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MAJORIZATION BOUNDS FOR SIGNLESS LAPLACIAN EIGENVALUES∗

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Abstract. It is known that, for a simple graph $G$ and a real number $\alpha$, the quantity $s'_{\alpha}(G)$ is defined as the sum of the $\alpha$-th power of non-zero singless Laplacian eigenvalues of $G$. In this paper, first some majorization bounds over $s'_{\alpha}(G)$ are presented in terms of the degree sequences, and number of vertices and edges of $G$. Additionally, a connection between $s'_{\alpha}(G)$ and the first Zagreb index, in which the Hölder’s inequality plays a key role, is established. In the last part of the paper, some bounds (included Nordhauss-Gaddum type) for signless Laplacian Estrada index are presented.

Key words. Signless Laplacian matrix, Signless Laplacian-Estrada index, (First) Zagreb index, Majorization, Strictly Schur-convex.

AMS subject classifications. 05C50, 15C12, 15F10.

1. Introduction and preliminaries. Let $G$ be a simple graph with $n$ vertices. The Laplacian matrix of $G$ is defined by $L(G) = \Delta - A$, where $A$ and $\Delta$ are the $(0,1)$-adjacency matrix and the diagonal matrix of the vertex degrees of $G$, respectively. We know that Laplacian spectrum of $G$ consists of the eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ (arranged in non-increasing order) of $L(G)$. It is also known that $\mu_n = 0$ and the multiplicity of 0 is equal to the number of connected components of $G$. We may refer [26] and its citations for detailed properties of the Laplacian spectrum. On the other hand, the signless Laplacian matrix of $G$ is defined by $Q(G) = \Delta + A$. We denote the eigenvalues of $Q(G)$ by $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$.

Let $\alpha$ be a real number. Then we denote by $s_\alpha(G)$ the sum of the $\alpha$-th power of non-zero Laplacian eigenvalues of $G$. In other words,

$$s_\alpha = s_\alpha(G) = \sum_{i=1}^{t} \mu_i^\alpha,$$

where $t$ is the number of non-zero Laplacian eigenvalues of $G$. In a similar manner,
the α-th power of the non-zero signless Laplacian eigenvalues of $G$ is denoted by

$$s'_\alpha = s'_\alpha(G) = \sum_{i=1}^{h} q_i^\alpha,$$

where $h$ is the number of non-zero signless Laplacian eigenvalues of $G$. In here, the cases $\alpha = 0$ and $\alpha = 1$ are trivial since $s'_0(G) = h$ and $s'_1(G) = 2m$, where $m$ is the number of edges of $G$. In the literature, the bounds over quantities $s_\alpha$ and $s'_\alpha$ have been studied largely. For instance, in [30], Zhou established some properties of $s_\alpha$ for $\alpha \neq 0$ and $\alpha \neq 1$. He also discussed further properties by taking into account $s_2$ and $s'_2$. In fact, some of the results obtained in [30] are improved in [27]. Additionally, some bounds for $s_\alpha(G)$ related to degree sequences have been established in [31]. On the other hand, in [33], by taking $G$ as a bipartite graph, some new bounds over $s_\alpha(G)$ have been given. In detail, lower and upper bounds for incidence energy, and also lower bounds for Kirchhoff index and Laplacian Estrada index have been deduced. Furthermore, Akbari et al. [1] obtained some relations between $s_\alpha$ and $s'_\alpha$ for the ranges $0 < \alpha \leq 1$, $1 < \alpha < 2$ and $2 \leq \alpha < 3$.

The Estrada index of a graph $G$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ is defined as $EE = EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$. It is very useful descriptors in a large variety of problems, including those in biochemistry and $m$ complex networks [10, 11, 12]. (We also refer [14, 20, 29] for some recent results.) Further, in [13], the Laplacian-spectral counterpart of the Estrada index is defined as

$$LEE = LEE(G) = \sum_{i=1}^{n} e^{\mu_i}.$$ (One can also look at the studies [4, 8, 21, 31, 32] for more details on the theory of the Laplacian Estrada index.) The next step of $LEE$ is termed as the signless Laplacian Estrada index $SLEE$ of $G$ with $n$ vertices which is defined as

$$SLEE = SLEE(G) = \sum_{i=1}^{n} e^{q_i}.$$

By [15, 16], since the Laplacian and signless Laplacian spectra of bipartite graphs coincide, we easily say that $LEE$ and $SLEE$ coincide in the case of bipartite graphs. Therefore, since the vast majority of molecular graphs are bipartite, $SLEE$ gives nothing new outcomes relative to the previously studied $LEE$. On the other hand, chemically interesting case in which $SLEE$ and $LEE$ differ are the fullerenes, fluoranthenes and other non-alternant conjugated species (see [3, 9, 21, 22, 23]).

