2013

Bounds for an operator concave function

Sever S. Dragomir
sever.dragomir@wits.ac.za

Jun Ichi Fujii

Yuki Seo

Follow this and additional works at: http://repository.uwyo.edu/ela

Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.1649

This Article is brought to you for free and open access by Wyoming Scholars Repository. It has been accepted for inclusion in Electronic Journal of Linear Algebra by an authorized editor of Wyoming Scholars Repository. For more information, please contact scholcom@uwyo.edu.
Abstract. Let $\Phi$ and $\Psi$ be unital positive linear maps satisfying some conditions with respect to positive scalars $\alpha$ and $\beta$. It is shown that if a real valued function $f$ is operator concave on an interval $J$, then
$$\beta (f(\Psi(A)) - \Psi(f(A))) \leq f(\Phi(A)) - \Phi(f(A)) \leq \alpha (f(\Psi(A)) - \Psi(f(A)))$$
for every selfadjoint operator $A$ with spectrum $\sigma(A) \subset J$. Moreover, an external version of estimates above is presented.

Key words. Operator concave function, Davis-Choi-Jensen inequality, Positive linear map.

AMS subject classifications. 47A63.

1. Introduction. Let $\Phi$ be a unital positive linear map from $B(H)$ to $B(K)$, where $B(H)$ is the $C^*$-algebra of all bounded linear operators on a Hilbert space $H$. The Davis-Choi-Jensen inequality [1, 2] states that if a real-valued function $f$ is operator concave on an interval $J$, then

$$(1.1) \quad \Phi(f(A)) \leq f(\Phi(A))$$

for every selfadjoint operator $A$ with spectrum $\sigma(A) \subset J$. Though in the case of concave function the inequality (1.1) does not hold in general, we have the following estimate [7]: If $f$ is concave and $A$ is a selfadjoint operator on $H$ such that $mI \leq A \leq MI$ for some scalars $m < M$, then

$$(1.2) \quad -\mu(m, M, f)I \leq f(\Phi(A)) - \Phi(f(A)) \leq \mu(m, M, f)I$$

for all unital positive linear maps $\Phi$ where the bound $\mu(m, M, f)$ of $f$ is defined by

$$(1.3) \quad \mu(m, M, f) = \max \left\{ f(t) - \frac{f(M) - f(m)}{M - m} (t - m) - f(m) : t \in [m, M] \right\}.$$
In [3], the first author showed the following estimate for the normalized Jensen functional: If a real-valued function \( f \) is concave on a convex set \( C \), then for each positive \( n \)-tuples \((p_1, \ldots, p_n)\) and \((q_1, \ldots, q_n)\) with \( \sum_{i=1}^{n} p_i = 1 \) and \( \sum_{i=1}^{n} q_i = 1 \)

\[
\min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left( f \left( \sum_{i=1}^{n} q_i x_i \right) - \sum_{i=1}^{n} q_i f(x_i) \right) \leq \left( f \left( \sum_{i=1}^{n} p_i x_i \right) - \sum_{i=1}^{n} p_i f(x_i) \right)
\]

\[
\leq \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left( f \left( \sum_{i=1}^{n} q_i x_i \right) - \sum_{i=1}^{n} q_i f(x_i) \right)
\]

for all \((x_1, \ldots, x_n) \in C^n\).

In this note, based on the idea of [3], we shall provide new bounds for the difference of the Davis-Choi-Jensen inequality. Among others, we show that if \( \Phi \) is a unital positive linear map and \( f \) is operator concave on an interval \([m, M]\), then

\[
f(\Phi(A)) - \Phi(f(A)) \leq 2 \left( f \left( \frac{m + M}{2} \right) - f\left( \frac{m}{2} \right) \right) I \]

for every selfadjoint operator \( A \) such that \( mI \leq A \leq MI \) for some scalars \( m < M \). Moreover, we discuss an external version of the Davis-Choi-Jensen inequality.

