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ON THE MAIN SIGNLESS LAPLACIAN EIGENVALUES
OF A GRAPH∗

HANYUAN DENG† AND HE HUANG†

Abstract. A signless Laplacian eigenvalue of a graph G is called a main signless Laplacian
eigenvalue if it has an eigenvector the sum of whose entries is not equal to zero. In this paper, some
necessary and sufficient conditions for a graph with one main signless Laplacian eigenvalue or two
main signless Laplacian eigenvalues are given. And the trees and unicyclic graphs with exactly two
main signless Laplacian eigenvalues are characterized, respectively.

Key words. Signless Laplacian eigenvalue, Main eigenvalue, Tree, Unicyclic graph.

AMS subject classifications. 05C50.

1. Introduction. Let $M$ be a square matrix of order $n$. An eigenvalue $\lambda$ of $M$
is said to be a main eigenvalue if the eigenspace $\mathcal{E}(\lambda)$ of $\lambda$ is not orthogonal to the
all-1 vector $j$, i.e., if it has an eigenvector the sum of whose entries is not equal to zero. An eigenvector $x$ is a main eigenvector if $x^Tj \neq 0$. Specially, if $M = A$ is
the $(0, 1)$-adjacency matrix of a graph $G$, then the main eigenvalues of $A$ are said
to be main eigenvalues of $G$. A graph with exactly one main eigenvalue is regular.
Cvetković [5] proposed the problem of characterizing graphs with exactly $k$ main
eigenvalues, $k > 1$. Hagos [9] gave a characterization of graphs with exactly two main
eigenvalues. Recently, Hou and Zhou [11] characterized the tree with exactly two
main eigenvalues. Hou and Tian [10] determined all connected unicyclic graphs with
exactly two main eigenvalues. Zhu and Hu [12] characterized all connected bicyclic
graphs with exactly two main eigenvalues. Rowlinson [13] surveyed results relating
main eigenvalues and main angles to the structure of a graph, and discussed graphs
with just two main eigenvalues in the context of measures of irregularity and in the
context of harmonic graphs.

In this paper, we assume that $G$ is a simple connected graph, and consider the
main eigenvalues of the signless Laplacian matrix $Q$ of $G$, where $Q = D + A$ and $D$
is the diagonal matrix of vertex degrees. The main eigenvalues of $Q$ are said to be

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the main signless Laplacian eigenvalues of $G$. The signless Laplacian appears very rarely in published papers before 2003. Recently, the signless Laplacian has attracted the attention of researchers, see, e.g. \cite{1,2,3,6,7}. Here, we obtain necessary and sufficient conditions for a graph with one main signless Laplacian eigenvalue or two main signless Laplacian eigenvalues, and then characterize the trees and unicyclic graphs with exactly two main signless Laplacian eigenvalues, respectively.

2. The graphs with one or two main signless Laplacian eigenvalues. In this section, we show that a graph with exactly one main signless Laplacian eigenvalue is regular, and give a characterization of graphs with exactly two main signless Laplacian eigenvalues.

Note that if $G$ is a simple connected graph with signless Laplacian matrix $Q$, then there is an eigenvector $x > 0$ of the largest eigenvalue $\mu_1$ of $Q$ such that $Qx = \mu_1 x$, and $x^T j \neq 0$ by the Perron-Frobenius theorem. This shows that the largest eigenvalue $\mu_1$ of $Q$ is a main signless Laplacian eigenvalue. So, $G$ has at least one main signless Laplacian eigenvalue.

The following result gives a characterization of graphs with exactly one main signless Laplacian eigenvalue.

**Theorem 2.1.** A graph $G$ with exactly one main signless Laplacian eigenvalue if and only if $G$ is regular.

**Proof.** If $G$ is $k$-regular, then $Qj = 2kj$. This shows that $\mu_1 = 2k$ is an eigenvalue of $Q$ with an eigenvector $j$. Since $Q$ is a non-negative irreducible symmetric matrix, $\mu_1 = 2k$ is the largest eigenvalue of $Q$ with the multiplicity 1 by the Perron-Frobenius theorem. And the eigenvectors of other eigenvalues of $Q$ are orthogonal with $j$. So, $Q$ has exactly one main eigenvalue.