In the next section, we present some majorization bounds for $s'_\alpha$ in terms of the degree sequences, and number of vertices and edges of a simple graph $G$. Moreover,
we establish a connection between $s'_\alpha(G)$ and the first Zagreb index. In the final section, we give some bounds (included Nordhauss-Gaddum type) for $SLEE$.

2. New bounds over $s'_\alpha$. In this first main section, we will give some bounds on the quantities presented in (1.1) and (1.2).

For any two non-increasing sequences 

$$x = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad y = (y_1, y_2, \ldots, y_n),$$

we say that $x$ is majorized by $y$, denoted by $x \preceq y$, if

$$\sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} y_i \quad \text{for} \quad j = 1, 2, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

For a real-valued function $f$ defined on a set in $\mathbb{R}^n$, if $f(x) < f(y)$ whenever $x \preceq y$ but $x \neq y$, then $f$ is said to be strictly Schur-convex [25]. The following lemma makes clear the property of strictly Schur-convexity for the function $f$.

**Lemma 2.1** ([25]). Let $\alpha$ be a real number such that $\alpha \neq 0$ and $\alpha \neq 1$.

(i) For $i = 1, 2, \ldots, h$, suppose $x_i \geq 0$. Then the following hold:

- $f(x) = \sum_{i=1}^{h} x_i^\alpha$ is strictly Schur-convex if $\alpha > 1$.
- $f(x) = -\sum_{i=1}^{h} x_i^\alpha$ is strictly Schur-convex if $0 < \alpha < 1$.

(ii) For $i = 1, 2, \ldots, h$, suppose $x_i > 0$. Then $f(x) = \sum_{i=1}^{h} x_i^\alpha$ is strictly Schur-convex if $\alpha < 0$.

We also remind that the degree sequence of $G$ is a list of the degrees of the vertices in non-increasing order which is denoted by $(d) = (d_1, d_2, \ldots, d_n)$. We let denote $\overline{(d)} = (d_1 + 1, d_2, \ldots, d_{n-1}, d_n - 1)$, where $d_1$ is the maximum vertex degree of $G$. We similarly denote $(\mu) = (\mu_1, \mu_2, \ldots, \mu_n)$ and $(q) = (q_1, q_2, \ldots, q_n)$ as the spectrums of the Laplacian and signless Laplacian matrices, respectively.

**Remark 2.2.** In [17], Grone proved that if $G$ has at least one edge, then $\overline{(d)} \preceq (\mu)$ is correct while $\overline{(d)} \preceq (q)$ is not. For example, if $G$ is a connected non-bipartite graph with at least one pendant vertex, then

$$d_1 + 1 + d_2 + \cdots + d_{n-1} = 2m > q_1 + q_2 + \cdots + q_{n-1}$$

since $G$ is non-bipartite (which implies $q_n > 0$).
It is well known that (see, for example, [25, p. 218]) the spectrum of a positive semi definite Hermitian matrix majorizes its main diagonal (when both are rearranged in non-increasing order). As a result of this, we can give the following proposition.

**Proposition 2.3.** For the graph $G$ with signless Laplacian spectrum $(q) = (q_1, q_2, \ldots, q_n)$ and degree sequence $(d) = (d_1, d_2, \ldots, d_n)$, it is always true that $(d) \preceq (q)$.

We also need the following preliminary results for our main theorems in this paper.

**Proposition 2.4 ([1]).** Let $G$ be a graph of order $n$ and $\alpha$ be a real number. Then we have the following relations for the quantities in (1.1) and (1.2):

i) If $0 < \alpha \leq 1$ or $2 \leq \alpha \leq 3$, then $s'_{\alpha} \geq s_{\alpha}$.

ii) If $1 \leq \alpha \leq 2$, then $s'_{\alpha} \leq s_{\alpha}$.

On the other hand, for the quantity $s_{\alpha}$ in (1.1), it has been obtained the following lower and upper bounds in [31] by considering degree sequences.