2. Davis-Choi-Jensen inequality. Let \( \Phi \) and \( \Psi \) be two positive linear maps from \( \mathcal{B}(H) \) to \( \mathcal{B}(K) \). \( \Phi \) is said to be \( \alpha \)-upper dominant by \( \Psi \) if there exists \( \alpha > 0 \) such that \( \alpha \Psi \geq \Phi \). Similarly \( \Phi \) is said to be \( \beta \)-lower dominant by \( \Psi \) if there exists \( \beta > 0 \) such that \( \Phi \geq \beta \Psi \). Moreover, \( \Phi \) is \( (\alpha, \beta) \)-dominant by \( \Psi \) if \( \Phi \) is \( \alpha \)-upper and \( \beta \)-lower dominant by \( \Psi \). The vector \((p_1, \ldots, p_n)\) is said to be a weight vector if \( p_i > 0 \) for all \( i = 1, \ldots, n \) and \( \sum_{i=1}^{n} p_i = 1 \). For example, we put two positive linear maps \( \Phi \) and \( \Psi : \mathcal{B}(H) \oplus \cdots \oplus \mathcal{B}(H) \mapsto \mathcal{B}(H) \) as follows:

\[
\Phi(A_1 \oplus \cdots \oplus A_n) = \sum_{i=1}^{n} p_i A_i \quad \text{and} \quad \Psi(A_1 \oplus \cdots \oplus A_n) = \sum_{i=1}^{n} q_i A_i,
\]

where \((p_1, \ldots, p_n)\) and \((q_1, \ldots, q_n)\) are weight vectors. If we put \( \alpha = \max_{1 \leq i \leq n} \{ \frac{p_i}{q_i} \} \) and \( \beta = \min_{1 \leq i \leq n} \{ \frac{p_i}{q_i} \} \), then it follows that \( \Phi \) is \( (\alpha, \beta) \)-dominant by \( \Psi \). In fact, we have

\[
(\alpha \Psi - \Phi)(A_1 \oplus \cdots \oplus A_n) = \sum_{i=1}^{n} (\alpha - \frac{p_i}{q_i}) q_i A_i
\]

and

\[
(\Phi - \beta \Psi)(A_1 \oplus \cdots \oplus A_n) = \sum_{i=1}^{n} (\frac{p_i}{q_i} - \beta) q_i A_i.
\]

Therefore, \( \alpha \Psi - \Phi \) and \( \Phi - \beta \Psi \) are positive linear maps.
Firstly, we provide the following estimates associated with the Davis-Choi-Jensen inequality for two unital positive linear maps:

**Theorem 2.1.** Let $\Phi$ and $\Psi$ be two unital positive linear maps from $B(H)$ to $B(K)$ such that $\Phi$ is $(\alpha, \beta)$-dominant by $\Psi$. If $f$ is operator concave on an interval $J$, then

\[
\beta (f(\Psi(A)) - \Psi(f(A))) \leq f(\Phi(A)) - \Phi(f(A)) \leq \alpha (f(\Psi(A)) - \Psi(f(A)))
\]

for every selfadjoint operator $A$ with spectrum $\sigma(A) \subset J$.

**Proof.** If we put $\Phi_0(X) = \frac{1}{\alpha}X$, then $\Phi_0$ is a positive linear map. Since $\Phi$ is $\alpha$-upper dominant by $\Psi$, we have $\Psi - \frac{1}{\alpha} \Phi$ is positive and $(\Psi - \frac{1}{\alpha} \Phi)(I) + \Phi_0(I) = I$. Therefore, by (1.1), we have

\[
f(\Psi(A)) = f(\Psi(A) - \frac{1}{\alpha} \Phi(A) + \frac{1}{\alpha} \Phi(A)) = f\left((\Psi - \frac{1}{\alpha} \Phi)(A) + \Phi_0(\Phi(A))\right)
\geq (\Psi - \frac{1}{\alpha} \Phi)(f(A)) + \Phi_0(f(\Phi(A)))
= \Psi(f(A)) - \frac{1}{\alpha} \Phi(f(A)) + \frac{1}{\alpha} f(\Phi(A))
\]

and this implies the second inequality of (2.1).

