The Dirichlet spectral radius of trees

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THE DIRICHLET SPECTRAL RADIUS OF TREES

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Abstract. In this paper, the trees with the largest Dirichlet spectral radius among all trees with a given degree sequence are characterized. Moreover, the extremal graphs having the largest Dirichlet spectral radius are obtained in the set of all trees of order \( n \) with a given number of pendant vertices.

Key words. Dirichlet spectral radius, Degree sequence, Tree.

AMS subject classifications. 05C50.

1. Introduction. In this paper, we only consider simple connected graphs. Let \( G = (V(G), E(G)) \) be a graph of order \( n \) with vertex set \( V(G) = \{v_0, v_1, \ldots, v_{n-1}\} \) and edge set \( E(G) \). Let \( d(v) \) denote the degree of a vertex \( v \). Then \( \pi(G) = (d(v_0), d(v_1), \ldots, d(v_{n-1})) \) is called the degree sequence of \( G \). A sequence of positive integers \( \pi = (d_0, d_1, \ldots, d_{n-1}) \) is said to be a tree degree sequence if there exists at least one tree whose degree sequence is \( \pi \). The Laplacian matrix of \( G \) is defined as \( L(G) = D(G) - A(G) \), where \( D(G) \) and \( A(G) \) denote the diagonal matrix of vertex degrees and the adjacency matrix of \( G \), respectively. Let \( T_\pi \) be the set of all the trees with a given degree sequence \( \pi \), where \( \pi = (d_0, d_1, \ldots, d_{n-1}) \) satisfies \( d_0 \geq d_1 \geq \cdots \geq d_{n-1} \). The adjacency eigenvalues and Laplacian eigenvalues of graphs have been intensively investigated during the last decades. Bıyıkoğlu et al. [1] and Zhang [12] determined all graphs with the maximal spectral radius and Laplacian spectral radius among all trees with a given degree sequence, respectively. Tan [10] determined the trees with the largest Laplacian spectral radius among all weighted trees with a given degree sequence and positive weight set. A graph \( G = (V_0 \cup \partial V, E_0 \cup \partial E) \) with boundary consists of a set of interior vertices \( V_0 \), boundary vertices \( \partial V \), interior edges \( E_0 \) that connect interior vertices, and boundary edges \( \partial E \) that join interior vertices with boundary vertices (see [2]). A real number \( \lambda \) is called a Dirichlet eigenvalue of \( G \) if there exists a function \( f \neq 0 \) such that they satisfy the Dirichlet eigenvalue

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problem:

\[
\begin{align*}
L(G)f(u) &= \lambda f(u), \quad u \in V_0, \\
f(u) &= 0, \quad u \in \partial V.
\end{align*}
\]

The function \( f \) is called a Dirichlet eigenfunction corresponding to \( \lambda \) (see [7]). The largest Dirichlet eigenvalue of \( G \) is called Dirichlet spectral radius, denoted by \( \lambda(G) \).

Recently, there is an increasing interest in the Dirichlet eigenvalues of graphs (see [2], [3] and [7]), since it can be regarded to be analogous to the Dirichlet eigenvalues of Laplacian operator on a manifold. The related eigenvalue problems have been occasionally occurred in fields like game theory [4], network analysis [5], and Pattern Recognition [8]. In this paper, we regard pendant vertices as boundary vertices and assume that both the set \( V_0 \) and the set \( \partial V \) are not empty. Motivated by the above results, we will study the Dirichlet spectral radius of graphs in \( T_\pi \). The main result of this paper is as follows:

**Theorem 1.1.** For a given tree degree sequence \( \pi \), \( T^*_\pi \) (see in Section 3) is the unique tree with the largest Dirichlet spectral radius in \( T_\pi \).

The rest of the paper is organized as follows. In Section 2, some notations and results are presented. In Section 3, we present the proof of Theorem 1.1 and some corollaries.

**2. Preliminaries.** Let \( R_G(f) \) be the Rayleigh quotient of Laplace operator \( L \) on real-valued function \( f \) on \( V(G) \), where

\[
R_G(f) = \frac{< Lf, f >}{< f, f >} = \frac{\sum_{uv \in E} (f(u) - f(v))^2}{\sum_{v \in V} f^2(v)}.
\]

Let \( \mathcal{F} \) denote the set of all real-valued functions \( f \) on \( V(G) \) with \( f(u) = 0 \) for any boundary vertex \( u \). The following proposition states a well-known fact about the Rayleigh quotients.

