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A NOTE ON CHARACTERIZATION OF THE SHORTED OPERATION*

EDWARD L. PEKAREV†

Abstract. In this article, an alternative characterization of the shorted operation defined on the class of all positive operators on a Hilbert space is presented.

Key words. Hilbert space, Positive operator, Shorted operator.

AMS subject classifications. 47A63.

1. Introduction. Let $\mathcal{H}$ be a Hilbert space and A be a (bounded linear) positive operator in $\mathcal{H}$. Given a closed subspace $\mathcal{L}$ of $\mathcal{H}$ the *shorted operator* $A_{\mathcal{L}}$ of a positive operator A to a subspace $\mathcal{L}$ is defined as the maximum among all the positive operators belonging to the set

$$\{X; O \leq X \leq A, ranX \subseteq \mathcal{L}\},$$

where $ranX$ denotes the range of X.

The existence of such a maximum was pointed out in 1947 by M.G. Krein [14] in connection with an extension problem of Hermitian positive semidefinite forms. Some properties of the correspondence $A \mapsto A_{\mathcal{L}}$ (*shorted operation*) were studied in [24], where various applications to the theory of characteristic operator-functions were found [6, 21, 25]. In 1971, W.N. Anderson, Jr. rediscovered this operation, and investigated its fundamental properties for positive matrices [1]. In 1975, W.N. Anderson, Jr. and G.E. Trapp considered the shorted operation for positive operators in association with electrical network analysis [2]. The motivation for the further study of the shorted operation is clear because of its significance for operator theory. The notion of the shorted operator (or Schur complement) was generalized to selfadjoint operators on Hilbert spaces in 1979 by T. Ando [4], and later (2001–2004) by G. Corach together with his colleagues A. Maestripieri, P. Massey, and D. Stojanoff [7, 8, 17]. In 2007, the shorted operation was extended to J-selfadjoint operators in Krein spaces by A. Maestripieri and F.M. Peria [16], and in 2012 to nonnegative forms by Z. Sebestyen and T. Titkos [26]. The notion of the shorted operator for a not

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necessarily closed operator range was motivated by Lebesgue-type decompositions, due to T. Ando [3].

Some applications of the shorted operation to convex analysis, theory of operator means, quantum mechanical measurement, etc. were considered in [11, 12, 15, 18, 27].

In 1976, K. Nishio and T. Ando [19] characterized the shorted operation as well as the so-called parallel addition of positive operators. In particular, they described the shorted operation in terms of some concave type inequalities for a mapping defined on the class of all positive operators in Hilbert space. In this note our purpose is to present a characteristic property of this mapping in terms of the set of its fixed points.

2. Preliminaries. Given a Hilbert space $\mathcal{H}$ we denote by $\mathcal{B}_+ = \mathcal{B}_+(\mathcal{H})$ the set of all (linear bounded) positive operators on $\mathcal{H}$, i.e., such Hermitian operators A that $(Af, f) \geq 0$ for all $f \in \mathcal{H}$. For $A, B \in \mathcal{B}_+$ we denote $A \leq B$ if $B - A \in \mathcal{B}_+$, and in such a case, we denote $[A, B] = \{X \in \mathcal{B}_+ : A \leq X \leq B\}$.

Let $A^{\frac{1}{2}}$ stand for the unique positive square root of $A \in \mathcal{B}_+$, $A^{-\frac{1}{2}}\mathcal{L}$ – for the preimage of a set $\mathcal{L} \subseteq \mathcal{H}$ under $A^{\frac{1}{2}}$, $ranA = A\mathcal{H}$ and let $\mathcal{L}^-$ denotes the closure of $\mathcal{L}$ in $\mathcal{H}$. For $A, B \in \mathcal{B}_+$ we use without further references the following equivalence relations arising from the well known results of R.G. Douglas [9] (see also [10]):

$$ranA^{\frac{1}{2}} \subseteq ranB^{\frac{1}{2}} \;\;\Leftrightarrow\;\; A \leq \alpha B \text{ for some positive scalar } \alpha,$$

$$X \in [O, A] \;\;\Leftrightarrow\;\; X = A^{\frac{1}{2}}KA^{\frac{1}{2}} \text{ for some } K \in [O, I].$$

