Turán Numbers of Vertex-disjoint Cliques in \( r \)-partite Graphs
Final Honors Project, University of Wyoming

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History and Motivation

- Graph theory is the math of connections
- Applications in other fields, both abstract and applied
- Historically began with Euler: "The Seven Bridges of Königsberg" (1736)
- Erdős is considered the father of the field (early 20th century)
Graph: A graph \( G \) is a pair of sets \( G = (V, E) \), where \( V \) is a fixed set of vertices, and the edge set \( E \) is a set of pairs of distinct elements from \( V \). We often write \( V \) as \( V(G) \) and \( E \) as \( E(G) \).
Definitions

- Graph

- Subgraph: A **subgraph** $H$ of $G$ is a pair of sets $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$, which is itself a graph. If $H$ is a subgraph of $G$, we write $H \subseteq G$. 
Definitions

- Graph
- Subgraph
- Clique: A **clique** is a graph or subgraph in which every vertex is adjacent to every other vertex. A clique of size $r$ is a **complete** graph on $r$ vertices.
Definitions

- Graph
- Subgraph
- Clique

More than one cliques present that do not share vertices are called **vertex-disjoint cliques**.
Definitions

- Graph
- Subgraph
- Clique
- r-Partite: A graph $G$ is called **r-partite** if there are $r$ partitions of the vertex set $V(G) = V_1 \cup V_2 \cup \ldots \cup V_r$ such that if $y$ and $y'$ are both in the same $V_i$, then $xy \notin E(G)$. r-Partite graphs can also be **complete**.
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How can we count edges?
A complete graph is often denoted with a $K$.

$k$ copies of $K$ is denoted $kK$.

A complete $r$-partite graph is denoted $K_{n_1,...,n_r}$, where there $n_1, ... n_r$ are the number of vertices in each part.

We denote $k$ vertex-disjoint cliques of size $r$ as $kK_r$. 
Turán Numbers

$\text{ex}(G,H)$

What is the maximum number of edges which a subgraph of $G$ may have and still contain no copy of $H$?
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Let $G = K_{2,3}$ and $H = 2K_2$
What is $\text{ex}(K_{2,3}, 2K_2)$?
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Therefore, \( ex(K_{2,3}, 2K_2) = 3 \)
Our Theorem

For any integers $1 \leq k \leq n_1 \leq \ldots \leq n_r$, we have

$$\text{ex}(K_{n_1, n_2, \ldots, n_r}, kK_r) = \sum_{1 \leq i < j \leq r} n_i n_j - n_1 n_2 + n_2(k - 1)$$
Proof Ideas

- First half of the proof is the lower bound
- Second half of the proof is the upper bound
  - Proof strategy - **induction**
  - Double induction on $n_1 + k$
  - Two lemmas as base cases where we are only looking for one clique, and where each part size is equal
The extremal number is \textit{at least} the number we claim (lower bound)

The extremal number is \textit{at most} the number we claim (upper bound)

Therefore, the extremal number is \textit{exactly} the number we claim

Characteristics of host and forbidden graph allow us to have equality
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Questions?