Similarly, if we put $\Phi_1(X) = \beta X$, then it follows that

\[
f(\Phi(A)) = f (\Phi(A) - \beta \Phi(A) + \beta \Psi(A)) = f ((\Phi - \beta \Psi)(A) + \Phi_1(\Psi(A)))
\geq (\Phi - \beta \Psi)(f(A)) + \Phi_1(f(\Psi(A))) = \Phi(f(A)) - \beta \Psi(f(A)) + \beta f(\Psi(A)).
\]

By Theorem 2.1, we have the following corollary as an operator concave version of [3, Theorem 1], see also [4].

**Corollary 2.2.** Let $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ be weight vectors. If $f$ is operator concave on an interval $J$, then

\[
\beta \left( f \left( \sum_{i=1}^{n} q_i A_i \right) - \sum_{i=1}^{n} q_i f(A_i) \right)
\leq f \left( \sum_{i=1}^{n} p_i A_i \right) - \sum_{i=1}^{n} p_i f(A_i)
\leq \alpha \left( f \left( \sum_{i=1}^{n} q_i A_i \right) - \sum_{i=1}^{n} q_i f(A_i) \right)
\]

for all selfadjoint operators $A_1, \ldots, A_n$ such that $\sigma(A_i) \subset J$ for all $i = 1, \ldots, n$, where $\alpha = \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\}$ and $\beta = \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\}$.

In particular,

\[
n \min_{1 \leq i \leq n} \left\{ p_i \right\} \left( f \left( \sum_{i=1}^{n} \frac{1}{p_i} A_i \right) - \sum_{i=1}^{n} \frac{1}{p_i} f(A_i) \right)
\]
Bounds for an Operator Concave Function

Let $\Phi$, $\Psi$, $\Phi'$ and $\Psi'$ be positive linear maps from $\mathcal{B}(H)$ to $\mathcal{B}(K)$ such that $\Phi(I) + \Psi(I) = I$ and $\Phi'(I) + \Psi'(I) = I$ and $\Phi$ is $(\alpha, \beta)$-dominant by $\Phi'$ and $\Psi$ is $(\alpha, \beta)$-dominant by $\Psi'$. If a real-valued function $f$ is operator concave on an interval $J$, then

$$
\beta \left( f(\Phi'(A)) + \Psi'(B) \right) - (\Phi'(f(A)) + \Psi'(f(B))) \\
\leq f(\Phi(A) + \Psi(B)) - (\Phi(f(A)) + \Psi(f(B))) \\
\leq \alpha \left( f(\Phi'(A) + \Psi'(B)) - (\Phi'(f(A)) + \Psi'(f(B))) \right)
$$

for all selfadjoint operators $A$ and $B$ with $\sigma(A)$, $\sigma(B)$, $\sigma(\Phi(A) + \Psi(B))$ and $\sigma(\Phi'(A) + \Psi'(B)) \subset J$.

Remark 2.4. Similarly, we have the following $n$-variable version of Corollary 3. Let $\{\Phi_i\}$ and $\{\Phi'_i\}$ be positive linear maps from $\mathcal{B}(H)$ to $\mathcal{B}(K)$ such that $\sum_{i=1}^n \Phi_i(I) = \sum_{i=1}^n \Phi'_i(I) = I$ and $\Phi_i$ is $(\alpha, \beta)$-dominant by $\Phi'_i$ for $i = 1, \ldots, n$. If a real-valued function $f$ is operator concave on an interval $J$, then

$$
\beta \left( \sum_{i=1}^n \Phi'_i(A_i) - \sum_{i=1}^n \Phi'_i(f(A_i)) \right) \leq \sum_{i=1}^n \Phi_i(A_i) - \sum_{i=1}^n \Phi_i(f(A_i)) \\
\leq \alpha \left( \sum_{i=1}^n \Phi'_i(A_i) - \sum_{i=1}^n \Phi'_i(f(A_i)) \right)
$$

for all selfadjoint operators $A$ and $B$ with $\sigma(A)$, $\sigma(B)$, $\sigma(\sum_{i=1}^n \Phi_i(A))$ and $\sigma(\sum_{i=1}^n \Phi'_i(A)) \subset J$.