We show, for the sake of completeness, that for given $A \in \mathcal{B}_+$ and a (closed) subspace $\mathcal{L} \subseteq \mathcal{H}$, the operator $A_{\mathcal{L}} := A^{\frac{1}{2}}P_{\mathcal{M}}A^{\frac{1}{2}}$, where $P_{\mathcal{M}}$ is the orthoprojection onto $\mathcal{M} = (A^{-\frac{1}{2}}\mathcal{L})^-$, really is the shorted operator of A to $\mathcal{L}$ (it was first proved by M.G. Krein [14]).

In fact, the operator $A_{\mathcal{L}}$ satisfies

$$O \leq A_{\mathcal{L}} \leq A \text{ and } ranA_{\mathcal{L}} \subseteq ranA^{\frac{1}{2}}P_{\mathcal{M}} \subseteq \mathcal{L}.$$

Besides, since for arbitrary $X \in [O, A]$ there exists such an operator

$$K \in [O, I] \text{ that } X = A^{\frac{1}{2}}KA^{\frac{1}{2}},$$

the inclusion $ranX \subseteq \mathcal{L}$ for a closed subspace $\mathcal{L}$ leads to

$$ranA^{\frac{1}{2}}K^{\frac{1}{2}} = ran(A^{\frac{1}{2}}KA^{\frac{1}{2}})^{\frac{1}{2}} \subseteq \mathcal{L}.$$

That means $ranK \subseteq \mathcal{M}$, and therefore, $K \leq P_{\mathcal{M}}$, or $X \leq A_{\mathcal{L}}$. Thus,

$$A_{\mathcal{L}} = A^{\frac{1}{2}}P_{\mathcal{M}}A^{\frac{1}{2}} = \max\{X; O \leq X \leq A, ranX \subseteq \mathcal{L}\}. \tag{2.1}$$

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Now, it is easy to verify the following well-known properties (see [3, 13, 22, 23]) of the correspondence $\pi_{\mathcal{L}} : A \mapsto A_{\mathcal{L}}$:

1) $O \le A_{\mathcal{L}} \le A$;

2) $(\alpha A)_{\mathcal{L}} = \alpha A_{\mathcal{L}}$ for any nonnegative scalar α;

3) $(A_{\mathcal{L}})_{\mathcal{L}} = A_{\mathcal{L}}$;

4) $A_{\mathcal{L}} + B_{\mathcal{L}} \le (A+B)_{\mathcal{L}}$, where both $A, B \ge O$;

5) $(A^2)_{\mathcal{L}} \le (A_{\mathcal{L}})^2$.

Indeed, properties 1), 2) and 3) follow immediately from (2.1), as well as the property 4), in view of

$$A_{\mathcal{L}} + B_{\mathcal{L}} \le A + B, \quad ran(A_{\mathcal{L}} + B_{\mathcal{L}}) \subseteq \mathcal{L}.$$

Finally, if $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ are orthoprojectors onto the spaces $\mathcal{M} = (A^{-\frac{1}{2}}\mathcal{L})^-$ and $\mathcal{N} = (A^{-1}\mathcal{L})^-$, respectively, then $(I - P_{\mathcal{M}})A^{\frac{1}{2}}P_{\mathcal{N}} = O$. This implies that

$$A^{\frac{1}{2}}P_{\mathcal{N}}A^{\frac{1}{2}} = P_{\mathcal{M}}A^{\frac{1}{2}}P_{\mathcal{N}}A^{\frac{1}{2}}P_{\mathcal{M}} \le P_{\mathcal{M}}AP_{\mathcal{M}},$$

and hence,

$$(A^2)_{\mathcal{L}} = AP_{\mathcal{N}}A \le A^{\frac{1}{2}}P_{\mathcal{M}}AP_{\mathcal{M}}A^{\frac{1}{2}} = (A_{\mathcal{L}})^2.$$

The following necessary and sufficient conditions characterizing the shorted operation were obtained in [19].