If $G$ has exactly one main signless Laplacian eigenvalue, then the largest eigenvalue $\mu_1$ is the unique main eigenvalue of $Q$. Let $\xi$ be an eigenvector of $\mu_1$, $V_1 = \varepsilon(\mu_1)$ be the eigenspace of $\mu_1$. Then $V_1$ is the space spanning by $\xi$. If $V_2$ is the space spanning by eigenvectors of all eigenvalues of $Q$ different from $\mu_1$, and $V_3$ is the space spanning by $j$, then $\text{dim}(V_2) = n - 1$ and $\text{dim}(V_3) = 1$. Since $Q$ is a real symmetric matrix, $V_1$ is the orthogonal complement of $V_2$. And $V_3$ is also the orthogonal complement of $V_2$ since $\mu_1$ is the unique main eigenvalue of $Q$. So, $V_1 = V_3$, and $\xi = aj$ for some real $a \neq 0$. From $Q\xi = \mu_1 \xi$, the row sums of $Q$ are equal, and $G$ is regular. $\square$

Now, we discuss the characterization of graphs with exactly two main signless Laplacian eigenvalues.

For any positive semi-definite matrix $M$ of order $n$, all its eigenvalues are non-negative. Let $\mu_1 > \mu_2 > \cdots > \mu_r$ be the eigenvalues of $M$ with multiplicities
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Let \( Q_i = PE_iP^T \). Then

\[
Q_i j = PE_iP^T j = P_i P^T j,
\]

and \( M \) has the spectral decomposition

\[
M = \mu_1 Q_1 + \mu_2 Q_2 + \cdots + \mu_r Q_r,
\]

where

\[
Q_i Q_j = \begin{cases} 0 & i \neq j; \\ Q_i & i = j. \end{cases}
\]

And, for any polynomial \( f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, \)

\[
f(M) &= a_0(M)^n + a_1(M)^{n-1} + \cdots + a_{n-1}M + a_n I \\
&= a_0 \sum_{i=1}^{r} \mu_i^n Q_i + a_1 \sum_{i=1}^{r} \mu_i^{n-1} Q_i + \cdots \\
&\quad + a_{n-1} \sum_{i=1}^{r} \mu_i Q_i + a_n \sum_{i=1}^{r} Q_i \\
&= \sum_{i=1}^{r} f(\mu_i) Q_i.
\]

**Lemma 2.2.** Let \( \mu_1, \mu_2, \ldots, \mu_t \) \((1 \leq t \leq r)\) be the main eigenvalues of a positive semi-definite matrix \( M \) of order \( n \), and \( m(x) = (x - \mu_1)(x - \mu_2)\cdots(x - \mu_t) \).

(i) \( m(M) j = 0 \);
(ii) If \( f(x) \) is a polynomial with real coefficients and \( f(M)j = 0 \), then \( m(x)|f(x) \).

Proof. (i) From (2.2), we know that
\[
m(M)j = \sum_{i=1}^{r} m(\mu_i)Q_i j = \sum_{i=t+1}^{r} m(\mu_i)Q_i j.
\]
Since \( \mu_{t+1}, \ldots, \mu_r \) are not the main eigenvalues of \( M \), and from (2.1), \( Q_i j = 0 \) for \( i=t+1, \ldots, r \). So, \( m(M)j = \sum_{i=t+1}^{r} m(\mu_i)Q_i j = 0 \).

(ii) From (2.2) and \( Q_i j = 0 \), for \( i=t+1, \ldots, r \),
\[
f(M)j = \sum_{i=1}^{r} f(\mu_i)Q_i j = \sum_{i=1}^{t} f(\mu_i)Q_i j.
\]
Since \( f(M)j = 0 \), \( \sum_{i=1}^{t} f(\mu_i)Q_i j = 0 \). For \( k = 1, 2, \ldots, t \), we have
\[
Q_k(\sum_{i=1}^{t} f(\mu_i)Q_i j) = f(\mu_k)Q_k j = 0.
\]
So, \( f(\mu_k) = 0 \) for \( k = 1, 2, \ldots, t \) and \( m(x)|f(x) \).

A number \( \alpha \) is an algebraic integer if there is a monic polynomial \( f(x) \) with integral coefficients such that \( f(\alpha) = 0 \).

**Lemma 2.3.** [8] An \( \alpha \in \mathbb{Q} \) is an algebraic integer if and only if \( \alpha \) is an integer.

**Lemma 2.4.** [8] If \( \alpha \) and \( \beta \) are algebraic integers, then \( \alpha \pm \beta \) and \( \alpha \beta \) are also algebraic integers.

