Signless Laplacian spectral characterization of some joins

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SIGNLESS LAPLACIAN SPECTRAL CHARACTERIZATION OF SOME JOINS

XIAOGANG LIU† AND PENGLI LU‡

Abstract. The join of two disjoint graphs $G$ and $H$, denoted by $G \vee H$, is the graph obtained by joining each vertex of $G$ to each vertex of $H$. In this paper, the signless Laplacian characteristic polynomial of the join of two graphs is first formulated. And then, a lower bound for the $i$th largest signless Laplacian eigenvalue of a graph is given. Finally, it is proved that $G \vee K_m$, where $G$ is an $(n-2)$-regular graph on $n$ vertices, and $K_n \vee K_2$ except for $n = 3$, are determined by their signless Laplacian spectra.

Key words. Signless Laplacian spectrum, Q-cospectral, Join, Determined by the signless Laplacian spectrum.

AMS subject classifications. 05C50.

1. Introduction. All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $v_1, v_2, \ldots, v_n$ are indexed in the non-increasing order of degrees. Let $d_i = d_i(G) = d_G(v_i)$ be the degree of the vertex $v_i$, and $\text{deg}(G) = (d_1, d_2, \ldots, d_n)$ be the degree sequence of $G$. The complement of $G$, denoted by $\overline{G}$, is the graph with the same vertex set as $G$ such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. The adjacency matrix of $G$, denoted by $A(G)$, is the $n \times n$ matrix whose $(i, j)$-entry is 1 if $v_i$ and $v_j$ are adjacent and 0 otherwise. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of $G$, where $D(G)$ is the $n \times n$ diagonal matrix with $d_1, d_2, \ldots, d_n$ as diagonal entries. Given an $n \times n$ symmetric matrix $M$, denote by $\phi(M; x) = \det(xI_n - M)$, or simply $\phi(M)$, the characteristic polynomial of $M$, where $I_n$ is the identity matrix of size $n$. The roots of the equation $\phi(M; x) = 0$, denoted by $\theta_1(M) \geq \theta_2(M) \geq \cdots \geq \theta_n(M)$, are called the eigenvalues of $M$. The eigenvalues of $A(G)$ and $Q(G)$ are called the adjacency eigenvalues and signless Laplacian eigenvalues of $G$, respectively. Denote

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by \( \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \) and \( \nu_1(G) \geq \nu_2(G) \geq \cdots \geq \nu_n(G) \) the adjacency eigenvalues and the signless Laplacian eigenvalues of \( G \), respectively. The multi-set of eigenvalues of \( A(G) \) (respectively, \( Q(G) \)) is called the adjacency (respectively, signless Laplacian) spectrum of \( G \). Two graphs are said to be \( A \)-cospectral (respectively, \( Q \)-cospectral) if they have the same adjacency (respectively, signless Laplacian) spectrum. A graph is said to be determined by its signless Laplacian (respectively, adjacency) spectrum if there is no other non-isomorphic graph which is \( Q \)-cospectral (respectively, \( A \)-cospectral) with it.

Which graphs are determined by their spectra? This is a classical question in spectral graph theory, which was raised by Günthard and Primas [12] in 1956 with motivations from chemistry. For its background, please refer to [1, 5, 7, 9, 10]. It is well known that this question is still far from being completely solved, since it is often very challenging to check whether or not an arbitrary given graph is determined by its spectrum, even for some simple-looking graphs.

Let \( G \) and \( H \) be two disjoint graphs. Denote by \( G \cup H \) the disjoint union of \( G \) and \( H \). Especially, \( mG \) means the disjoint union of \( m \) copies of \( G \). The join of two disjoint graphs \( G \) and \( H \), denoted by \( G \lor H \), is the graph obtained by joining each vertex of \( G \) to each vertex of \( H \). Until now only several joins have been proven to be determined by their signless Laplacian spectra, stated as follows:

1. Any star \( K_{1,m} \cong mK_1 \lor K_1 \), except for \( K_{1,3} \), is determined by its signless Laplacian spectrum [3, 18, 19], where \( K_n \) denotes a complete graph on \( n \) vertices.
2. \( K_n \) is determined by its signless Laplacian spectrum [7].
3. \( K_1 \lor (cK_2 \cup (n-2c-1)K_1) \) with \( n \geq 2c+1 \) and \( c \geq 0 \) is determined by its signless Laplacian spectrum [13].
4. Any wheel graph \( K_1 \lor C_n \) is determined by its signless Laplacian spectrum [16], where \( C_n \) denotes a cycle on \( n \) vertices.
5. Let \( G \) be an \( r \)-regular graph on \( n \) vertices. For \( r = 1 \), \( r = n-2 \), or \( r = 2 \) and \( n \geq 11 \), \( G \lor K_1 \) is determined by its signless Laplacian spectrum. For \( r = n-3 \), \( G \lor K_1 \) is determined by its signless Laplacian spectrum if and only if \( G \) has no triangles [2].
6. Let \( G \) be an \( r \)-regular graph on \( n \) vertices. For \( r = 1 \) or \( r = n-2 \), \( G \lor K_2 \) is determined by its signless Laplacian spectrum. For \( r = n-3 \), \( G \lor K_2 \) is determined by its signless Laplacian spectrum if and only if \( G \) has no triangles [20].