In the case of a concave function, we have no relation between $\Phi(f(A))$ and $f(\Phi(A))$. Though we have the estimate of $[1, 2]$, we provide new bounds for the difference of the Davis-Choi-Jensen inequality by means of the difference of concavity.

Theorem 2.5. Let $\Phi$ be a unital positive linear map from $\mathcal{B}(H)$ to $\mathcal{B}(K)$. If a real-valued function $f(t)$ is concave on $[m, M]$, then

$$
- \frac{2}{M - m} \left( f \left( \frac{m + M}{2} \right) - \frac{f(m) + f(M)}{2} \right) \Phi(F(A)) \\
\leq f(\Phi(A)) - \Phi(f(A)) \leq \frac{2}{M - m} \left( f \left( \frac{m + M}{2} \right) - \frac{f(m) + f(M)}{2} \right) F(\Phi(A))
$$

for every selfadjoint operator $A$ such that $mM \leq A \leq MM$ for some scalars $m < M$.
where a real-valued function $F(t)$ on $[m,M]$ is defined by

$$F(t) = \frac{M - m}{2} + \left| t - \frac{M + m}{2} \right|.$$

**Proof.** Since $\Phi$ is a unital positive linear map and $f$ is concave on $[m,M]$, we have

$$\Phi(f(A)) \geq \Phi \left( \frac{f(M) - f(m)}{M - m} A + \frac{Mf(m) - mf(M)}{M - m} I \right) = \frac{f(M) - f(m)}{M - m} \Phi(A) + \frac{Mf(m) - mf(M)}{M - m} I.$$

On the other hand, it follows from (1.4) that

$$f(t) = \frac{f(M) - f(m)}{M - m} t + \frac{Mf(m) - mf(M)}{M - m}$$

$$= f \left( \frac{M - t}{M - m} + \frac{t - m}{M - m} M \right) - \frac{M - t}{M - m} f(m) + \frac{t - m}{M - m} f(M)$$

$$\leq \frac{2}{M - m} \max \{M - t, t - m\} \left( f \left( \frac{m + M}{2} \right) - f \left( \frac{m + M}{2} \right) \right)$$

$$= \frac{2}{M - m} \left( f(m) + f(M) - f \left( \frac{m + M}{2} \right) \right) F(t),$$

and this implies

$$f(\Phi(A)) - \Phi(f(A)) \leq \frac{2}{M - m} \left( f(m) + f(M) \right) - f \left( \frac{m + M}{2} \right) F(\Phi(A)).$$

For the first half of Theorem 2.5 we have

$$f(\Phi(A)) - \Phi(f(A)) \geq \frac{f(M) - f(m)}{M - m} \Phi(A) + \frac{Mf(m) - mf(M)}{M - m} I - \Phi(f(A))$$

$$= \Phi \left( \frac{f(M) - f(m)}{M - m} A + \frac{Mf(m) - mf(M)}{M - m} I - f(A) \right)$$

$$\geq - \frac{2}{M - m} \left( f(m) + f(M) \right) - f \left( \frac{m + M}{2} \right) \Phi(F(A)). \quad \square$$

Since $\frac{1}{M - m} F(\Phi(A)) \leq I$ in Theorem 2.5 we have an evaluation type by the middle point:

**Corollary 2.6.** Let $\Phi$, $f$ and $A$ be as in Theorem 2.5. Then

$$f(\Phi(A)) - \Phi(f(A)) \leq 2 \left( f \left( \frac{m + M}{2} \right) - f(m) + f(M) \right) I.$$
The following corollary is another expression of \[22\].

**Corollary 2.7.** Let \( \Phi, f \) and \( A \) be as in Theorem 2.5. Then

\[
- \left( \hat{f}_{\max} - \frac{f(m) + f(M)}{2} \right) I \leq f(\Phi(A)) - \Phi(f(A)) \leq \left( \hat{f}_{\max} - \frac{f(m) + f(M)}{2} \right) I,
\]

where \( \hat{f}(t) = f(t) - \frac{(f(M) - f(m))}{M - m} t + \frac{(M + m)(f(M) - f(m))}{2(M - m)} \) and \( \hat{f}_{\max} = \max \{ \hat{f}(t) : m \leq t \leq M \} \).