**Theorem 2.5.** Let \( G \) be non-regular. Then \( G \) has exactly two main signless Laplacian eigenvalues \( \mu_1 \) and \( \mu_2 \) if and only if \( (Q - \mu_1 I)(Q - \mu_2 I)j = 0 \).

Proof. Let \( \mu_1, \ldots, \mu_t \) be the main eigenvalues of \( Q \), and \( m(x) = (x - \mu_1) \cdots (x - \mu_t) \).

If \( (Q - \mu_1 I)(Q - \mu_2 I)j = 0 \), then \( f(Q)j = 0 \) for \( f(x) = (x - \mu_1)(x - \mu_2) \), and \( m(x)|f(x) \) by Lemma 2.2. So, \( m(x) = (x - \mu_1)(x - \mu_2) \) or \( (x - \mu_1) \) or \( (x - \mu_2) \), and \( t \leq 2 \). But \( G \) is non-regular, \( t = 2 \) from Theorem 2.1.

If \( G \) has exactly two main signless Laplacian eigenvalues \( \mu_1 \) and \( \mu_2 \), then \( (Q - \mu_1 I)(Q - \mu_2 I)j = 0 \) from Lemma 2.2. [8]

In the following, we give an alternative characterization of graphs with exactly two main signless Laplacian eigenvalues.
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In order to find all graphs with exactly two main eigenvalues, Hou and Tian [10] introduced a 2-walk \((a, b)\)-linear graph. For a graph \(G\), the degree of vertex \(v\) is denoted by \(d(v)\), the number of walks of length 2 of \(G\) starting at \(v\) is \(s(v) = \sum_{u \in N_G(v)} d(u)\), i.e., the sum of the degrees of the vertices adjacent to \(v\), where \(N_G(v)\) is the set of all neighbors of \(v\) in \(G\). A graph \(G\) is called 2-walk \((a, b)\)-linear if there exist unique integer numbers \(a, b\) with \(a^2 - 4b > 0\) such that \(s(v) = ad(v) + b\) holds for every vertex \(v \in V(G)\). Hagos [9] showed that a graph \(G\) has exactly two main eigenvalues if and only if \(G\) is 2-walk linear.

Like a 2-walk \((a, b)\)-linear graph, we define a 2-walk \((a, b)\)-parabolic graph. A graph \(G\) is called 2-walk \((a, b)\)-parabolic if there are uniquely a positive integer \(a\) and a non-negative integer \(b\) with \(a^2 - 8b > 0\) such that \(s(v) = -d^2(v) + ad(v) - b\) holds for every vertex \(v \in V(G)\).

**Theorem 2.6.** A graph \(G\) has exactly two main signless Laplacian eigenvalues if and only if \(G\) is a 2-walk \((a, b)\)-parabolic graph.

**Proof.** If \(G\) is a 2-walk \((a, b)\)-parabolic graph, then there are uniquely a positive integer \(a\) and a non-negative integer \(b\) such that \(a^2 - 8b > 0\) and \(s(v) = -d^2(v) + ad(v) - b\) for any \(v \in V(G) = \{v_1, v_2, \ldots, v_n\}\). So, \(s(v_i) + d^2(v_i) - ad(v_i) + b = 0\), and

\[
\begin{align*}
\frac{1}{2}(A + D)^2 j - aA j + bj &= 0, \\
\frac{1}{2}Q^2 j - \frac{1}{2}aL^+ j + bj &= 0, \\
Q^2 j - aQ j + 2bj &= 0.
\end{align*}
\]

Let \(f(x) = x^2 - ax + 2b\). Then \(f(Q)j = 0\), and \(f(x) = 0\) has two real roots since \(a^2 - 8b > 0\). Moreover, \(G\) is non-regular since one has \(s(v) = -d^2(v) + 2kd(v) - 0\) and \(s(v) = -d^2(v) + (2k + 1)d(v) - k\) for a \(k\)-regular graph, i.e., \((a, b) = (2k, 0)\) or \((2k + 1, k)\) is not unique. From Theorem 2.5, \(G\) has exactly two main signless Laplacian eigenvalues.

