| = 1) exists if and only if $\text{Im}(\lambda \varphi) \geq 0$.

Consider the first equation in (3.4). Since $\bar{\varphi}(\lambda + \varepsilon_1) \neq 0$, $z \neq 0$ too. Thus,

$$\text{Im}(\bar{\varphi}(\lambda + \varepsilon_1)) = \text{Im}(\bar{\varphi}(\lambda + \varepsilon_1)) = \bar{\varphi} > 0$$

and so $\text{Im}(\bar{\varphi}(\lambda)) \geq 0$.

Conversely, if $\text{Im}(\bar{\varphi}(\lambda)) \geq 0$, then we put $\varepsilon_1 = 0$ and take $x, z$ such that the first equation in (3.4) holds. Taking arbitrarily small $y, t$ for which $S$ is nonsingular, we get arbitrarily small $\varepsilon_1, \varepsilon_2, \varepsilon_2$ for which the other equalities in (3.3) hold.

- The arrow $\left[ \begin{array}{c} \lambda \\ 0 \\ \lambda \end{array} \right] \rightarrow \left[ \begin{array}{c} \varphi \\ \tau \end{array} \right]$ (|\lambda| = |\tau| = 1) exists if and only if $\lambda = \pm \tau$.

The arrow $\left[ \begin{array}{c} \lambda \\ 0 \\ \lambda \end{array} \right] \rightarrow \left[ \begin{array}{c} \varphi \\ \tau \end{array} \right]$ (|\lambda| = |\tau| = 1) exists if and only if there exists an arbitrarily small perturbation $[\delta] + E$ of $[\delta]$, such that $S^* \tau = \left[ \begin{array}{c} \lambda \\ 0 \\ \lambda \end{array} \right] + E$.

Equating the determinants of both sides, we find that $-\tau^2 \det(S^* S)$ is arbitrarily close to $-\lambda^2$. Since

$$\det(S^* S) = \det(S) \det(S)$$

is a real positive number, $|\tau^2| \det(S^* S)$ is arbitrarily close to $|\lambda^2|$. Since $|\lambda| = |\tau| = 1$, $\det(S^* S)$ is arbitrarily close to 1. Hence, $-\tau^2 = -\lambda^2$, and so $\lambda = \pm \tau$. 

\[\text{Im}(\bar{\varphi}(\lambda + \varepsilon_1)) = \text{Im}(\bar{\varphi}(\lambda + \varepsilon_1)) = \bar{\varphi} > 0\]
Conversely, let $\lambda = \pm \tau$. Since
\[
\begin{bmatrix}
1 & 1 \\
1/2 & -1/2
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 1/2 \\
1 & -1/2
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix} = 
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix},
\]
$[\lambda 0 1 0]$ is *congruent to $\pm [0 1 1]$. Its arbitrarily small perturbation $\pm [\lambda 0 1 0] \ (\varepsilon \in \mathbb{R}, \varepsilon > 0)$ is *congruent to $\pm [0 1 1]$ via $\text{diag}(\sqrt{\varepsilon}, 1/\sqrt{\varepsilon})$. Therefore, $[\lambda 0 1 0] \rightarrow [\lambda 0 1 1]$, and so $[\lambda 0 1 0] \rightarrow [0 1 1]$. 

**Step 2:** Let us prove that we have not missed arrows in (2.3). We write $M \rightarrow N$ if the closure graph $G_2$ does not have the arrow $M \rightarrow N$; i.e., if each matrix obtained from $M$ by an arbitrarily small perturbation is not *congruent to $N$. Lemma 3.2 ensures that we need to prove only the absence of the arrows
\[
\begin{bmatrix}
\lambda & 0 \\
0 & \pm \lambda
\end{bmatrix} \rightarrow 
\begin{bmatrix}
\mu & 0 \\
0 & \nu
\end{bmatrix}, \quad
\begin{bmatrix}
\lambda & 0 \\
0 & \pm \lambda
\end{bmatrix} \rightarrow 
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
\lambda & 0 \\
0 & \pm \lambda
\end{bmatrix} \rightarrow 
\begin{bmatrix}
0 & \tau \\
0 & \tau
\end{bmatrix},
\]

- $\begin{bmatrix}
\lambda & 0 \\
0 & \pm \lambda
\end{bmatrix} \rightarrow 
\begin{bmatrix}
\mu & 0 \\
0 & \nu
\end{bmatrix}$ and $\begin{bmatrix}
\lambda & 0 \\
0 & \pm \lambda
\end{bmatrix} \rightarrow 
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}$, $(|\lambda| = |\mu| = |\nu| = 1, \mu \neq \pm \nu, |\sigma| < 1)$.

Suppose that there is an arbitrarily small perturbation $A := \begin{bmatrix}
\lambda & 0 \\
0 & \pm \lambda
\end{bmatrix} + E$ of $\begin{bmatrix}
\lambda & 0 \\
0 & \pm \lambda
\end{bmatrix}$ that is *congruent to $B := [0 0]$ or $C := [0 1 0]$. Then $A^*A := (A^{-1})^*A$ is similar to $B^*B$ or $C^*C$, which is impossible since the eigenvalues of $A^*A$ are arbitrarily close to $\lambda^{-1} = \lambda^2$, whereas $B^*B = \text{diag}(\mu^2, \nu^2)$ and $C^*C = \text{diag}(\sigma, \sigma^{-1})$.

- $\begin{bmatrix}
\lambda & 0 \\
0 & \pm \lambda
\end{bmatrix} \rightarrow 
\begin{bmatrix}
0 & \tau \\
0 & \tau
\end{bmatrix}$, $(|\lambda| = |\tau| = 1)$.

Let $\begin{bmatrix}
\lambda & 0 \\
0 & \pm \lambda
\end{bmatrix} \rightarrow \tau [0 1 1]$; i.e., there exists an arbitrarily small perturbation $A := [0 0] + E$ of $[0 1 0]$ that is *congruent to $B := \lambda^{-1}\tau [0 1]$. This means that there exists a nonsingular $S$ such that
\[
S^* \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} + E = \lambda^{-1}\tau \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
Equating the determinants of both sides, we find that
\[
r(1 + \varepsilon) = - (\lambda^{-1}\tau)^2, \quad r := \det(S^*S) > 0,
\]
in which $\varepsilon$ is arbitrarily small. Since $-(\lambda^{-1}\tau)^2$ is fixed and $|\lambda^{-1}\tau| = 1$, we have $-(\lambda^{-1}\tau)^2 = -1$, and so $\lambda^{-1}\tau = \pm i$. Then $\text{rank}(B + B^*) = 1$, which is impossible since $A + A^*$ is *congruent to $B + B^*$ and $\text{rank}(A + A^*) = 2$. 


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