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3-BY-3 MATRICES WITH ELLIPTICAL NUMERICAL RANGE REVISITED[*]

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Abstract. According to Kippenhahn's classification, numerical ranges $W(A)$ of unitarily irreducible 3×3 matrices A come in three possible shapes, an elliptical disk being one of them. The known criterion for the ellipticity of $W(A)$ consists of several equations, involving the eigenvalues of A. It is shown herein that the set of 3×3 matrices satisfying these conditions is nowhere dense, i.e., one of the necessary conditions can be violated by an arbitrarily small perturbation of the matrix, and therefore by an insufficiently good numerical approximation of the eigenvalues.

Moreover, necessary and sufficient conditions for a real A to have an elliptical $W(A)$ are derived, involving only the matrix coefficients and not requiring the knowledge of the eigenvalues. A particular case of real companion matrices is considered in detail.

Key words. Numerical range, Companion matrices.

AMS subject classifications. 15A60.

1. Introduction. Given a square $n \times n$ matrix with complex entries, we can compute an associated *numerical range* (also known as *field of values*), which is a subset of the complex plane $\mathbb{C}$. This numerical range is the set of all outputs of the inner product $\langle Ax, x \rangle$ over unit vectors x; we will repeat this definition in more detail in the next section.

Numerical ranges have been a subject of ongoing research for almost 100 years, starting with the pioneering papers [5, 11]. In 1951, Kippenhahn established in [8] many of the early results in the field, including a complete classification of numerical ranges of 3×3 matrices (see [9] for an English translation). As it happens, for a unitarily irreducible 3×3 matrix its numerical range is either an ellipse, or has one flat portion on its boundary (the rest of the boundary being an algebraic curve of 4th degree), or has an ovular shape, bounded by an algebraic curve of 6th degree.

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More recently, tests were developed allowing to determine which type of the numerical range occurs for a given 3×3 matrix [7], with the flat portion boundary situation further treated in [10]. Note that the exact knowledge of the eigenvalues is crucial for these tests.

In the current paper, we return to the case of 3×3 matrices with elliptical numerical ranges. Corollary 3.1 shows that the set of matrices with this type of numerical range is nowhere dense. More precisely, given a matrix whose numerical range is the convex hull of an ellipse and a point, a small perturbation of the eigenvalues is likely to drastically change the shape of the numerical range, turning it into an ovular one. When using a numerical approximation to check the formulas for explicit matrices, a small inaccuracy in the calculation can perturb the answer just enough to give an incorrect conclusion.

Further, we establish formulas, equivalent to the ones derived in [7], to determine whether the numerical range of a given 3×3 real matrix is the convex hull of an ellipse and a point. These formulas are given completely in terms of trace invariants which can be computed directly from the entries of the original matrix. See Theorem 3.2 in Section 3. From there, a necessary and sufficient condition for a 3×3 matrix to have an elliptical numerical range is derived.

In the final Section 4, we illustrate these results using the set of all real 3×3 companion matrices as an example. Such matrices form a 3-parameter family, and the set of matrices which have elliptic numerical range is described by a single equation, the graph of which is a surface in $\mathbb{R}^3$. It is clear that any perturbation away from the surface will cause the numerical range to change from an elliptical disc to a different type of shape.

The auxiliary Section 2 contains definitions of each of the previously discussed terms and statements of the relevant theorem from [7].

2. Preliminaries. With $\mathbb{F}$ standing for either the field $\mathbb{R}$ of real or $\mathbb{C}$ of complex numbers, we denote by $\mathbb{F}^{n \times m}$ the linear space (algebra, if $m = n$) of the $n \times m$ matrices with their entries in $\mathbb{F}$. We also use the standard abbreviation $\mathbb{F}^n := \mathbb{F}^{n \times 1}$, and supply $\mathbb{C}^n$ with the inner product $\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y}_j$ and the respective norm $\|x\| := \langle x, x \rangle^{1/2}$.

DEFINITION 2.1. Let $A \in \mathbb{C}^{n \times n}$. Then the *numerical range* of A, $W(A)$, is given by

$$W(A) = \{ \langle Ax, x \rangle \colon \|x\| = 1 \}.$$

The numerical range is a compact convex subset of $\mathbb{C}$ containing the spectrum

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$\sigma(A)$ of A; see e.g. [6, Chapter 1] for these and other properties of $W(A)$. Further, recall the following:

DEFINITION 2.2. A matrix $A \in \mathbb{C}^{n \times n}$ is *unitarily reducible* if there exists a unitary matrix U such that U^*AU has a block diagonal form

$$U^*AU = \begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & A_2 \end{pmatrix} = A_1 \oplus A_2,$$

where the sizes of A_1 and A_2 are smaller than n.