**Proposition 2.1.** For a graph \( G = (V_0 \cup \partial V, E_0 \cup \partial E) \) with boundary, we have

\[
\lambda(G) = \max_{f \in \mathcal{F}} R_G(f) = \max_{f \in \mathcal{F}} \frac{< Lf, f >}{< f, f >}.
\]

Moreover, if \( R_G(f) = \lambda(G) \) for a function \( f \in \mathcal{F} \), then \( f \) is a Dirichlet eigenfunction of \( \lambda(G) \).

Let \( Q(G) = D(G) + A(G) \) be the signless Laplacian matrix of \( G \) and its Rayleigh
is the adjacency matrix of the graph $G$, function $f$ is a Dirichlet Perron vector of $G$ if $f(v) > 0$ for all $v \in V(G)$ or $f(v) < 0$ for all $v \in V(G)$.

Let $f$ be a unit eigenvector of $\mu(G)$. We call $f$ a Dirichlet Perron vector of $G$ if $f(v) > 0$ for any $v \in V_0(G)$.

**Lemma 2.3.** Let $G$ be a tree. Then the signless Dirichlet spectral radius $\mu(G)$ of $G$ is positive. Moreover, if $f$ is a signless Dirichlet eigenfunction of $\mu(G)$, then $f(v) > 0$ for all $v \in V(G)$ or $f(v) < 0$ for all $v \in V(G)$.

**Proof.** Without loss of generality, assume that $G = (V_1, V_2, E(G))$ is a tree with bipartition $V_1$ and $V_2$. Let $f$ be Dirichlet Perron vector of $G$. Define $f_1(x) = \text{sign}(x)f(x)$, where $\text{sign}(x) = 1$ if $x \in V_1$ and $\text{sign}(x) = -1$ if $x \in V_2$. Then we have

$$\mu(G) = \Delta_G(f) = \sum_{uv \in E} (f(u) + f(v))^2 = \sum_{uv \in E} (f_1(u) - f_1(v))^2 = R_G(f_1) \leq \lambda(G).$$

The condition $\lambda(G) \leq \mu(G)$ follows analogously. \[\square\]
3. The trees with the largest Dirichlet spectral radius in $\mathcal{T}_\pi$. In this section, we will characterize the trees with the largest Dirichlet spectral radius in $\mathcal{T}_\pi$. Let $G + uv$ (resp. $G - uv$) denote the graph obtained from $G$ by adding (resp. deleting) an edge $uv$ in $G$. The following lemmas will be used in our proof.

**Lemma 3.1.** (See also [12]) Let $T \in \mathcal{T}_\pi$ and $uv, xy \in E(T)$ such that $v$ and $y$ are not in the path from $u$ to $x$. Let $f$ be the Dirichlet Perron vector of $T$ and $T' = T - uv - xy + uy + xv$. Then $T' \in \mathcal{T}_\pi$ and $\lambda(T') \geq \lambda(T)$ if $f(u) \geq f(x)$ and $f(y) \geq f(v)$. Moreover, $\lambda(T') \geq \lambda(T)$ if one of the two inequalities is strict.

**Proof.** Let $f$ be the Dirichlet Perron vector of $T$. Clearly, $T' \in \mathcal{T}_\pi$. Then we have

$$\lambda(T') - \lambda(T) \geq \Delta_T(f) - \Delta_T(f)$$

$$= (f(u) + f(y))^2 + (f(x) + f(v))^2 - (f(u) + f(v))^2 - (f(x) + f(y))^2$$

$$= 2(f(u) - f(x))(f(y) - f(v))$$

$$\geq 0.$$ 

If $\lambda(T') = \lambda(T)$, then $f$ also must be a signless Dirichlet eigenfunction of $\lambda(T')$. By

$$\lambda(T)f(u) = \sum_{z,zu \in E(T) \setminus \{uv\}} (f(u) + f(z)) + (f(u) + f(v))$$

$$= \lambda(T')f(u)$$

$$= \sum_{z,zu \in E(T) \setminus \{uv\}} (f(u) + f(z)) + (f(u) + f(y)),$$

we have $f(y) = f(v)$. Similarly, we have $f(u) = f(x)$. So, the assertion holds. \hfill \Box

Let $\text{dist}(v)$ denote the distance between $v$ and $v_0$, where $v_0$ is the root of $G$. We call $y$ a child of $x$ and $x$ the parent of $y$, if $xy \in E(G)$ with $\text{dist}(y) = \text{dist}(x) + 1$.