THEOREM 2.1. *An operation $\pi(\cdot)$ on the class of all positive operators turns out to be the shorted operation to some closed subspace $\mathcal{L}$, i.e., $\pi(A) = A_{\mathcal{L}}$ for every positive operator A, if and only if it satisfies the following conditions:*

1) $O \le \pi(A) \le A$;

2) $\pi(\alpha A) = \alpha\pi(A)$ for any nonnegative scalar α;

3) $\pi(\pi(A)) = \pi(A)$;

4) $\pi(A) + \pi(B) \le \pi(A+B)$ for $A, B \in \mathcal{B}_+$;

5) $\pi(A^2) \le \pi(A)^2$;

6) $\pi(A + \pi(B)C\pi(B)) \le \pi(A) + \pi(B)C\pi(B)$.

The shorted operation given by a subspace $\mathcal{L}$ is denoted as follows:

$$\pi_{\mathcal{L}}(\cdot) : \pi_{\mathcal{L}}(A) = A_{\mathcal{L}} \quad (A \in \mathcal{B}_+).$$

In what follows, $\Pi(\mathcal{B}_+)$ mean the set of all operations $\pi(\cdot)$ on $\mathcal{B}_+$, that satisfy conditions 1)–5) of Theorem 2.1. We collect several simple consequences of that

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conditions in the following:

LEMMA 2.2. *Let an operation $\pi(\cdot) \in \Pi(\mathcal{B}_+)$ and let $\mathcal{F}_\pi (\subseteq \mathcal{B}_+)$ denote the set of all its fixed points. Then*

(i) $\pi(A_1) \leq \pi(A_2)$ whenever $O \leq A_1 \leq A_2$,

(ii) $\pi(P)$ is an orthogonal projector whenever P is an orthogonal projector,

(iii) $\mathcal{F}_\pi = \{\pi(A) : A \in \mathcal{B}_+\} \neq \emptyset$,

(iv) if operators $A, B \in \mathcal{F}_\pi$ and scalars $\alpha, \beta \geq 0$, then $\alpha A + \beta B \in \mathcal{F}_\pi$.

Proof. (i) In view of the condition 4) under assumptions $A = A_1,\ B = A_2 - A_1$, we have:

$$\pi(A_1) \leq \pi(A_1) + \pi(A_2 - A_1) \leq \pi(A_1 + (A_2 - A_1)) = \pi(A_2).$$

(ii) In view of conditions 1) and 5), we obtain that if P is an orthogonal projector, then $\pi(P) \leq P \leq I$ and

$$\pi(P) = \pi(P^2) \leq \pi(P)^2 = \pi(P)^{\frac{1}{2}}\pi(P)\pi(P)^{\frac{1}{2}} \leq \pi(P)^{\frac{1}{2}} P \pi(P)^{\frac{1}{2}} \leq \pi(P).$$

Therefore, since $O \leq \pi(P) = \pi(P)^2$, the operator $\pi(P)$ is an orthogonal projector.