On the other hand, if \(G\) has exactly two main signless Laplacian eigenvalues \(\mu_1\) and \(\mu_2\), then by Theorem 2.5,

\[(Q^2 - (\mu_1 + \mu_2)Q + \mu_1\mu_2 I)j = 0,
\]
i.e.,

\[(D + A)^2 j - (\mu_1 + \mu_2)(D + A)j + \mu_1\mu_2 j = 0.
\]

So, \(d^2(v) + s(v) - (\mu_1 + \mu_2)d(v) + \frac{\mu_1 + \mu_2}{2} = 0\) for all \(v \in V(G)\). Let \(\mu_1 + \mu_2 = a\) and \(\mu_1\mu_2 = 2b\), then \(s(v) = -d^2(v) + ad(v) - b\), and \(a > 0, b \geq 0\) and \(a^2 - 8b > 0\) since \(\mu_1 \neq \mu_2\) are the eigenvalues of the positive semi-definite matrix \(Q\). Note that \(G\) is...
non-regular by Theorem 2.1, there are \( u, v \in V(G) \) such that \( d(u) \neq d(v) \). From \( s(u) = -d^2(u) + ad(u) - b \) and \( s(v) = -d^2(v) + ad(v) - b \), we have

\[
a = \frac{s(u) - s(v)}{d(u) - d(v)} + d(u) + d(v),
\]

(2.3)

\[
b = \frac{s(u) - s(v)}{d(u) - d(v)} d(v) + d(u)d(v) - s(v),
\]

and \( a, b \) are rational numbers and unique. Because \( \mu_1, \mu_2 \) are the roots of monic polynomial \( \det(\lambda I - Q) = 0 \) with integral coefficients, \( \mu_1, \mu_2 \) are algebraic integers.

By Lemmas 2.3 and 2.4, \( a, b \) are integers.

3. Trees with exactly two main signless Laplacian eigenvalues. In this section, we determine all trees with exactly two main signless Laplacian eigenvalues.

Let \( G = (V, E) \) be a tree with \( n \geq 3 \) vertices and the maximum degree \( \Delta \). If \( G \) has exactly two main signless Laplacian eigenvalues, then from Theorem 2.6, there exist uniquely a positive integer \( a \) and a non-negative integer \( b \) such that such that

\[
a^2 - 8b > 0 \quad \text{and} \quad s(v) = -d^2(v) + ad(v) - b
\]

(3.1)

for any \( v \in V(G) \).

Case 1. \( b = 0 \).

Let \( v_1 \in V \) with degree \( d(v_1) = 1 \), and \( v_2 \) is its unique adjacent vertex. Then \( d(v_2) = s(v_1) = -1 + a \) by (3.1), and

\[
a = d(v_2) + 1 \leq \Delta + 1.
\]

(3.2)

Let \( v_0 \in V \) with degree \( d(v_0) = \Delta \), then \( \Delta = d(v_0) \leq s(v_0) = -d^2(v_0) + ad(v_0) = -\Delta^2 + a\Delta \), and \( \Delta \leq a - 1 \), i.e.,

\[
a \geq \Delta + 1
\]

(3.3)

with equality if and only if \( G \) is a star with the center \( v_0 \). From (3.2) and (3.3), we have \( a = \Delta + 1 \). So, \( G = S_n \) is a star.

Case 2. \( b = 1 \).

Let \( P_k = v_1v_2\cdots v_k \) is a longest path of \( G \). Then \( d(v_2) = s(v_1) = -1 + a - 1 = a - 2 \) by (3.1), and

\[
s(v_2) = -d^2(v_2) + ad(v_2) - 1 = -a^2 + a(a - 2) - 1 = 2a - 5.
\]
By (3.1), we have
\[
d(v_3) = s(v_2) - d(v_2) + 1
= (2a - 5) - (a - 2) + 1
= a - 2 = d(v_2).
\]
By (3.1), we have \( s(v_3) = s(v_2) = 2a - 5 = d(v_2) + d(v_3) - 1 \). This shows that the adjacent vertices of \( v_3 \) are pendant vertices except \( v_2 \). So, \( G = S_{\frac{a+b}{2}} \) is a double star.

**Case 3.** \( b \geq 2 \).

Let \( P_k = v_1v_2\cdots v_k \) is a longest path of \( G \), then \( d(v_2) \geq 2 \). By (3.1), \( d(v_2) = s(v_1) = -1 + a - b \), and \( a - b \geq 3 \).