**Proposition 2.5 ([31]).** Let $G$ be a connected graph with $n \geq 2$ vertices.

(i) If $\alpha > 1$, then $s_{\alpha} \geq (d_1 + 1)^{\alpha} + \sum_{i=2}^{n-1} d_i^{\alpha} + (d_n - 1)^{\alpha}$.

(ii) If $0 < \alpha < 1$, then $s_{\alpha} \leq (d_1 + 1)^{\alpha} + \sum_{i=2}^{n-1} d_i^{\alpha} + (d_n - 1)^{\alpha}$.

Moreover, equalities hold in both cases if and only if $(q) = (d)$.

Proof. Suppose that $\alpha > 1$. Then, by Lemma 2.1-(i), $f(x) = \sum_{i=1}^{h} x_i^{\alpha}$ (such that
$x_i \geq 0$) is strictly Schur-convex which together with Proposition 2.3 implies that

$$s'_\alpha = \sum_{i=1}^{n} q_i^\alpha \geq d_1^\alpha + d_2^\alpha + \cdots + d_n^\alpha,$$

where equality holds if and only if $(q_1, q_2, \ldots, q_n) = (d_1, d_2, \ldots, d_n)$.

Now let us assume that $0 < \alpha < 1$. Then again, by Lemma 2.1(i), $f(x) = -\sum_{i=1}^{n} x_i^\alpha$ (such that $x_i \geq 0$) is strictly Schur-convex which implies that

$$-s'_\alpha = -\sum_{i=1}^{n} q_i^\alpha \geq -[d_1^\alpha + d_2^\alpha + \cdots + d_n^\alpha],$$

or equivalently,

$$s'_\alpha = \sum_{i=1}^{n} q_i^\alpha \leq d_1^\alpha + d_2^\alpha + \cdots + d_n^\alpha.$$

It is clear that the equality holds if and only if $(q_1, q_2, \ldots, q_n) = (d_1, d_2, \ldots, d_n)$.

Hence, the result. \(\Box\)

Using Lemma 2.1, it is easy to see the following result.

**Corollary 2.7.** Let $G$ be a graph. Then the following inequalities hold:

a) If $\alpha > 1$ or $\alpha < 0$, then

$$\sum_{i=1}^{n} d_i^\alpha < (d_1 + 1)^\alpha + \sum_{i=2}^{n-1} d_i^\alpha + (d_n - 1)^\alpha.$$

(We should note that for the case $\alpha < 0$, we also need the assumption $d_i > 1$ or equivalently, $G$ has no pendant vertices.)

b) If $0 < \alpha < 1$, then

$$\sum_{i=1}^{n} d_i^\alpha > (d_1 + 1)^\alpha + \sum_{i=2}^{n-1} d_i^\alpha + (d_n - 1)^\alpha.$$

We note that Theorem 2.6 can be converted to bipartite graphs as in the following.

**Theorem 2.8.** Let $G$ be a connected bipartite graph with $n \geq 3$ vertices. If $\alpha > 1$, then

$$s'_\alpha \geq (d_1 + 1)^\alpha + \sum_{i=2}^{n-1} d_i^\alpha + (d_n - 1)^\alpha > \sum_{i=1}^{n} d_i^\alpha.$$
Proof. For the first inequality, by Lemma 2.1(i), \( f(x) = \sum_{i=1}^{h} x_i^\alpha \) is strictly Schur-convex for \( x_i \geq 0 \), where \( i = 1, 2, \ldots, h \). Since \( G \) is bipartite, it is known that \( q_n = 0 \), and \( Q(G) \) and \( L(G) \) share the same eigenvalues. Thus, by Remark 2.2 we obtain
\[
(d_1 + 1, d_2, \ldots, d_{n-1}, d_n - 1) \preceq (q_1, q_2, \ldots, q_n).
\]
Hence,
\[
s'_\alpha = \sum_{i=1}^{n} q_i^\alpha \geq (d_1 + 1)^\alpha + \sum_{i=2}^{n-1} d_i^\alpha + (d_n - 1)^\alpha,
\]
as required. We note that the second inequality follows immediately from Corollary 2.7(a).

From Lemma 2.1, Remark 2.2, Proposition 2.4 and Corollary 2.7, we get the following result.