**Proof.** Since

\[
\hat{f}(t) = f(t) - \frac{(f(M) - f(m))}{M - m} t - \frac{M f(m) - m f(M)}{M - m} = \frac{f(m) + f(M)}{2} - \frac{f(m) + f(M)}{2} - \frac{M f(m) - m f(M)}{M - m} \leq \hat{f}_{\max} - \frac{f(m) + f(M)}{2},
\]

it follows from the concavity of \( f \) that

\[
f(\Phi(A)) - \Phi(f(A)) \leq f(\Phi(A)) - \frac{f(M) - f(m)}{M - m} \Phi(A) - \frac{M f(m) - m f(M)}{M - m} I \leq \left( \hat{f}_{\max} - \frac{f(m) + f(M)}{2} \right) I.
\]

On the other hand, by the Stinespring decomposition theorem \[10\], \( \Phi \) restricted to a \( C^* \)-algebra \( C^*(A) \) generated by \( A \) and \( I \) admits a decomposition \( \Phi(X) = C^* \phi(X)C \) for all \( X \in C^*(A) \), where \( \phi \) is a \( * \)-representation of \( C^*(A) \subset B(H) \) and \( C \) is a bounded linear operator from \( K \) to a Hilbert space \( K' \). Since \( \Phi \) is unital, we have \( C^*C = I \).

For every unit vector \( x \in K \),

\[
(f(\Phi(A))x, x) - \langle \Phi(f(A))x, x \rangle = (f(C^* \phi(A)C)Cx, x) - \langle C^* \phi(f(A))Cx, x \rangle = \langle (f(\phi(A))Cx, Cx) - \langle f(C^* \phi(A)C)x, x \rangle \leq f((\phi(A)Cx, Cx)) - \langle f(C^* \phi(A)C)x, x \rangle \leq f(C^* \phi(A)Cx, x) - \frac{f(M) - f(m)}{M - m} (C^* \phi(A)Cx, x) - \frac{M f(m) - m f(M)}{M - m} \leq \hat{f}_{\max} - \frac{f(m) + f(M)}{2}.
\]

Therefore, we have

\[
\Phi(f(A)) - f(\Phi(A)) \leq \left( \hat{f}_{\max} - \frac{f(m) + f(M)}{2} \right) I
\]

and this implies the first half part of the desired inequality. \( \square \)
3. External version of Davis-Choi-Jensen inequality. In this section, we consider bounds of operator concavity in terms of an external formula. A real-valued continuous function \( f \) on \( J \) is operator concave if and only if

\[
(f((1 + p)A - pB) \leq (1 + p)f(A) - pf(B)
\]

for all \( p > 0 \) and all selfadjoint operators \( A \) and \( B \) with \( \sigma(A), \sigma(B) \) and \( \sigma((1 + p)A - pB) \subset J \). Then we have the following external version of the Jensen operator inequality: If \( f \) is operator concave, then

\[
f \left( (1 + \sum_{i=1}^{n} p_i)A - \sum_{i=1}^{n} p_iB_i \right) \leq (1 + \sum_{i=1}^{n} p_i)f(A) - \sum_{i=1}^{n} p_i f(B_i)
\]

for all selfadjoint operators \( A \) and \( B_i \ (i = 1, \ldots, n) \) with \( \sigma(A), \sigma(B_i) \) and \( \sigma((1 + \sum_{i=1}^{n} p_i)A - \sum_{i=1}^{n} p_iB_i) \subset J \), also see [5, 9].

For a real-valued continuous function \( f \), we define the following notation

\[
A \sigma_f B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) A^{\frac{1}{2}}
\]

for positive invertible \( A \) and selfadjoint \( B \), also see [8].