(iii) Note that 2) implies $\pi(O) = O$, and in view of 3) of Theorem 2.1, we arrive at

$$\mathcal{F}_\pi = \{\pi(A) : A \in \mathcal{B}_+\} \neq \emptyset.$$

(iv) Given $A, B \in \mathcal{F}_\pi$ and $\alpha, \beta \geq 0$, and having in view 1), 2), 4), we get:

$$\alpha A + \beta B = \alpha\pi(A) + \beta\pi(B) \leq \pi(\alpha A + \beta B) \leq \alpha A + \beta B.$$

Hence, $\alpha A + \beta B = \pi(\alpha A + \beta B) \in \mathcal{F}_\pi$. □

We use the partial order $\prec_r$ on $\mathcal{B}_+$, defining $A \prec_r B$ iff $ranA^{\frac{1}{2}} \subseteq ranB^{\frac{1}{2}}$. If $A \prec_r B$ and $B \prec_r A$, that is $\lambda A \leq B \leq \mu A$ for some positive scalars λ, μ, such operators A, B are in the equivalence relation $A \sim_r B$, or, in other words, they belong to the same so-called "Thompson component". For $\mathcal{F}_\pi$, we study some of its order-convex subsets, which can be divided into equivalence classes.

3. Results. Let an operation $\pi(\cdot) \in \Pi(\mathcal{B}_+)$, $\mathcal{F}_\pi$ be the set of all its fixed points, and $\mathcal{F}_\pi^{co} \subseteq \mathcal{F}_\pi$ consists exactly of such operators $A \in \mathcal{F}_\pi$, that $[O, A] \subseteq \mathcal{F}_\pi$. As it is shown in the next section, $\mathcal{F}_\pi^{co} \neq \mathcal{F}_\pi$ in general. Clearly, if $A \sim_r B$, then $\pi(A) \sim_r \pi(B)$, because using inequalities $\lambda A \leq B \leq \mu A$ $(\lambda, \mu \geq 0)$ one can easily get the inequalities $\lambda\pi(A) \leq \pi(B) \leq \mu\pi(A)$ $(\lambda, \mu \geq 0)$.

Besides, if $A \in \mathcal{F}_\pi^{co}$ and $B \sim_r A$, then also $B \in \mathcal{F}_\pi^{co}$. Indeed, if $[O, A] \subseteq \mathcal{F}_\pi^{co}$, then

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for any scalar $\mu > 0$ we get $[O, \mu A] \subseteq \mathcal{F}_\pi^{co}$ as well, and from $(\lambda A \leq) B \leq \mu A$ it follows that $B \in \mathcal{F}_\pi^{co}$. Thus, $\mathcal{F}_\pi^{co}$ is divided into some components, and it is clear, that $\mathcal{F}_\pi^{co}$ is an ordered convex subset of $\mathcal{F}_\pi$. In the case of finite dimensional $\mathcal{H}$, every chain $A_1 \prec_r A_2 \prec_r A_3 \prec_r \cdots$ in $\mathcal{F}_\pi$ or in $\mathcal{F}_\pi^{co}$ is stable, that is for some i_0 the relationship $A_{i_0} \sim_r A_{i_0+1} \sim_r A_{i_0+2} \sim_r \cdots$ holds. So, there exists at least one maximal element in $\mathcal{F}_\pi$ or in $\mathcal{F}_\pi^{co}$, correspondingly. While in general case $\mathcal{F}_\pi$ and $\mathcal{F}_\pi^{co}$ may have different or the same largest elements, the next theorem holds.

THEOREM 3.1. *Let an operation $\pi(\cdot) \in \Pi(\mathcal{B}_+)$, and $\mathcal{F}_\pi$, $\mathcal{F}_\pi^{co}$ be the sets defined above. Then the following statements are equivalent:*

1) $\pi(\cdot) = \pi_{\mathcal{L}}(\cdot)$ for some closed subspace $\mathcal{L} \subseteq \mathcal{H}$;

2) $\pi(I) \in \mathcal{F}_\pi^{co}$;

3) $\mathcal{F}_\pi$ and $\mathcal{F}_\pi^{co}$ have the same largest element.