**Lemma 3.2.** Let $T = (V_0 \cup \partial V, E_0 \cup \partial E)$ be a tree with the largest Dirichlet spectral radius in $\mathcal{T}_\pi$ and $f$ be the Dirichlet Perron vector of $T$. Then the vertices of $T$ can be relabelled $\{v_0, v_1, \ldots, v_{n-1}\}$ such that the following hold:

(1) $f(v_0) \geq f(v_1) \geq \cdots \geq f(v_{n-1})$;

(2) Let $v_0$ be the root of $T$, then $\text{dist}(v_0) \leq \text{dist}(v_1) \leq \cdots \leq \text{dist}(v_{n-1})$;

(3) If $v_i, v_s \in V(T)$ with $i < s$, then for any child $v_j$ of $v_i$ and any child $v_t$ of $v_s$, there must be $j < t$.

**Proof.** Let $V(T) = \{v_0, v_1, \ldots, v_{n-1}\}$ such that $f(v_0) \geq f(v_1) \geq \cdots \geq f(v_{n-1})$. We start with the vertex $v_0$. If $v_0 v_1 \in V(T)$, there is nothing to do. Otherwise, there exists a child $x_0$ of $v_0$ with $f(v_1) \geq f(x_0)$. If $f(v_1) = f(x_0)$, we exchange the labelling of $v_1$ and $x_0$. In the following, we assume $f(v_1) > f(x_0)$. Then $v_1$ is not a pendant
vertices. Since $T$ is connected, there exist a path $P_{0,1}$ from $v_0$ to $v_1$ and a parent $v_1$ of $v_0$ which is in $P_{0,1}$ and can not be $v_0$. Since $v_1$ is an interior vertex, there is also some child $v_1$ of $v_1$ which is not in $P_{0,1}$. If $x_0 \in P_{0,1}$, let $T_1 = T - v_0x_0 - v_1w_1 + v_0v_1 + x_0w_1$. Otherwise, let $T_1 = T - v_0x_0 - v_1u_1 + v_0v_1 + x_0u_1$. Since $f(v_0) \geq f(w_1)$, $f(v_1) > f(x_0)$ and $f(v_0) \geq f(u_1)$, we have $\lambda(T_1) > \lambda(T)$ by Lemma 3.1. It is a contradiction to our assumption that $T$ has the largest Dirichlet spectral radius in $T_x$. Let $s_0 = d(v_0)$. By the same way, we can prove that $v_0$ is also adjacent to $v_2, v_3, \ldots, v_{s_0}$.

Next we proceed in an analogous way with all children of $v_1$ and prove that the vertices $v_{d(v_0)+1}, v_{d(v_0)+2}, \ldots, v_{s_1}$ are adjacent to $v_1$, where $s_1 = d(v_0) + d(v_1) - 1$. Let $s_{r-1} = d(v_0) + d(v_1) + \cdots + d(v_{r-1}) - r + 1$. Now assume that $v_{r-1}$ has already been adjacent to the respective vertices $v_{s_{r-2}+1}, v_{s_{r-2}+2}, \ldots, v_{s_{r-2}+d(v_{r-1})-1}$. In the following, we observe the vertex $v_r$. If $v_r$ is adjacent to $v_{s_{r-1}+1}$, there is nothing to do. Otherwise, there exist a child $x_r$ of $v_r$ with $f(v_{s_{r+1}+1}) \geq f(x_r)$ and a path $P_{r,s_{r+1}+1}$ from $v_r$ to $v_{s_{r+1}+1}$. Without loss of generality, assume $f(v_{s_{r+1}+1}) > f(x_r)$. Then there exist a parent $u_r$ of $v_{s_{r+1}+1}$ in $P_{r,s_{r+1}+1}$ and some child $w_r$ which is not in $P_{r,s_{r+1}+1}$. Let $T_r = T - v_r x_r - v_{s_{r+1}+1} w_r + v_r v_{s_{r+1}+1} + x_r w_r (if x_r \in P_{r,s_{r+1}+1})$ or $T_r = T - v_r x_r - v_{s_{r+1}+1} w_r + v_r v_{s_{r+1}+1} + x_r w_r (if x_r \notin P_{r,s_{r+1}+1})$. Since $f(v_r) \geq f(u_r)$, $f(v_r) \geq f(w_r)$ and $f(v_{s_{r+1}+1}) > f(x_r)$, we have $\lambda(T_r) > \lambda(T)$ by Lemma 3.1. It is a contradiction to our assumption that $T$ has the largest Dirichlet spectral radius in $T_x$. By the same procedure, we can prove that $v_r$ is adjacent to the respective vertices $v_{s_{r-2}+1}, v_{s_{r-2}+2}, \ldots, v_{s_{r-2}+d(v_{r-1})-1}$.