Since \( P_k = v_1v_2\cdots v_k \) is a longest path of \( G \), the adjacent vertices of \( v_2 \) are pendant vertices except \( v_3 \).

\[
d(v_3) = s(v_2) - (d(v_2) - 1)
= (-a - b - 1)^2 + a(a - b - 1) - (d(v_2) - 1)
= (-a - b - 1)^2 + a(a - b - 1) - (a - b - 2)
= ab - b^2 - 2b + 1
\]

and

\[
d(v_3) - d(v_2) = (ab - b^2 - 2b + 1) - (a - b - 1)
= ab - b^2 - b + a - 2
= (b - 1)(a - b - 2) > 0.
\]

So, \( d(v_3) > d(v_2) = a - b - 1 \geq 2 \). And no pendant vertex is adjacent to \( v_3 \); Otherwise, let \( u \) be a pendant vertex adjacent to \( v_3 \). Then \( d(v_3) = s(u) = -1 + a - b \) by (3.1), contradicting with \( d(v_3) > a - b - 1 \).

For any \( x \in N_G(v_3) \setminus \{v_2, v_4\} \), since \( x \) is not a pendant vertex, there is \( y \in V(G) \setminus \{v_3\} \) such that \( xy \in E(G) \), and \( y \) is a pendant vertex by the longest path \( P_k = v_1v_2\cdots v_k \). Then

\[
d(x) = s(y) = -1 + a - b = d(v_2), \quad \forall x \in N_G(v_3) \setminus \{v_2, v_4\}.
\]

So,

\[
s(v_3) = \sum_{x \in N_G(v_3)} d(x) = \sum_{x \in N_G(v_3) \setminus \{v_2, v_4\}} d(x) + d(v_2) + d(v_4)
= (d(v_3) - 2)d(v_2) + d(v_2) + d(v_4)
\]

and

\[
d(v_4) = s(v_3) - (d(v_3) - 1)d(v_2).
\]
By (3.1), \( s(v_3) = -d^2(v_3) + ad(v_3) - b \). Note that \( d(v_2) = a - b - 1 \) and \( d(v_4) = ab - b^2 - 2b + 1 \),

\[
\begin{align*}
    d(v_4) &= -d^2(v_3) + ad(v_3) - b - (d(v_3) - 1)d(v_2) \\
    &= d(v_3)(-d(v_3) + a - 3) - b + d(v_2) \\
    &= d(v_3)(-ab + b^2 + 3b) + a - 2b - 1 \\
    &= d(v_3)b(b - a + 3) + (a - 2b - 1).
\end{align*}
\]

If \( a - b = 3 \), then \( d(v_4) = a - 2b - 1 = 2 - b \leq 0 \). If \( a - b \geq 4 \), then \( d(v_4) = d(v_3)(b - a + 4) - bd(v_3) + (a - 2b - 1) \leq -bd(v_3) + (a - 2b - 1) = -b(ab - b^2 - 2b + 1) + (a - 2b - 1) = -a(b^2 - 1) + b^2 + 2b^2 - 3b - 1 \leq -(b + 4)(b^2 - 1) + b^3 + 2b^2 - 3b - 1 = -2b^2 - 2b + 3 < 0 \).

This is impossible.

On the other hand, it is easy to check that \( G = S_n \) and \( G = S_{\frac{a+2}{2}, \frac{b}{2}} \) are 2-walk \((n,0)\)-parabolic graph and \((\frac{a}{2}, 2, 1)\)-parabolic graph, respectively.

From above, we have

**Theorem 3.1.** A tree with \( n \geq 3 \) vertices has exactly two main signless Laplacian eigenvalues if and only if \( G \) is the star \( S_n \) or the double star \( S_{\frac{a+2}{2}, \frac{b}{2}} \).

It was showed in [11] that the trees with \( n \geq 3 \) vertices has exactly two main eigenvalues (of adjacent matrix) are \( S_n, S_{\frac{a+2}{2}, \frac{b}{2}} \) and \( T_a \). But from Theorem 3.1, we know that \( T_a \) is not a tree with exactly two main signless Laplacian eigenvalues, where \( T_a \) (\( a \geq 2 \)) is defined in [11] to be the tree with one vertex \( v \) of degree \( a^2 - a + 1 \) while every neighbor of \( v \) has degree \( a \) and all remaining vertices are pendant.

**4. Unicyclic graphs with exactly two main signless Laplacian eigenvalues.** In this section, we determine all unicyclic graphs with exactly two main signless Laplacian eigenvalues.