Proof. 1) $\Rightarrow$ 2). If a closed subspace $\mathcal{L} \subseteq \mathcal{H}$ is such that $\pi(A) = A_{\mathcal{L}}$ for every positive operator A, then $\pi(I) = I_{\mathcal{L}} = P_{\mathcal{L}}$, orthoprojection on $\mathcal{L}$, and it is clear in view of (2.1) that $[O, \pi(I)] = [O, P_{\mathcal{L}}] \subseteq \mathcal{F}_\pi$. This means that $\pi(I) \in \mathcal{F}_\pi^{co}$.

2) $\Rightarrow$ 3). If $A \in \mathcal{F}_\pi$ and $\alpha A = \pi(\alpha A) \leq \pi(I)$ for some $\alpha > 0$ then $A \sim_r \alpha A \prec_r \pi(I)$. Thus, $\pi(I)$ is the largest element in $\mathcal{F}_\pi$ and as $\pi(I) \in \mathcal{F}_\pi^{co}$, also $\pi(I)$ is the largest element in $\mathcal{F}_\pi^{co}$.

3) $\Rightarrow$ 1). Note that $\pi(I)$ is an orthogonal projector in view of Lemma 2.2 (ii). Therefore, denoting $\mathcal{L}=ran\pi(I)$ we have $\pi(I) = P_{\mathcal{L}}$. To prove the equality $\pi(A) = A_{\mathcal{L}}$ for a given $A \in \mathcal{B}_+$, find such $\alpha > 0$ that $\alpha A \leq I$ and so $\alpha\pi(A) \leq \pi(I) = P_{\mathcal{L}}$. It follows that $ran\pi(A) \subseteq \mathcal{L}$, and as $\pi(A) \leq A$ for $\pi(\cdot) \in \Pi(\mathcal{B}_+)$ it implies that $\pi(A) \leq A_{\mathcal{L}}$. But from $O \leq A_{\mathcal{L}} \leq A$ it follows $\pi(A_{\mathcal{L}}) \leq \pi(A)$ and it is sufficient to prove that $\pi(A_{\mathcal{L}}) = A_{\mathcal{L}}$. Actually, $(\alpha A)_{\mathcal{L}} \leq P_{\mathcal{L}} = \pi(I)$, and therefore, $(\alpha A)_{\mathcal{L}} \in [O, \pi(I)]$. But $\pi(I)$ is the largest element in $\mathcal{F}_\pi$, as it was shown in the part 2) $\Rightarrow$ 3). Therefore, $\pi(I)$ is the largest element in $\mathcal{F}_\pi^{co}$ as well. Hence, $(\alpha A)_{\mathcal{L}} \in [O, \pi(I)] \subseteq \mathcal{F}_\pi$, $\pi(\alpha A_{\mathcal{L}}) = \alpha A_{\mathcal{L}}$ or $\pi(A_{\mathcal{L}}) = A_{\mathcal{L}}$. This completes the proof. $\square$

COROLLARY 3.2. *Let an operation $\pi(\cdot) \in \Pi(\mathcal{B}_+)$. Then $\pi(\cdot) = \pi_{\mathcal{L}}(\cdot)$ for some closed subspace $\mathcal{L} \subseteq \mathcal{H}$ if and only if the conditions $O \neq A \leq \pi(I)$ imply the inequality $\pi(A) \neq O$ for every positive operator A.*

Proof. Let $\pi(\cdot) = \pi_{\mathcal{L}}(\cdot)$ with $\mathcal{L}=ran\pi(I)$ and $O \neq A \leq \pi(I)$. Then $ranA \subseteq \mathcal{L}$, so $A_{\mathcal{L}} = A$, and hence, $\pi(A) = A \neq O$.