Let \( \Phi \) be a positive linear map from \( B(H) \) to \( B(K) \). In \( [6] \), we show the following external version of the Davis-Choi-Jensen inequality: Let \( f \) be a real-valued continuous function on an interval \( J \). Then \( f \) is operator concave if and only if

\[
f(\Phi(A) - \Psi(B)) \leq \Phi(I) \sigma_f \Phi(A) - \Psi(f(B))
\]

for all positive linear maps \( \Phi, \Psi \) such that \( \Phi(I) - \Psi(I) = I \) and for all selfadjoint operators \( A \) and \( B \) with \( \sigma(A), \sigma(B) \) and \( \sigma(\Phi(A) - \Psi(B)) \subset J \). The invertibility of \( \Phi(I) \) guarantees the formulation of (3.3). In this case, we have

\[
\Phi(f(A)) \leq \Phi(I) \sigma_f \Phi(A).
\]

In fact, in the Stinespring decomposition theorem \( \Phi(X) = C^* \phi(X)C \), we have the polar decomposition \( C = V|C| \) such that \( |C| \) is invertible, because \( C^*C = \Phi(I) = I + \Psi(I) > 0 \). Since \( V^*V = I \), it follows that

\[
\Phi(f(A)) = |C|V^* f(\phi(A))V|C| = |C|f(V^* \phi(A)V)|C| = |C|f(|C|^{-1}C^* \phi(A)C|C|^{-1})|C| = \Phi(I) \sigma_f \Phi(A).
\]

If moreover \( C \) is invertible, then we have \( \Phi(f(A)) = \Phi(I) \sigma_f \Phi(A) \), and hence,

\[
f(\Phi(A) - \Psi(B)) \leq \Phi(f(A)) - \Psi(f(B)),
\]

see [5].
Based on the external version (3.4) of the Davis-Choi-Jensen inequality, we have the following bounds for the difference of the operator concavity.

**Theorem 3.1.** Let $\Phi, \Psi, \Phi'$ and $\Psi'$ be positive linear maps from $\mathcal{B}(H)$ to $\mathcal{B}(K)$ such that $\Phi(I) - \Psi(I) = I$ and $\Phi'(I) - \Psi'(I) = I$ and $\Phi$ is $(\beta, \alpha)$-dominant by $\Phi'$ and $\Psi$ is $(\alpha, \beta)$-dominant by $\Psi'$. If a real-valued function $f$ is operator concave on an interval $J$, then

$$
\beta (\Phi'(f(A)) - \Psi'(f(B))) - f(\Phi'(A) - \Psi'(B)) \\
\leq \Phi(f(A)) - \Psi(f(B)) - f(\Phi(A) - \Psi(B)) \\
\leq \alpha (\Phi(f(A)) - \Psi(f(B)) - f(\Phi'(A) - \Psi'(B)))
$$

for all selfadjoint operators $A$ and $B$ with $\sigma(A), \sigma(B), \sigma(\Phi(A) - \Psi(B))$ and $\sigma(\Phi'(A) - \Psi'(B)) \subset J$.

**Proof.** Put $\Phi_1(X) = \alpha X$. Since $\Phi$ is $\alpha$-lower dominant by $\Phi'$ and $\Psi$ is $\alpha$-upper dominant by $\Psi'$, and $(\Phi - \alpha \Phi')(I) + (\alpha \Psi' - \Psi)(I) + \Phi_1(I) = I$, it follows from the operator concavity of $f$ that

$$
f(\Phi(A) - \Psi(B)) = f(\Phi - \alpha \Phi')(A) + (\alpha \Psi' - \Psi)(B) + \Phi_1(\Phi'(A) - \Psi'(B)) \\
\geq (\Phi - \alpha \Phi')(f(A)) + (\alpha \Psi' - \Psi)(f(B)) + \Phi_1(f(\Phi'(A) - \Psi'(B))) \\
= \Phi(f(A)) - \Psi(f(B)) - \alpha (f(\Phi'(A) - \Psi'(B)) - (\Psi'(f(A)) - \Psi'(f(B)))
$$

for all selfadjoint operators $A$ and $B$ with $\sigma(A), \sigma(B), \sigma(\Phi(A) - \Psi(B))$ and $\sigma(\Phi'(A) - \Psi'(B)) \subset J$. This fact implies the second half of Theorem 3.1. Similarly, we obtain the first half of Theorem 3.1. □