By the induction, the assertion holds.

**Lemma 3.3.** Let $G = (V_0 \cup \partial V, E_0 \cup \partial E)$ be a graph with boundary and $P$ be a path from an interior vertex $v_1$ to another interior vertex $v_2$. Suppose that $v_1 u_i \notin E(G)$, $v_2 u_i \notin E(G)$ and $u_i$ is not on the path $P$ for $i = 1, 2, \ldots, t$ with $t < d(v_1) - 2$. By deleting the $t$ edges $v_1 u_1, v_1 u_2, \ldots, v_1 u_t$ and adding the $t$ edges $v_2 u_1, v_2 u_2, \ldots, v_2 u_t$, we get a new tree $G'$. Let $f$ be the Dirichlet Perron vector of $G$. Then if $f(v_1) \leq f(v_2)$, we have

$$\lambda(G') > \lambda(G).$$

**Proof.** By

$$\lambda(G') - \lambda(G) \geq \Delta_{G'}(f) - \Delta_G(f)$$

$$= \sum_{i=1}^{t} (f(v_2) + f(u_i))^2 - \sum_{i=1}^{t} (f(v_1) + f(u_i))^2$$

$$\geq 0,$$

we have $\lambda(G') \geq \lambda(G)$. If $\lambda(G') = \lambda(G)$, then $f$ also must be a signless Dirichlet.
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eigenfunction of \( \lambda(G') \). By

\[
\lambda(G') f(v_1) = \sum_{x, v_1 \in G'} (f(v_1) + f(x)) = \lambda(G) f(v_1) = \sum_{x, v_1 \in G'} (f(v_1) + f(x)) + \sum_{i=1}^{t}(f(v_1) + f(u_i)),
\]

we have \( \sum_{i=1}^{t}(f(v_1) + f(u_i)) = 0 \). This is a contradiction with \( f(v_1) > 0 \) and \( f(u_i) \geq 0 \).

So, the assertion holds. The proof is completed. \( \square \)

In the following, we use the method of [12] to define a special tree \( T_\pi^* \) with a given nonincreasing tree degree sequence \( \pi = (d_0, d_1, \ldots, d_n) \) as follows. Select a vertex \( v_{0,1} \) as the root of \( T_\pi^* \) and begin with \( v_{0,1} \) of the zero-th layer. Let \( s_1 = d_0 \) and select \( s_1 \) vertices \( v_{1,1}, v_{1,2}, \ldots, v_{1,s_1} \) of the first layer as the children of \( v_{0,1} \). Next we construct the second layer as follows. Let \( s_2 = s_1 - d_1 + 1 \) and select \( s_2 \) vertices \( v_{2,1}, v_{2,2}, \ldots, v_{2,s_2} \) such that \( v_{2,1}, \ldots, v_{2,d_1-1} \) are the children of \( v_{1,1} \), and \( v_{2,d_1}, \ldots, v_{2,d_1+d_2-2} \) are the children of \( v_{1,2}, \ldots, v_{1,4} \) and \( v_{2,d_i+\cdots+d_{i-1}-s_i+2}, \ldots, v_{2,d_i+\cdots+d_{i-1}-s_i} \) are the children of \( v_{1,s_i} \).

Assume that all vertices of the \( t \)-th layer have been constructed and are denoted by \( v_{t,1}, v_{t,2}, \ldots, v_{t,s_t} \). We construct all the vertices of the \( (t+1) \)-st layer by the induction hypothesis. Let \( s_{t+1} = s_t + \cdots + s_{t-1} + d_t - s_t \) and select \( s_{t+1} \) vertices \( v_{t+1,1}, v_{t+1,2}, \ldots, v_{t+1,s_{t+1}} \) of the \( (t+1) \)-st layer such that \( v_{t+1,1}, \ldots, v_{t+1,d_{t}+\cdots+d_{t-1}-s_{t-1}} \) are the children of \( v_{t,1} \), \ldots, and \( v_{t+1,s_{t+1}-d_{t}+\cdots+d_{t-1}+2}, \ldots, v_{t+1,s_{t+1}} \) are the children of \( v_{t,s_t} \).

In this way, we obtain only one tree \( T_\pi^* \) with degree sequence \( \pi \) such that \( v_{0,1} \) has the maximum degree in all interior vertices (see Fig. 3.1 for an example).