Conversely, it is enough to show, that $[O, \pi(I)] \subseteq \mathcal{F}_\pi$ under the assumption $O \neq A \leq \pi(I) \;\Rightarrow\; \pi(A) \neq O$. On the other hand, for any X, $O \leq X \leq \pi(I)$, in view of conditions 3) and 4) of Theorem 2.1, the following inequalities hold:

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$$\pi(X) = \pi(X-\pi(X)+\pi(X)) \geq \pi(X-\pi(X))+\pi(\pi(X)) = \pi(X-\pi(X))+\pi(X).$$

This implies that $\pi(X-\pi(X)) = O$, and hence, $\pi(X) = X$. ◻

4. Remarks. Given operators $A, B \in \mathcal{B}_+$ T. Ando in [3] have considered the so-called Lebesgue-type decomposition $A = C+D$ $(C, D \in \mathcal{B}_+)$, where $ranD^{\frac{1}{2}} \cap ranB^{\frac{1}{2}} = \{0\}$ and C is a limit of an increasing sequence $\{C_n\}_{n=1}^{\infty} \subset \mathcal{B}_+$ satisfying the condition $C_n \prec_r B$ $(n = 1, 2, \ldots)$. Among all of such operators C there exists the maximum. It depends only on the operator A and (not necessarily closed) the operator range $\mathcal{L} = ranB^{\frac{1}{2}}$. This maximum operator was called in [22] shorted operator (or convolution) of A to $\mathcal{L}$ and denoted, as before, by $A_{\mathcal{L}}$. Basic properties of the correspondence $\pi_{\mathcal{L}} : A \mapsto A_{\mathcal{L}}$ were stated in [3, 22] and then in [13, 23]. In particular, the properties 1)–5) of Theorem 2.1 were established as well as the equalities

$$A_{\mathcal{L}} = A^{\frac{1}{2}} P_{\mathcal{M}} A^{\frac{1}{2}} = \max\{X; O \leq X \leq A, (X^{-\frac{1}{2}}\mathcal{L})^- = \mathcal{H}\}, \tag{4.1}$$

where $P_{\mathcal{M}}$ – the orthoprojector onto $\mathcal{M} = (A^{-\frac{1}{2}}\mathcal{L})^-$. At the same time, it is easy to see, that condition 6) of that theorem may be violated if the operator range $\mathcal{L}$ is not closed.

Indeed, let for some operator ranges $\mathcal{L}$ and $\mathcal{M}$ the equality $A_{\mathcal{L}} = A_{\mathcal{M}}$ hold for all $A \in \mathcal{B}_+$. Then $\mathcal{L} = \mathcal{M}$ because if $(0 \neq) f \in \mathcal{L}, f \bar{\in} \mathcal{M}$ and $A = P$ – the orthoprojection on the subspace spanned by f, one can find in view of (4.1) that $A_{\mathcal{L}} = P$, $A_{\mathcal{M}} = O$. So, $\mathcal{L} \subseteq \mathcal{M}$, and as similarly $\mathcal{M} \subseteq \mathcal{L}$, it follows $\mathcal{L} = \mathcal{M}$. Therefore, if an operator range $\mathcal{L}$ is not closed and $\mathcal{M}$ is any closed subspace of $\mathcal{H}$, then shorted operations to $\mathcal{L}$ and $\mathcal{M}$ are not equal. Hence, the condition 6) of Theorem 2.1 fails.

Note also, that if the operator range $\mathcal{L} \subset \mathcal{H}$, $\mathcal{L} \neq \mathcal{H}$, is dense in $\mathcal{H}$, and $\pi(A) = A_{\mathcal{L}}$ $(A \in \mathcal{B}_+)$, then $\pi(I) = I$ is the maximum element of $\mathcal{F}_\pi$, but $[O, I]$ is not contained in $\mathcal{F}_\pi$, so $\mathcal{F}_\pi^{co} \neq \mathcal{F}_\pi$ (compare with Theorem 3.1). Besides, if $A, B \in \mathcal{B}_+$ and $A \in \mathcal{F}_\pi^{co}, ranB^{\frac{1}{2}} = \mathcal{L}$, then $B \in \mathcal{F}_\pi^{co}$ and $A \prec_r B$.

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