Finally, we show an application of Theorem 3.1. Put positive linear maps $\Phi, \Psi, \Phi'$ and $\Psi'$ : $\mathcal{B}(H) \oplus \cdots \oplus \mathcal{B}(H) \mapsto \mathcal{B}(H)$ as follows:

$$
\Phi(A_1 \oplus \cdots \oplus A_n) = \sum_{i=1}^{n} \frac{1 + \sum_{i=1}^{n} p_i}{n} A_i \quad \text{and} \quad \Psi(A_1 \oplus \cdots \oplus A_n) = \sum_{i=1}^{n} p_i A_i,
$$

$$
\Phi'(A_1 \oplus \cdots \oplus A_n) = \sum_{i=1}^{n} \frac{1 + \sum_{i=1}^{n} q_i}{n} A_i \quad \text{and} \quad \Psi'(A_1 \oplus \cdots \oplus A_n) = \sum_{i=1}^{n} q_i A_i,
$$

where $p_i, q_i > 0$ for $i = 1, \ldots, n$. Then it follows that $\Phi(I) - \Psi(I) = I$ and $\Phi'(I) - \Psi'(I) = I$. If we put $\alpha = \max\{\frac{p_i}{n}\}$ and $\frac{1 + \sum_{i=1}^{n} p_i}{1 + \sum_{i=1}^{n} q_i} > \alpha$, then $\Phi$ is $\alpha$-lower dominant of $\Phi'$ and $\Psi$ is $\alpha$-upper dominant of $\Psi'$. If we put $\beta = \min\{\frac{q_i}{n}\}$ and $\frac{1 + \sum_{i=1}^{n} p_i}{1 + \sum_{i=1}^{n} q_i} < \beta$, then $\Phi$ is $\beta$-upper dominant of $\Phi'$ and $\Psi$ is $\beta$-lower dominant of $\Psi'$. Hence, by Theorem 3.1 we obtain the following external version of Corollary 2.2

**Corollary 3.2.** Let $f$ be operator convex on an interval $J$ and $A$ and $B_i$ ($i = 1, \ldots, n$) selfadjoint operators with $\sigma(A), \sigma(B_i)$ and $\sigma((1 + \sum_{i=1}^{n} p_i)A - \sum_{i=1}^{n} p_i B_i)$
J. Let $\alpha = \max\{\frac{p_i}{q_i}\}$ and $\beta = \min\{\frac{p_i}{q_i}\}$. If $\beta > \frac{1 + \sum_{i=1}^{n} p_i}{1 + \sum_{i=1}^{n} q_i}$, then

$$\beta \left( f \left( 1 + \sum_{i=1}^{n} q_i \right) A - \sum_{i=1}^{n} q_i B_i \right) - \left( 1 + \sum_{i=1}^{n} q_i \right) f(A) - \sum_{i=1}^{n} q_i f(B_i)$$

$$\leq f \left( 1 + \sum_{i=1}^{n} p_i \right) A - \sum_{i=1}^{n} p_i B_i - \left( 1 + \sum_{i=1}^{n} p_i \right) f(A) - \sum_{i=1}^{n} p_i f(B_i)$$

and if $\frac{1 + \sum_{i=1}^{n} p_i}{1 + \sum_{i=1}^{n} q_i} > \alpha$, then

$$f \left( 1 + \sum_{i=1}^{n} p_i \right) A - \sum_{i=1}^{n} p_i B_i - \left( 1 + \sum_{i=1}^{n} p_i \right) f(A) - \sum_{i=1}^{n} p_i f(B_i)$$

$$\leq f \left( 1 + \sum_{i=1}^{n} q_i \right) A - \sum_{i=1}^{n} q_i B_i - \left( 1 + \sum_{i=1}^{n} q_i \right) f(A) - \sum_{i=1}^{n} q_i f(B_i)$$.

REFERENCES