The unique unicyclic graph with \( n \) vertices and the minimum degree \( \delta \geq 2 \) is the cycle \( C_n \), and it is regular. By Theorem 2.1, it has exactly one main signless Laplacian eigenvalues. So, we only need to consider the unicyclic graphs with the minimum degree \( \delta = 1 \).

**Remark 4.1.** If \( G \) is a 2-walk \((a, b)\)-parabolic graph with \( \delta(G) = 1 \), then \( a - b \geq 3 \) since there is a pendent vertex \( x \) with the only incident edge \( xy \) in \( G \) and \( d(y) = s(x) = -1 + a - b \geq 2 \).

Let \( \mathcal{G}_{a,b} = \{ G : G \) is a 2-walk \((a, b)\)-parabolic unicyclic graph with \( \delta(G) = 1 \}\), and for each \( G \in \mathcal{G}_{a,b} \), let \( G_0 \) be the graph obtained from \( G \) by deleting all pendent vertices. If \( v \in V(G_0) \), we use \( d_{G_0}(v) \) to denote the degree of the vertex \( v \) in \( G_0 \).

**Lemma 4.2.** If \( G \in \mathcal{G}_{a,b} \) and \( v \in V(G_0) \), then \( d(v) = d_{G_0}(v) \) or \( d(v) = a - b - 1 \).
Proof. If there is a pendant $x$ adjacent to $v$ in $G$, then $d(v) = s(x) = a - b - 1$ by (3.1). Otherwise, $d(v) = d_{G_0}(v)$. \qed

Lemma 4.3. If $G \in G_{a,b}$, then (i) $\delta(G_0) \geq 2$; (ii) $a - b \geq 4$ and $a \geq 5$.

Proof. (i) If $\delta(G_0) = 1$, then there is a $y \in V(G_0)$ such that $d_{G_0}(y) = 1$, and there must exist a pendant vertex $x$ adjacent to $y$ in $G$. By (3.1), $d(y) = s(x) = -1 + a - b$, this shows that there are $a - b - 2$ pendant vertices adjacent to $y$ in $G$. Let $z$ be the unique non-pendant vertex adjacent to $y$ in $G$, then $s(y) = \sum_{w \in N_G(y)} d(w) = d(z) + (a - b - 2)$. By (3.1), we know

$$s(y) = -d^2(y) + ad(y) - b = -(a - b - 1)^2 + a(a - b - 1) - b$$

and $d(z) = s(y) - (a - b - 2) = -(a - b - 1)^2 + a(a - b - 1) - b - (a - b - 2)$, i.e.,

$$d(z) = ab - b^2 - 2b + 1.$$ \hfill (4.1)

So,

$$d(z) - d(y) = (ab - b^2 - 2b + 1) - (a - b - 1) = (b - 1)(a - b - 2).$$ \hfill (4.2)

(II) If $b = 1$, then by (4.1) yields $d(z) = 1$. This is impossible since $z$ is a non-pendant vertex adjacent to $y$ in $G$.

(II) If $b = 1$, then by (4.1) and (4.2), $d(z) = a - 2$ and $d(y) = d(z) = a - 2$.

From (3.1), $s(z) = -(a - 2)^2 + a(a - 2) - 1 = 2a - 5 = d(y) + d(z) - 1$. This shows that all the vertices adjacent to $z$, except $y$, are pendant vertices. So, $G$ is a double star with the centers $z$ and $y$. This is impossible since $G \in G_{a,b}$.

(III) If $b \geq 2$, then no pendant vertex is adjacent to $y$ in $G$; Otherwise, $d(z) = s(u) = -1 + a - b \geq 2$, where $u$ is a pendant vertex is adjacent to $z$. This implies that $d(z) = d(y)$ and $a - b \geq 3$, contradicting with (4.2).

By (3.1), we have $s(z) = -d^2(z) + ad(z) - b$. And

$$s(z) = \sum_{w \in N_G(z)} d(w) = \sum_{w \in N_{G_0}(z)} d(w) = d(y) + \sum_{w \in N_{G_0}(z) \setminus \{y\}} d(w) \geq d(y) + 2(d(z) - 1) = a - b - 1 + 2(d(z) - 1).$$

So,

$$-d^2(z) + ad(z) - b \geq a - b - 1 + 2(d(z) - 1),$$
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d^2(z) - (a - 2)d(z) + a - 3 \leq 0, \\
and

1 \leq d(z) \leq a - 3.