**Example 3.4.** Let \( \pi = (4, 3, 3, 3, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1) \). Then \( T_\pi^* \) is as follows:

![Fig. 3.1. \( T_\pi^* \) with degree sequence \( \pi \) (o ... boundary vertices).](image-url)
Proof of Theorem 1.1. Let $T$ be a tree with the largest Dirichlet spectral radius in $\mathcal{T}$. Suppose $V(T) = \{v_0, v_1, \ldots, v_{n-1}\}$ such that they satisfy the three assertions in Lemma 3.2. Let $f$ be the Dirichlet Perron vector of $T$.

In the following, we will prove that $d(v_0) \geq d(v_1) \geq \cdots \geq d(v_{n-1})$. If the assertion does not hold, there exists the smallest integer $t \in \{0, 1, \ldots, n-1\}$ such that $d(v_t) < d(v_{t+1})$. Since $f(v_t) \geq f(v_{t+1})$, $v_t$ and $v_{t+1}$ are interior vertices. Let $u_1, u_2, \ldots, u_{d(v_{t+1})-1}$ be all the children of $v_{t+1}$. Then we have $f(v_t) \geq f(v_{t+1}) \geq f(u_i)$ for $1 \leq i \leq d(v_{t+1})-1$ by Lemma 3.2. Let $T_1 = T - v_{t+1}u_1 - v_{t+1}u_2 - \cdots - v_{t+1}u_s + v_{t+1}u_1 + v_{t+1}u_2 + \cdots + v_{t+1}u_s$, where $s = d(v_{t+1}) - d(v_t)$. Then $T_1 \in \mathcal{T}_\pi$ and $\lambda(T_1) > \lambda(T)$ by Lemma 3.3. This is a contradiction to our assumption that $T$ has the largest Dirichlet spectral radius in $\mathcal{T}$. So, we have $d(v_i) = d_i$ for $0 \leq i \leq n-1$. Clearly, $T$ is isomorphic to $T^*_\pi$. The proof is completed.

Let $\pi = (d_0, d_1, \ldots, d_{n-1})$ and $\pi' = (d'_0, d'_1, \ldots, d'_{n-1})$ be two nonincreasing positive sequences. If $\sum_{i=0}^t d_i' \leq \sum_{i=0}^t d_i$ for $t = 0, 1, \ldots, n-2$ and $\sum_{i=0}^{n-1} d_i = \sum_{i=0}^{n-1} d_i'$, then $\pi'$ is said to majorize $\pi$, and is denoted by $\pi \preceq \pi'$ (see [12]).

Lemma 3.5. (12) Let $\pi = (d_0, d_1, \ldots, d_{n-1})$ and $\pi' = (d'_0, d'_1, \ldots, d'_{n-1})$ be two nonincreasing graphic degree sequences. If $\pi \preceq \pi'$, then there exist a series of graphic degree sequences $\pi_1, \pi_2, \ldots, \pi_k$ such that $\pi \preceq \pi_1 \preceq \pi_2 \preceq \cdots \preceq \pi_k \preceq \pi'$, and only two components of $\pi$, and $\pi+i$ are different by 1.

Theorem 3.6. Let $\pi$ and $\pi'$ be two tree degree sequences such that they have same frequency of the number 1. If $\pi \preceq \pi'$, then $\lambda(T^*_\pi) \leq \lambda(T^*_\pi')$ with equality holds if and only if $\pi = \pi'$.

Proof. Let $T$ be a tree with $k$ pendant vertices and degree sequence $\pi = (d_0, d_1, \ldots, d_{n-1})$. Then $d_{n-k} = d_{n-k+1} = \cdots = d_{n-1} = 1$ and $d_{n-k-1} \geq 2$. Clearly,
π ≤ ω. By Theorem 3.6, the assertion holds.

**Corollary 3.8.** Let \( T \) be a tree of order \( n \) with \( k \) pendant vertices. If \( n ≤ 2k+1 \), then \( \lambda(T) \leq \frac{2k+\sqrt{k^2-8k+4n}}{2} \) with equality if and only if \( T \) is \( T_0^* \).

**Proof.** Let \( f \) be the Dirichlet Perron vector of \( T_0^* \) and \( u \in V(T_0^*) \) with \( d(u) = k \). Since \( n ≤ 2k + 1 \), the vertex \( u \) is adjacent to any vertex \( v \) with \( d(v) = 2 \). By \( \lambda(T_0^*)f(u) = kf(u) + (n - k - 1)f(v) \) and \( \lambda(T_0^*)f(v) = 2f(v) + f(u) \), we have \( \lambda(T_0^*) = \frac{2k+\sqrt{k^2-8k+4n}}{2} \). The assertion holds by Corollary 3.7.

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**REFERENCES**