We will repeatedly be using the following:

(i) $W(A)$ is unitarily invariant: $W(U^*AU) = W(A)$ for any unitary $U \in \mathbb{C}^{n \times n}$.
(ii) If A is unitarily reducible, that is, unitarily similar to the direct sum $A_1 \oplus A_2$, then $W(A)$ is the convex hull of $W(A_1) \cup W(A_2)$.

From these properties it follows in particular that $W(A)$ is the convex hull of $\sigma(A)$ whenever A is normal. On the other hand, if $A \in \mathbb{C}^{2 \times 2}$ and is not normal, by the Elliptical Range Theorem, $W(A)$ is an ellipse with foci at the eigenvalues of A. The case $n = 3$ has also been treated fully, see [8] and its English translation [9] for the classification and [7] for the pertinent tests.

All possible shapes of the numerical ranges $W(A)$ of 3-by-3 matrices A were described by Kippenhahn in [8, 9]. It is possible, in particular, for $W(A)$ to be the convex hull of an ellipse and a point:

$$W(A) = \text{conv}\{E, p\}. \tag{2.1}$$

If p lies in the interior of E, then of course $W(A)$ is the elliptical disk bounded by E, and the matrix A in this case may or may not be unitarily reducible. Necessary and sufficient conditions on A to have $W(A)$ of this shape were obtained in [7]. They can be stated as follows.

THEOREM 2.3. *The numerical range $W(A)$ of a 3×3 matrix A with the eigenvalues λ_j, $j = 1, 2, 3$, is of the form (2.1) if and only if*

(i) *A is not normal (that is, $d := \text{trace}(A^*A) - \sum_{j=1}^{3} |\lambda_j|^2 \neq 0$), and*
(ii) *the number*

$$\lambda := \text{trace } A + \frac{1}{d} \left(\sum_{j=1}^{3} |\lambda_j|^2 \lambda_j - \text{trace}(A^*A^2) \right) \tag{2.2}$$

coincides with one of λ_j.

If these conditions hold, then in (2.1), $p = \lambda$, while the minor axis of E has length $\sqrt{d}$ and its foci coincide with the two remaining eigenvalues, λ_1 and λ_2, of A. In

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particular, $W(A)$ *is a true ellipse if and only if*

$$(|\lambda_1 - \lambda| + |\lambda_2 - \lambda|)^2 - |\lambda_1 - \lambda_2|^2 \le d. \qquad (2.3)$$

3. Main results. It is a simple observation, not explicitly recorded earlier, that the set of matrices satisfying the conditions of Theorem 2.3 (and thus the set $\mathcal{E}_3$ of 3×3 matrices with elliptical numerical ranges) is nowhere dense. For the sake of completeness (and convenience of future references), we provide the statement and its proof below. Note that the set $\mathcal{F}_3$ of 3×3 unitarily irreducible matrices A with a flat portion on the boundary of $W(A)$ is nowhere dense as well; this can be easily seen from the proof of Theorem 1.2 in [10].

PROPOSITION 3.1. *The set of* 3×3 *matrices satisfying* (2.1) *is nowhere dense.*

Proof. Clearly, condition (i) of Theorem 2.3 is stable, so that we need to show that its condition (ii) can be violated by an arbitrarily small perturbation of A. Without loss of generality, put A in an upper triangular form

$$\begin{bmatrix} \lambda_1 & x & y \\ 0 & \lambda_2 & z \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Then (2.2) equals $(\lambda_1 |z|^2 + \lambda_2 |y|^2 + \lambda_3 |x|^2 - x\overline{y}z)/d$. It will be shown below that if x, y, z are modified in an appropriate way, (small) perturbations of x, y, z yield small perturbations of λ while the eigenvalues λ_j remain unchanged. This will lead to failure to satisfy condition (ii) of Theorem 2.3.

Case 1. $xyz \ne 0$. Keeping x, y unchanged, replace z by ωz with $|\omega| = 1$. The resulting value of λ will differ from the original one unless $\omega = 1$, and condition (ii) of Theorem 2.3 will fail for all $\omega \ne 1$ sufficiently close to 1.

Case 2. $xyz = 0$. Given $\epsilon > 0$, first perturb x, y, z in such a way that all of x, y, and z become non-zero while the resulting matrix A_1 lies in the $\epsilon/2$-neighborhood of A. If condition (ii) of Theorem 2.3 fails for A_1, we are done. Otherwise, A_1 is as in Case 1, and thus, the desired matrices exist in its $\epsilon/2$-neighborhood. $\square$

Because of Proposition 3.1, the exact values of λ_j are seemingly needed to determine whether or not (2.1) actually holds. It is therefore desirable to recast formula (2.2) exclusively in terms of the entries of A. We show here how this can be done for real A.