**Corollary 2.9.** Let \( G \) be a graph. Therefore,

a) if \( 0 < \alpha < 1 \), then \( s_\alpha < \sum_{i=1}^{n} d_i^\alpha \),

b) if \( 1 < \alpha \), then \( s_\alpha > \sum_{i=1}^{n} d_i^\alpha \),

c) if \( 2 \leq \alpha \leq 3 \), then \( s'_\alpha \geq (d_1 + 1)^\alpha + \sum_{i=2}^{n-1} d_i^\alpha + (d_n - 1)^\alpha \).

Recall that the first Zagreb index \( M_1(G) \) of the graph \( G \) is the sum of the squares of the degrees of vertices of \( G \). (One can find the details about this graph invariant in [19] and the references cited therein.) This index actually has been found in many applications, specially, in chemistry [19] and received wide investigations, and lots of properties of which have been reported (see [5, 7, 18, 28] for the details).

Now, by using the Hölder’s inequality, we only establish a lower bound for the quantity \( s'_\alpha \) given in (1.2) in terms of the first Zagreb index \( M_1(G) \). By using this theorem, we may have a chance to derive lots of bounds over \( s'_\alpha \) for a connected (molecular) graph \( G \) related to its number of vertices (atoms) and edges (bonds).

**Theorem 2.10.** Let \( G \) be a graph with \( n \) vertices and \( m \geq 1 \) edges. If \( \alpha < 0 \) or \( 0 < \alpha < 1 \) or \( \alpha > 2 \), then
\[
s'_\alpha \geq \frac{(2m)^{2-\alpha}}{(M_1(G) + 2m)^{1-\alpha}}.
\]
Equality holds in (2.1) if and only if \( q_1 = q_2 = \cdots = q_n \). Furthermore, if \( 1 < \alpha < 2 \), then the inequality in (2.1) is reversed.

Proof. Let \( x_1, x_2, \ldots, x_s \) be positive real numbers, and let \( p \) be a real number with \( p \neq 0, p \neq \frac{1}{2}, p \neq 1 \). If \( p < 0 \) or \( p > 1 \), then \( \frac{2p-1}{p-1} > 1 \). By Hölder's inequality, we have

\[
\sum_{i=1}^{s} x_i^p = \sum_{i=1}^{s} x_i^{\frac{p}{p-1}} \frac{2p-1}{p} x_i^{1-\frac{p}{p-1}} \leq \left( \sum_{i=1}^{s} x_i^p \right)^{\frac{p-1}{p}} \left( \sum_{i=1}^{s} x_i^{2p} \right)^{\frac{p}{p-1}} \cdot
\]

Shortly, we get

\[
\left( \sum_{i=1}^{s} x_i^p \right)^{\frac{p}{p-1}} \geq \left( \sum_{i=1}^{s} x_i^{2p} \right)^{\frac{p-1}{p}} 
\]

Thus,

\[
(2.2) \quad \sum_{i=1}^{s} x_i \geq \frac{\left( \sum_{i=1}^{s} x_i^{p} \right)^{\frac{2p-1}{p}}}{\left( \sum_{i=1}^{s} x_i^{2p} \right)^{\frac{p-1}{p}}}
\]

where the equality holds if and only if \( x_1 = x_2 = \cdots = x_s \).

Now if we write \( s = n, x_i = q_i^\alpha \) and \( p = \frac{1}{\alpha} \) in (2.2), then (2.1) follows immediately when \( \alpha < 0 \) or \( 0 < \alpha < 1 \) since \( s_1' (G) = 2m \) and \( s_2' (G) = M_1 (G) + 2m \). Furthermore the equality in (2.1) holds if and only if \( q_1 = q_2 = \cdots = q_n \). The proofs for the cases \( \alpha > 2 \) and \( 1 < \alpha < 2 \) are similar (here, take \( 0 < p < \frac{1}{2} \) and \( \frac{1}{2} < p < 1 \), respectively).

