Lattice-points enumeration in polytopes: Study of the coefficients of the Ehrhart quasi-polynomial
Outline:

1. Polytopes, convex hull
2. Why do we care?
3. Counting lattice points in polytopes
4. The periods of the Ehrhart quasi polynomial
1. Polytopes, convex hull
Integer dilates of polytopes: family of polytopes

The unit square and its 3rd dilate. If we let $P$ to be the unit square (side length of 1), then $3P$ is the square of side length of 3.
2. Why do we care?

- A lot of problems boil down to finding integer solutions to a problem or knowing if there is a solution at all.

- A typical example:
  Suppose you want to load elephants and giraffes on a plane

![Elephant](image1.png) 3,000 kg  
![Giraffe](image2.png) 1,000 kg  
![Plane](image3.png) 8,000 kg

How many elephant and giraffes can we put on the plane? Or, how many integer solutions do we have to:

\[ 3,000 \times E + 1,000 \times G \leq 8,000 \]

where \( E, G \geq 0 \).
In general we have:

\[ k_1x_1 + k_2x_2 + \cdots + k_jx_j \leq n \]

with \( x_1, x_2, \ldots, x_j \) the number of \( j \) types of animals of weight \( k_1, k_2, \ldots, k_j \) with; and \( n \) the weight capacity of the plane.
3. Counting lattice points in polytopes

a. Integer polytopes

- The vertices have all integer coordinates

- The Ehrhart function:
  \[ ehr_P(n) = \text{number of lattice points in } nP \]
Example

We have:
\[ ehr_p(1) = \text{# of lattice points in the unit square } C = 4 \]
\[ ehr_p(2) = \text{# of lattice points in } 2C = 9 \]
\[ ehr_p(3) = \text{# of lattice points in } 3C = 16 \]
\[ ehr_p(4) = 25 \]
\[ ehr_p(5) = 36 \]

In general,
\[ ehr_p(n) = (n + 1)^2 = n^2 + 2n + 1 \]

Theorem:

If \( P \) is an integer polytope, then \( ehr_p(n) \) is a polynomial in \( n \). That is,
\[ ehr_p(n) = a_t n^t + a_{t-1} n^{t-1} + \cdots + a_1 n + a_0 \]
for some constants \( a_0, a_1, \ldots, a_{t-1}, a_t \in \mathbb{R} \).
b. Rational polytopes

- The vertices have all rational coordinates. That is, they can be written as fractions.

- Again, $ehr_P(n) = \text{number of lattice points in } nP$
Let $l$ be the segment $[0, \frac{1}{3}]$. Then

$$ehr_l(n) = \left\lfloor \frac{n}{\frac{1}{3}} \right\rfloor + 1$$

where $[x]$ is the biggest integer smaller or equal to $x$. Then $[3.5] = 3$ and $[3] = 3$.

$$ehr_l(1) = 1 = \left\lfloor \frac{1}{\frac{1}{3}} \right\rfloor + 1$$
$$ehr_l(2) = 1 = \left\lfloor \frac{2}{\frac{1}{3}} \right\rfloor + 1$$
$$ehr_l(3) = 2 = \left\lfloor \frac{3}{\frac{1}{3}} \right\rfloor + 1 = [1] + 1$$
$$ehr_l(4) = 2 = \left\lfloor \frac{4}{\frac{1}{3}} \right\rfloor + 1$$
$$ehr_l(5) = 2 = \left\lfloor \frac{5}{\frac{1}{3}} \right\rfloor + 1$$
$$ehr_l(6) = 3 = \left\lfloor \frac{6}{\frac{1}{3}} \right\rfloor + 1 = [2] + 1$$

Then, $ehr_l(n) = \frac{1}{3} n + a(n)$

where $a(n) = \left\lfloor \frac{n}{\frac{1}{3}} \right\rfloor - \frac{n}{\frac{1}{3}} + 1$
Theorem:

If $P$ is an rational polytope, then $ehr_P(n)$ is a quasi-polynomial in $n$.
That is,

$$ehr_P(n) = a_t(n)n^t + a_{t-1}(n)n^{t-1} + \cdots + a_1(n)n + a_0(n)$$
for some periodic functions $a_0(n), a_1(n), \ldots, a_{t-1}(n), a_t(n)$

Definition

In $\mathbb{R}^3$ we say that $P$ has period sequence $(s_3, s_2, s_1, s_0)$ if the minimum period of
the coefficients $a_3(n), a_2(n), a_1(n), a_0(n)$ is $s_i$ for $i = 0, 1, 2, 3$, where

$$ehr_P(n) = a_3(n)n^3 + a_2(n)n^2 + a_1(n)n + a_0(n)$$

Theorem:

In $\mathbb{R}^3$, $s_3 = 1$ for any polytope $P$. In general, in $\mathbb{R}^t$, $s_t = 1$ and $a_t(n) = constant$. 
4. The periods of the Ehrhart quasi-polynomials

a. Results!

**Theorem (Rochais, 2015)**

- Given positive integer $s$, and $t$ there exist convex polytope $Q, P \subseteq \mathbb{R}^n$ with period sequences $(1, 1, \ldots, 1, s, 1)$ and $(1, 1, \ldots, 1, t)$.

- Given positive integer $u$, there exists a non-convex polytope $B \subseteq \mathbb{R}^n$ with period sequence $(1, u, 1, 1, \ldots, 1)$ provided that $1 \leq n \leq 13$, and $n \neq 12$.

- In particular, in $\mathbb{R}^3$: given positive integers $s$, $t$ and $u$, there exist a non-convex polytope $H$ with period sequence $(1, u, s, t)$. 
b. A polytope of period sequence \((1, 1, 1, t)\)

Let \(P = \text{conv}\{(0,0,0), (1,0,0), (0,1,0), \left(0,0,\frac{1}{t}\right)\}\) Then \(P\) has period sequence \((1,1,1, t)\).
c. A polytope of period sequence \((1, 1, s, 1)\) in 3 dimensions
A polytope of period sequence \((1, 1, s, 1)\)
b. Combining \((1, 1, 1, t)\) with \((1, 1, s, 1)\) to get a polytope with period sequence \((1, 1, s, t)\)
d. A non convex polytope of period $(1, u, 1, 1)$ in 3 dimensions
e. Higher dimensions

- Defining $Q$ recursively (recall that $Q$ has period sequence $(1, 1, \ldots, 1, s, 1)$):

Constructing $Q_d$ in $d$ dimensions
Questions?
We use piece-wise affine unimodular transformations as a way to skew polytopes without changing the lattice so leaving the number of lattice points inside a polytope intact. This is a very useful way to geometrically manipulate a polytope without changing its Ehrhart polynomial or quasi-polynomial.