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THEOREM 3.2. *Let A be a 3×3 real matrix. Then, its numerical range $W(A)$ is of type* (2.1) *if and only if*

$$\mathrm{trace}\, A + \frac{\mathrm{trace}\, A^3 - \mathrm{trace}(A^T A^2)}{\mathrm{trace}(A^T A) - \mathrm{trace}\, A^2} \tag{3.1}$$

coincides with one of its eigenvalues.

Proof. Let us distinguish between the cases when all the eigenvalues of A are real (simple or multiple) and when exactly one is real and two are non-real complex conjugate (and thus simple). Recall that it is possible to determine which case we are in by the sign of the discriminant of the characteristic polynomial $\det(A - zI)$, without any additional knowledge concerning the eigenvalues themselves.

Case 1. The matrix A has three real eigenvalues, λ_j, $j = 1, \dots, 3$. Observe that in this case,

$$\sum |\lambda_j|^2 = \sum \lambda_j{}^2 = \mathrm{trace}\, A^2 \ \text{ and } \ \sum |\lambda_j|^2 \lambda_j = \sum \lambda_j{}^3 = \mathrm{trace}\, A^3,$$

and rewrite formula (2.2), noting in addition that $A^* = A^T$ since A is real. Non-normality of A is implicit in (3.1): otherwise the denominator would vanish.

Case 2. The case of one real and two complex conjugate roots is more interesting.

For convenience of notation, (re)label the real eigenvalue of A by λ and its complex conjugate non-real eigenvalues by μ and $\overline{\mu}$. The proof runs differently when $\lambda = 0$ and $\lambda \neq 0$ (equivalently, $\det A = 0$ and $\det A \neq 0$), so we consider these cases separately.

Case 2a. $\det A \neq 0$. Then,

$$\sum_{j=1}^{3} |\lambda_j|^2 = \lambda^2 + 2|\mu|^2 = \lambda^2 + 2\mu\overline{\mu} = \lambda^2 + 2\frac{\det A}{\lambda}$$

and

$$\sum_{j=1}^{3} |\lambda_j|^2 \lambda_j = \lambda^3 + |\mu|^2 (\mu + \overline{\mu}) = \lambda^3 + \frac{\det A}{\lambda}(\mathrm{trace}\, A - \lambda).$$

Consequently, (2.2) takes the form

$$\mathrm{trace}\, A + \frac{\lambda^3 + \frac{\det A}{\lambda}(\mathrm{trace}\, A - \lambda) - \mathrm{trace}(A^T A^2)}{\mathrm{trace}(A^T A) - \lambda^2 - 2\frac{\det A}{\lambda}}. \tag{3.2}$$

Since this quantity is real, it coincides with an eigenvalue of A if and only if it equals λ. The latter requirement can be written as

$$2\lambda^4 - \mathrm{trace}\, A \lambda^3 - \mathrm{trace}(A^T A)\lambda^2$$
$$+ (\det A + \mathrm{trace}\, A \,\mathrm{trace}(A^T A) - \mathrm{trace}(A^T A^2))\lambda - \mathrm{trace}\, A \det A = 0. \tag{3.3}$$

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Applying the Euclidean division algorithm to (3.3) and the characteristic equation

$$\lambda^3 - (\operatorname{trace} A)\lambda^2 + \frac{(\operatorname{trace} A)^2 - \operatorname{trace} A^2}{2}\lambda - \det A = 0 \qquad (3.4)$$

of A, we find a remainder of $(\operatorname{trace}(A^2) - \operatorname{trace}(A^T A))x^2 + (-\operatorname{trace}(A)\operatorname{trace}(A^2) + \operatorname{trace}(A^3) + \operatorname{trace}(A)\operatorname{trace}(A^T A) - \operatorname{trace}(A^T A^2))x$. Since (3.1) is the nonzero root of this remainder, we conclude that equations (3.3) and (3.4) share a common solution if and only if (3.2) equals (3.1).