Example 2.11. Together with some known bounds for the first Zagreb index [5], [7], [18], [28], the bound in (2.1) may directly yield a lot different bounds for \( s_\alpha' \). For example, let us consider the bound [5]

\[
M_1 (G) \leq m \left( \frac{2m}{n-1} + n - 2 \right)
\]
for the first Zagreb index. This bound implies that, if \( \alpha < 0 \) or \( 0 < \alpha < 1 \) (resp., \( 1 < \alpha < 2 \)), then

\[
\langle s' \rangle_A \geq (\text{resp., } \leq) \frac{2m}{\left(\frac{m}{n-1} + \frac{n}{2}\right)^{1-\alpha}}.
\]

**Example 2.12.** Suppose that \( G \) is \( K_{r+1} \)-free with \( 2 \leq r \leq n-1 \). Then

\[
M_1(G) \leq \frac{2r-2}{r} nm
\]

such that equality holds if and only if \( G \) is complete bipartite graph for \( r = 2 \), and a regular complete \( r \)-bipartite graph for \( r \geq 3 \) (cf. [28]). This yields that if \( \alpha < 0 \) or \( 0 < \alpha < 1 \) (resp., \( 1 < \alpha < 2 \)), then

\[
\langle s' \rangle_A > (\text{resp., } <) \frac{2m}{\left(\frac{r-1}{r}n + 1\right)^{1-\alpha}}.
\]

We need the next lemma for another result over \( s' \).

**Lemma 2.13.** We have

\[
\sum_{i=1}^{n} d_i^\alpha \geq n^{1-\alpha} \left(2m\right)^\alpha \quad \text{if } \alpha < 0 \text{ or } \alpha > 1
\]

\[
\leq n^{1-\alpha} \left(2m\right)^\alpha \quad \text{if } 0 < \alpha < 1
\]

**Proof.** Observe that for \( x > 0 \), the function \( x^\alpha \) is strictly convex if and only if \( \alpha < 0 \) or \( \alpha > 1 \). Hence, let us suppose that \( \alpha < 0 \) or \( \alpha > 1 \). Then

\[
\left(\sum_{i=1}^{n} \frac{1}{d_i}\right)^\alpha \leq \sum_{i=1}^{n} \left(\frac{1}{d_i}\right)^\alpha,
\]

and in other words,

\[
\sum_{i=1}^{n} d_i^\alpha \geq \frac{1}{n^{\alpha-1}} \left(\sum_{i=1}^{n} d_i\right)^\alpha = n^{1-\alpha} \left(2m\right)^\alpha,
\]

as required. \( \square \)

From Theorem 2.6 and Lemma 2.13 we obtain the following consequence.

**Corollary 2.14.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Then for the ranges \( 0 < \alpha < 1 \) and \( \alpha > 1 \), we obtain the bounds

\[
s'_{\alpha} \leq n \left(\frac{2m}{n}\right)^\alpha \quad \text{and} \quad n \left(\frac{2m}{n}\right)^\alpha \leq s'_{\alpha},
\]
respectively.

The following lemma will be needed for a new bound over \( s'_{\alpha} \) (see Theorem 2.16 below).

**Lemma 2.15** ([6]). Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges. Then \( q_1 \geq \frac{4m}{n} \) with equality if and only if \( G \) is a regular graph.

**Theorem 2.16.** Let \( G \) be a connected graph with \( n \geq 3 \) vertices and \( m \) edges.

i) If \( \alpha < 0 \) or \( \alpha > 1 \), then

\[
s'_{\alpha} \geq \left( \frac{2m}{n} \right)^{\alpha} (2^{\alpha} + n - 2).
\]

ii) If \( 0 < \alpha < 1 \), then

\[
s'_{\alpha} \leq \left( \frac{2m}{n} \right)^{\alpha} (2^{\alpha} + n - 2).
\]

**Proof.** As in the proof of Lemma 2.13 it is clear that \( x^\alpha \) (where \( x > 0 \)) is a strictly convex function if and only if \( \alpha < 0 \) or \( \alpha > 1 \). Therefore, let us suppose that \( \alpha < 0 \) or \( \alpha > 1 \). We then have

\[
\sum_{i=2}^{n-1} q_i^{\alpha} \geq \frac{1}{(n-2)^{\alpha-1}} \left( \sum_{i=2}^{n-1} q_i \right)^{\alpha},
\]

where the equality holds if and only if \( q_2 = \cdots = q_{n-1} \). It follows that

\[
s'_{\alpha} \geq q_1^{\alpha} + q_n^{\alpha} + \frac{1}{(n-2)^{\alpha-1}} \left( \sum_{i=2}^{n-1} q_i \right)^{\alpha} = q_1^{\alpha} + q_n^{\alpha} + \left( \frac{2m - q_1 - q_n}{n-2} \right)^{\alpha}.
\]