By (4.1), 

\begin{align*}
ab &- b^2 - 2b + 1 \leq a - 3, \text{ i.e., } b^2 + (2 - a)b + a - 4 \geq 0. \\
\text{Then }

b \leq \frac{(a - 2) - \sqrt{(a - 4)^2 + 4}}{2} \quad \text{or} \quad b \geq \frac{(a - 2) + \sqrt{(a - 4)^2 + 4}}{2}
\end{align*}

From Theorem 2.6, \( a^2 > 8b \geq 16 \), i.e., \( a > 4 \), and

\begin{align*}
\frac{(a - 2) - \sqrt{(a - 4)^2 + 4}}{2} < \frac{(a - 2) - (a - 4)}{2} = 1.
\end{align*}

So, \( b \geq \frac{(a - 2) + \sqrt{(a - 4)^2 + 4}}{2} \). But

\begin{align*}
2 \leq d(y) = a - b - 1 \leq a - \frac{(a - 2) + \sqrt{(a - 4)^2 + 4}}{2} - 1 = a - \frac{\sqrt{(a - 4)^2 + 4}}{2}.
\end{align*}

We have \( a - 4 \geq \sqrt{(a - 4)^2 + 4} \). This is impossible.

Summarizing (I)-(III) above, we have \( \delta(G_0) \geq 2 \).

(ii) Because \( G \in \mathcal{G}_{a,b} \), \( \delta(G) = 1 \). There is a pendent vertex \( x \) and the only edge \( xy \) incident with \( x \) in \( G \). \( d(y) = s(x) = a - b - 1 \) by (3.3). From (i), \( d(y) \geq d_{G_0}(y) + 1 \geq d(G_0) + 1 \geq 3 \). So, \( a - b \geq 4 \).

Since

\begin{align*}
s(y) = d(y) - d_{G_0}(y) + \sum_{w \in N_{G_0}(y)} d(w) \\
\geq d(y) - d_{G_0}(y) + 2d_{G_0}(y) \\
\geq a - b - 1 + 2 = a - b + 1,
\end{align*}

by (2.3), we have

\begin{align*}
a = \frac{s(a) - s(x)}{d(y) - d(x)} + d(y) + d(x) \\
\geq \frac{(a - b + 1) - (a - b - 1)}{a - b - 2} + (a - b - 1) + 1 \\
= \frac{2}{a - b - 2} + a - b > a - b \geq 4.
\end{align*}

So, \( a \geq 5 \) since \( a \) is an integer from Theorem 2.6. \( \square \)

In the following, we determine all unicyclic graphs with exactly two main signless Laplacian eigenvalues.
Let $G_1$ be the unicyclic graph with $n$ vertices obtained by attaching $k \geq 1$ pendant vertices to each vertex of a cycle with length $r$, where $n = (k + 1)r$. It was showed in [10] that $G_1$ is the only connected $n$-vertex graph with exactly two main eigenvalues (of adjacent matrix). $G_2$ is the unicyclic graph with $n$ vertices obtained from the cycle $u_1u_2 \cdots u_{3t}$ by attaching one pendant vertex to the vertex $u_{3s+1}$ for $s = 0, 1, \ldots , t-1$, where $n = 4t$.

**Theorem 4.4.** Let $G$ be a unicyclic graphs with $n$ vertices different from the cycle $C_n$, $G$ has exactly two main signless Laplacian eigenvalues if and only if $G$ is isomorphic to one of graphs $G_1$ and $G_2$.

**Proof.** First, it is easy to check that $G_1$ is a 2-walk $(k + 5, 2)$-parabolic graph and $G_2$ is a 2-walk $(5, 1)$-parabolic graph.

Next, because $G \in G_{a,b}$ is unicyclic, $G_0$ is a cycle from Lemma 4.3(i). Let $G_0 = C_r = u_1u_2 \cdots u_ru_1$, then $d(u_i) \in \{d_{G_0}(u_i), a-b-1\} = \{2, a-b-1\}$ from Lemma 4.2, where $1 \leq i \leq r$.

(i) If $d(u_1) = d(u_2) = \cdots = d(u_r) = a-b-1$, then $u_i$ has $k = a-b-3$ pendant vertices for $1 \leq i \leq r$. So, $G \cong G_1$.