Case 2b. $\det A = 0$. Being real, the quantity (2.2) coincides with one of the eigenvalues of A if and only if it equals zero. Since

$$\sum_{j=1}^{3}|\lambda_j|^2 = 2|\mu|^2, \quad \sum_{j=1}^{3}|\lambda_j|^2\lambda_j = |\mu|^2(\mu + \overline{\mu}) = |\mu|^2\operatorname{trace} A,$$

a direct computation shows that this happens if and only if

$$|\mu|^2\operatorname{trace} A = \operatorname{trace} A\operatorname{trace}(A^T A) - \operatorname{trace}(A^T A^2). \qquad (3.5)$$

On the other hand, being the non-zero solutions to (3.4) with $\det A = 0$, μ and $\overline{\mu}$ satisfy the equation

$$x^2 - (\operatorname{trace} A)x + \frac{(\operatorname{trace} A)^2 - \operatorname{trace} A^2}{2} = 0.$$

Thus,

$$|\mu|^2 = \mu\overline{\mu} = \frac{(\operatorname{trace} A)^2 - \operatorname{trace} A^2}{2}. \qquad (3.6)$$

Plugging (3.6) into (3.5) yields

$$\operatorname{trace}(A^T A^2) - \operatorname{trace} A\operatorname{trace}(A^T A) - \frac{1}{2}\operatorname{trace} A\operatorname{trace} A^2 + \frac{1}{2}(\operatorname{trace} A)^3 = 0.$$

Recall the general trace representation of a 3×3 determinant:

$$\det A = 3\operatorname{trace}(A)\operatorname{trace}(A^2)/2 - \operatorname{trace}(A^3) - \operatorname{trace}(A)^3/2.$$

Since this quantity is zero, we see that the preceding equality holds if and only if (3.1) is an eigenvalue of A. □

4. Companion matrices. Consider the set of 3×3 *companion matrices*, i.e., matrices of the form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c & b & a \end{bmatrix}. \qquad (4.1)$$

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These matrices have an easily computable characteristic polynomial, $x^3 - ax^2 - bx - c$.

A direct computation shows that (3.1) is an eigenvalue of A if and only if (a, b, c) is a root of

$$
\begin{aligned}
p(a, b, c) := & - a^3 - 4ab + 2a^3b + 8ab^2 - a^3b^2 - 8ab^3 + 4ab^4 - ab^5 \\
& + (-8 - 3a^2 + 12b + 13a^2b - 8b^2 - 10a^2b^2 + 2a^2b^3 + 2b^4 - b^5)c \\
& + (9a - a^3 + 8ab - 16ab^2 + 6ab^3 - ab^4)c^2 \\
& + (15 - 6a^2 - 15b + 2a^2b + b^2 + b^3 - b^4)c^3 \\
& + (-9a + 5ab - ab^2)c^4 + (-6 + 3b - 2b^2)c^5 - c^7.
\end{aligned}
$$

The respective triples (a, b, c) form a surface S a portion of which is shown in Figure 4.1.
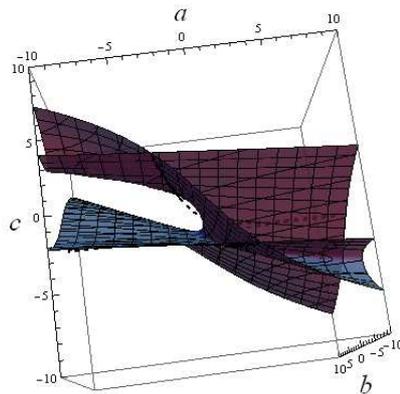

FIG. 4.1. *Graph of the relationship between a, b, and c for companion matrices whose numerical range is the convex hull of a point and an ellipse.*

On the other hand, it is easy to see which of the matrices (4.1) are unitarily (ir)reducible.

LEMMA 4.1. *A real companion matrix (4.1) is unitarily reducible if and only if*

$$
b = 1 - c^2, \; a = \frac{1}{c} - c, \quad c \in \mathbb{R} \setminus \{0\}. \tag{4.2}
$$

Proof. According to [4, Theorem 1.1] an $n \times n$ companion matrix is unitarily reducible if and only if its spectrum can be represented as $\eta\Omega_1 \cup \overline{\eta}^{-1}\Omega_2$ for some $\eta \in \mathbb{C} \setminus \{0\}$, with $\Omega_1 \cup \Omega_2$ being a partition of the set of all n-th roots of unity into two non-empty subsets. Thus, the eigenvalues have arguments $\arg \eta + \frac{2\pi j}{3}$, $j = 0, 1, 2$. Since in

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our setting at least one eigenvalue of A is real and two others are complex conjugate of each other, $\arg \eta = 0$, $\mod \frac{\pi}{3}$. Consequently, $\sigma(A) = \{ce^{2\pi i/3}, ce^{-2\pi i/3}, 1/c\}$, which, according to the Vieta theorem, happens if and only if (4.2) holds. ⬜

The curve Γ defined by (4.2) lies on the surface S, and all the triples $(a, b, c) \in S \backslash \Gamma$ correspond to unitarily irreducible matrices (4.1) with elliptical numerical ranges. Indeed, for unitarily irreducible matrices condition (2.3) is satisfied automatically, since the set (2.1) with p lying outside E would have p as its corner point. Consequently, p would then be a normal eigenvalue of A, making the latter unitarily reducible.