Now let us consider the function \( f(x, y) = x^\alpha + y^\alpha + \left( \frac{2m-x-y}{n-2} \right)^\alpha \), for \( x > 0, \ y > 0 \). In order to find its minimum, we have the following derivations for this function:

\[
f_x = \alpha \left[ x^{\alpha-1} - \frac{(2m-x-y)^{\alpha-1}}{(n-2)^{\alpha-1}} \right], \quad f_y = \alpha \left[ y^{\alpha-1} - \frac{(2m-x-y)^{\alpha-1}}{(n-2)^{\alpha-1}} \right],
\]

\[
f_{xx} = \alpha (\alpha - 1) \left[ x^{\alpha-2} + \frac{(2m-x-y)^{\alpha-2}}{(n-2)^{\alpha-1}} \right], \quad f_{xy} = f_{yx} = \alpha (\alpha - 1) \left( \frac{2m-x-y}{n-2} \right)^{\alpha-2}
\]

and

\[
f_{yy} = \alpha (\alpha - 1) \left[ y^{\alpha-2} + \frac{(2m-x-y)^{\alpha-2}}{(n-2)^{\alpha-1}} \right].
\]
A simple calculation implies that
\[ f_x = f_y = 0 \implies (n-1) x + y = 2m \text{ and } x + (n-1) y = 2m \implies x + y = \frac{4m}{n}. \]
For \( x + y = \frac{4m}{n} \), we clearly get \( f_{xx} > 0 \) and \( f_{xx} f_{yy} - f_{xy}^2 > 0 \).

From above, it is concluded that \( f(x, y) \) has a minimum value at \( x + y = \frac{4m}{n} \) and that the minimum value is
\[ x^\alpha + \left( \frac{4m}{n} - x \right)^\alpha + \left( \frac{2m - 4m}{(n-2)\alpha} \right)^\alpha. \]
By Lemma 2.15 \( q_1 \geq \frac{4m}{n} \).

Hence, the result. \( \square \)

**Remark 2.17.** One can easily see that the bounds in Theorem 2.16 are better than the bounds in Corollary 2.14.

One can also consider the next lemma.

**Lemma 2.18 ([2]).** Let \( G \) be a graph on \( n \) vertices. Then \( q_1 \geq 2(k-1) \) where \( k \) is the chromatic number. Equality holds if and only if \( G \cong K_n \) or \( G \) is the cycle \( C_n \) of odd length.

Hence, as a consequence of Theorem 2.16 and Lemma 2.18 we get the following corollary.

**Corollary 2.19.** Let \( G \) be a connected graph with \( n \geq 3 \) vertices and \( m \) edges.
\[ i) \] If \( \alpha < 0 \) or \( \alpha > 1 \), then
\[ s'_\alpha \geq 2^\alpha \left[ (k-1)^\alpha + \left( \frac{2m}{n} - k + 1 \right)^\alpha + \left( \frac{m}{n} \right)^\alpha (n-2) \right]. \]
\[ ii) \] If \( 0 < \alpha < 1 \), then
\[ s'_\alpha \leq 2^\alpha \left[ (k-1)^\alpha + \left( \frac{2m}{n} - k + 1 \right)^\alpha + \left( \frac{m}{n} \right)^\alpha (n-2) \right]. \]

3. **Bounds for signless Laplacian Estrada index.** Let \( G \) be a graph with \( n \) vertices. We recall that, for a non-negative integer \( k \) and the eigenvalues \( q_1 \geq q_2 \geq \cdots \geq q_n \) of \( Q(G) \),
\[
T_r(G) = \sum_{i=1}^{n} q_i^r
\]
denotes the \( r \)-th signless Laplacian spectral moment of \( G \). Obviously, \( T_0 \left( G \right) = n \) and \( T_r \left( G \right) = s'_r \) for \( r \geq 1 \). In fact, (3.1) will also be needed in our results.

By [2], it is obvious that \( \text{SLEE} \) in (1.4) can be written as

\[
\text{SLEE} = \sum_{i=0}^{\infty} \frac{T_r}{r!},
\]

where \( T_r \) is defined as in (3.1). In this section, we claim to convert the properties and obtained results in the previous section into the signless Laplacian Estrada index.