(ii) If there is $u_i \in V(G_0)$ such that $d(u_i) = 2$ for some $i \in \{1, 2, \ldots , r\}$, then by (3.1), we have

\begin{equation}
(4.3) \quad d(u_{i-1}) + d(u_{i+1}) = s(u_i) = -d^2(u_i) + ad(u_i) - b = -4 + 2a - b.
\end{equation}

Without loss of generality, we assume that $d(u_{i+1}) \geq d(u_{i-1})$. From Lemma 4.3(ii), $d(u_{i-1}) + d(u_{i+1}) = s(u_i) = (a-b-4) + a \geq 5$. Since $d(u_{i-1}), d(u_{i+1}) \in \{d_{G_0}(u_i), a-b-1\} = \{2, a-b-1\}$, we have $d(u_{i+1}) = d(u_{i-1}) = a-b-1$, or $d(u_{i+1}) = a-b-1$ and $d(u_{i-1}) = 2$.

If $d(u_{i+1}) = d(u_{i-1}) = a-b-1$, then $b = 2$ by (4.3). From (3.1), $s(u_{i-1}) = -(a-b-1)^2 + a(a-b-1) - b = 3a - 11$. And

\begin{equation}
s(u_{i-1}) = d(u_{i-2}) + d(u_i) + (d(u_{i+1}) - d_{G_0}(u_{i-1})),
\end{equation}

i.e., $d(u_{i-2}) = s(u_{i-1}) - d(u_i) - d(u_{i+1}) + d_{G_0}(u_{i-1}) = (3a - 11) - 2 - (a-b-1) + 2 = 2a - 8$. From Lemma 4.2, $d(u_{i-2}) = 2a - 8 \in \{a-b-1, d_{G_0}(u_{i-2})\} = \{a-3, 2\}$. We have $a = 5$, and $a-b = 3 < 4$, contradicting with Lemma 4.2(ii).

If $d(u_{i+1}) = a-b-1$ and $d(u_{i-1}) = 2$, then $a = 5$ from (4.3). By (3.1), $s(u_{i+1}) = -(a-b-1)^2 + a(a-b-1) - b = 4 + 2b - b^2$. And

\begin{equation}
s(u_{i+1}) = d(u_{i+2}) + d(u_i) + (d(u_{i+1}) - 2),
\end{equation}

i.e., $d(u_{i+2}) = s(u_{i+1}) - d(u_i) - d(u_{i+1}) + 2 = (4 + 2b - b^2) - 2 - (a-b-1) + 2 = 3b - b^2$. 


From Lemma 4.2, $d(u_{i+2}) = 3b - b^2 \in \{ a - b - 1, d_G(u_{i-2}) \} = \{ 4 - b, 2 \}$. We have $b = 1$ or $b = 2$. And $a - b \geq 4$ from Lemma 4.3. So, $b = 1$ and $d(u_{i+1}) = 3$, $d(u_{i+2}) = 2$.

By (3.1) again,

$$d(u_{i+3}) + d(u_{i+1}) = s(u_{i+2}) = -d^2(u_{i+2}) + 5d(u_{i+2}) - 1 = 5,$$
and $d(u_{i+3}) = 5 - d(u_{i+1}) = 2$;

$$d(u_{i+4}) + d(u_{i+2}) = s(u_{i+3}) = -d^2(u_{i+3}) + 5d(u_{i+3}) - 1 = 5,$$
and $d(u_{i+4}) = 5 - d(u_{i+2}) = 3$;

$$d(u_{i+5}) + d(u_{i+3}) + 1 = s(u_{i+4}) = -d^2(u_{i+4}) + 5d(u_{i+4}) - 1 = 5,$$
and $d(u_{i+5}) = 5 - d(u_{i+3}) - 1 = 2$.

Continuing like this, we have

$$d(u_k) = \begin{cases} 2, & k - i \equiv 0, 2 (mod 3); \\ 3, & k - i \equiv 1 (mod 3). \end{cases}$$

So, $r \equiv 0 (mod 3)$ and $G \cong G_2$. $lacksquare$

The results on main signless Laplacian eigenvalues presented in Section 2 are useful to the problem of characterizing graphs with a given number of main signless Laplacian eigenvalues. And Theorems 3.1 and 4.4 show that the set of graphs with a given number of main signless Laplacian eigenvalues is not identical with the set of graphs a given number of main eigenvalues (of adjacent matrix).

REFERENCES