On the other hand, for $(a, b, c) \in \Gamma$ condition (2.3) amounts to

$$c^4 - 3c^2 - 3 - 4c^{-2} \geq 0,$$

which holds if and only if $|c| \geq 2$. So, the unitarily reducible matrices (4.1) with $(a, b, c) \in \Gamma$ and $|c| \geq 2$ have elliptical numerical ranges, while for those with $|c| < 2$ the set $W(A)$ has a corner point. The values $c = \pm 1$, in particular, correspond to orthogonal matrices A, with $W(A)$ being an equilateral triangle.

The curve Γ is also plotted in Figure 4.1. Interestingly enough, a direct computation (using Mathematica) shows that the subset of S on which

$$\frac{\partial p}{\partial a} = \frac{\partial p}{\partial b} = \frac{\partial p}{\partial c} = 0$$

(that is, the set of singular points of S) coincides with Γ.

The surface S contains the axis $a = c = 0$, thus implying that all matrices of the form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & b & 0 \end{bmatrix} \tag{4.3}$$

have an elliptical numerical range. This is no surprise, however, since the matrix (4.3) is tridiagonal, and any such 3×3 matrix with equal $(1, 1)$ and $(3, 3)$ entries has this property [1, Theorem 4.2].

The rest of the cross-section of S by the coordinate plane $c = 0$ is described by the equation

$$a^2(b - 1)^2 + 4b - 8b^2 + 8b^3 - 4b^4 + b^5 = 0. \tag{4.4}$$

This curve is graphed in Figure 4.2.

According to Lemma 4.1, the matrices (4.1) with $c = 0$ are unitarily irreducible. Thus, if in addition (4.4) holds, the numerical range must be an ellipse.
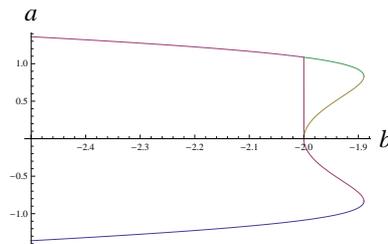
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FIG. 4.2. *Graph of relationship between a and b, when c = 0.*

EXAMPLE 4.2. Let in (4.1) $c = 0$ and $b = -2$. Then, $a = \sqrt{5(-2 + \sqrt{5})}$ is one of the two roots of (4.4). The respective matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sqrt{5(-2 + \sqrt{5})} & -2 & 0 \end{bmatrix}$$

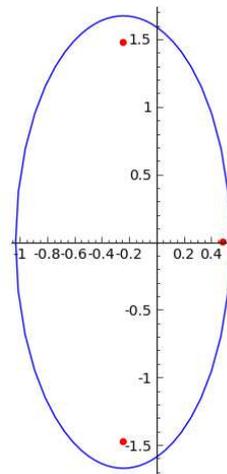has an elliptical numerical range, pictured in Figure 4.3.

FIG. 4.3. *Numerical range example when c = 0. The points are the eigenvalues of the matrix A.*

For completeness, let us mention that 3×3 companion matrices were considered in [2] from a different point of view. Namely, the question addressed there was, given z_1, z_2, how many z_3 exist such that the companion matrix with the eigenvalues z_1, z_2, z_3 has an elliptical numerical range. Companion matrices A for which $W(A)$ has a flat portion on the boundary were treated in [3].

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5. Conclusion. Necessary and sufficient conditions for the numerical range of a 3×3 matrix to be the convex hull of a point and an ellipse were derived herein. These conditions are equivalent to the ones derived in [7], however, they have the advantage of being expressed entirely in terms of the matrix coefficients, without requiring computation of the matrix eigenvalues. This type of reformulation of the conditions is necessary due to the fact that the set of matrices satisfying these conditions is nowhere dense, as shown in Proposition 3.1. Thus, even a small numerical error in approximating the eigenvalues of the matrix could lead to the conditions in Theorem 2.3 not being satisfied, and consequently to a wrong conclusion about the shape of the numerical range. Further, in Section 4, the necessary and sufficient conditions considered herein were applied to the set of companion matrices, several examples were computed explicitly, and the set of matrix coefficients yielding a numerical range of the desired shape (convex hull of a point and an ellipse) was analytically described and plotted (see Figure 4.1).

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