**Theorem 3.1.** Let \( G \) be a graph with \( n \geq 2 \) vertices. Then

\[
\text{SLEE} \geq \sum_{i=1}^{n} e^{d_i}.
\]

Moreover, the equality holds in above if and only if \((q) = (d)\).

**Proof.** We know that \( T_0 = n, T_1 = 2m, \)

\[
T_2 = \sum_{i=1}^{n} d_i (d_i + 1) = M_1 \left( G \right) + 2m
\]

and \( T_r = s'_r \left( G \right) \) for \( r \geq 1 \). By Theorem 2.6 (i),

\[
T_r \geq \sum_{i=1}^{n} d'_i \quad \text{for} \quad r = 0, 1
\]

such that equality holds for \( r = 0, 1 \). Thus,

\[
\text{SLEE} = \sum_{r \geq 0} \frac{T_r}{r!} \geq \sum_{r \geq 0} \frac{\sum_{i=1}^{n} d_r}{r!} = \sum_{i=1}^{n} e^{d_i},
\]

as desired. \( \square \)

**Theorem 3.2.** Let \( G \) be a graph with \( n \geq 2 \) vertices and \( m \) vertices. Then

\[
\text{SLEE} \leq n + 2m - 1 - \sqrt{M_1 \left( G \right) + 2m} + e^{\sqrt{M_1 \left( G \right) + 2m}}
\]

with equality holding if and only if at most one of \( q_1, q_2, \ldots, q_n \) is non-zero.

**Proof.** It is well known that \( \sum_{i=1}^{n} q_i^2 = M_1 \left( G \right) + 2m \). For an integer \( r \geq 3 \),

\[
\left( \sum_{i=1}^{n} q_i^2 \right)^r \geq \sum_{i=1}^{n} q_i^{2r} + \sum_{1 \leq i < j \leq n} \left( q_i^{2(r-1)} q_j^{2(r-1)} + q_i^{2(r-1)} q_j^{2} \right) \geq \sum_{i=1}^{n} q_i^{2r} + 2r \sum_{1 \leq i < j \leq n} q_i q_j^n \geq \left( \sum_{i=1}^{n} q_i^2 \right)^2,
\]

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and then,

\[ \sum_{i=1}^{n} q_i^r \leq \left( \sum_{i=1}^{n} q_i^2 \right)^{\frac{r}{2}} = (M_1(G) + 2m) \]

with equality holding if and only if at most one of \( q_1, q_2, \ldots, q_n \) is non-zero.

It is easily seen that

\[
SLEE = n + 2m + \sum_{r \geq 2} \frac{1}{r!} \left( \sum_{i=1}^{n} q_i^r \right) \leq n + 2m + \sum_{r \geq 2} \frac{1}{r!} \left( \sqrt{M_1(G) + 2m} \right)^r
\]

\[
= n + 2m - 1 - \sqrt{M_1(G) + 2m} + e \sqrt{M_1(G) + 2m}.
\]

Finally, we will give a Nordhauss-Gaddum type bound for \( SLEE \).

**Theorem 3.3.** Let \( G \) be a graph with \( n \geq 2 \) vertices and \( m \) edges. Also let \( \overline{G} \) be the complement of \( G \). Then

\[
SLEE(G) + SLEE(\overline{G}) > 2 \left[ e^{n-1} + (n-2) e^{\frac{n-1}{2}} \right].
\]

**Proof.** By the arithmetic-geometric inequality, we have

\[
SLEE = e^{q_1} + e^{q_2} + \cdots + e^{q_n} \geq e^{\frac{4m}{n}} + (n-2) e^{\frac{2m}{n}} + 1
\]

(cf. [2]). Let \( \overline{m} \) be the number of edges of \( \overline{G} \). Thus,

\[
SLEE(G) + SLEE(\overline{G}) \geq 2 + e^{\frac{4m}{n}} + e^{\frac{2m}{n}} + (n-2) \left( e^{\frac{4m}{n}} + e^{\frac{2m}{n}} \right)
\]

\[
\geq 2 + 2e^{\frac{2(m+\overline{m})}{n}} + 2(n-2) e^{\frac{m+\overline{m}}{n}}
\]

\[
= 2 + 2e^{(n-1)} + 2(n-2) e^{\frac{n-1}{2}}
\]

\[
> 2e^{n-1} + 2(n-2) e^{\frac{n-1}{2}}.
\]

Hence, the result. \( \square \)

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