The main purpose of this paper is to prove that some new families of joins are determined by their signless Laplacian spectra. Our paper is organized as follows. In Section 2 we formulate the signless Laplacian characteristic polynomial of the join of two graphs. Section 3 gives a lower bound for the \( i \)th largest signless Laplacian
eigenvalue of a graph. In Section 2 we prove that $G \lor K_n$, where $G$ is an $(n-2)$-regular graph on $n$ vertices, and $K_n \lor K_2$ except for $n = 3$, are determined by their signless Laplacian spectra.

2. The signless Laplacian characteristic polynomial of the join of two graphs. The $M$-coronal of an $n \times n$ square matrix $M$, denoted by $\Gamma_M(x)$, is defined to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is,

$$\Gamma_M(x) = 1^T(xI_n - M)^{-1}1_n,$$

where $1_n$ denotes the column vector of size $n$ with all the entries equal to one, and $1_n^T$ means the transpose of $1_n$. The characteristic polynomial of the signless Laplacian matrix of a graph $G$ is called the signless Laplacian characteristic polynomial of $G$.

In this section, we determine the signless Laplacian characteristic polynomial of the join of two graphs with the help of the coronal of the signless Laplacian matrix.

**Theorem 2.1.** Let $G_i$ be an arbitrary graph on $n_i$ vertices for $i = 1, 2$. Then

$$\phi(Q(G_1 \lor G_2); x) = \phi(Q(G_1); x - n_2) \cdot \phi(Q(G_2); x - n_1) \cdot (1 - \Gamma_Q(G_1)(x - n_2) \cdot \Gamma_Q(G_2)(x - n_1)),$$

where $Q$ denotes the signless Laplacian matrix of $G_1 \lor G_2$ can be written as

$$Q(G_1 \lor G_2) = \begin{pmatrix} n_2I_{n_1} + Q(G_1) & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & n_1I_{n_2} + Q(G_2) \end{pmatrix},$$

where $J_{s \times t}$ denotes the $s \times t$ matrix with all entries equal to one. Then the signless Laplacian characteristic polynomial of $G_1 \lor G_2$ is given by

$$\phi(Q(G_1 \lor G_2); x) = \det \begin{pmatrix} (x - n_2)I_{n_1} - Q(G_1) & -J_{n_1 \times n_2} \\ -J_{n_2 \times n_1} & (x - n_1)I_{n_2} - Q(G_2) \end{pmatrix}$$

$$= \det((x - n_2)I_{n_1} - Q(G_1)) \cdot \det(S)$$

$$= \phi(Q(G_1); x - n_2) \cdot \det(S),$$

where

$$S = (x - n_1)I_{n_2} - Q(G_2) - J_{n_2 \times n_1}((x - n_2)I_{n_1} - Q(G_1))^{-1}J_{n_1 \times n_2}$$

$$= (x - n_1)I_{n_2} - Q(G_2) - \Gamma_Q(G_1)(x - n_2)J_{n_2 \times n_2}$$

is the Schur complement of $(x - n_1)I_{n_2} - Q(G_2)$. Note that $J_{n_2 \times n_2} = 1_{n_2}1_{n_2}^T$, whose rank is equal to 1. Then the result follows from...
Here, in the penultimate step, we used the Sherman-Morrison formula for the determinant.

Theorem 2.1 enables us to construct infinitely many pairs of $Q$-cospectral graphs, as stated in the following corollary.

**Corollary 2.2.** If $G$ is an arbitrary graph, and $H_1$ and $H_2$ are $Q$-cospectral graphs with $\Gamma_{Q(H_1)}(x) = \Gamma_{Q(H_2)}(x)$, then $G \vee H_1$ and $G \vee H_2$ are $Q$-cospectral.

**Example 2.3.** (See [2, Theorem 3.6]) It is known that $K_{1,3}$ and $C_3 \cup K_1$ are $Q$-cospectral. By simple computations, we have $\Gamma_{Q(K_{1,3})}(x) = \Gamma_{Q(C_3 \cup K_1)}(x) = 4(x - 1)/x(x - 4)$. Then, by Corollary 2.2, we obtain that $G \vee K_{1,3}$ and $G \vee (C_3 \cup K_1)$ are $Q$-cospectral, where $G$ is an arbitrary graph.

Let $G$ be an $r$-regular with $n$ vertices and let $K_{p,q}$ denote the complete bipartite graph with $p, q \geq 1$ vertices in the two parts of its bipartition. The following formulas come from [1]:

\[
\Gamma_{Q(G)}(x) = \frac{n}{x - 2r}; \\
\Gamma_{Q(K_{p,q})}(x) = \frac{(p + q)x - (p - q)^2}{x^2 - (p + q)x}.
\]

Substituting these back into Theorem 2.1 we obtain the following results.

**Corollary 2.4.** (See [13, Theorem 2.1]) For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph on $n_i$ vertices. Then

\[
\phi(Q(G_1 \vee G_2); x) = \frac{\phi(Q(G_1); x - n_2)\phi(Q(G_2); x - n_1)}{(x - 2r_1 - n_2)(x - 2r_2 - n_1)} \cdot f(x),
\]

where $f(x) = x^2 - (2(r_1 + r_2) + (n_1 + n_2))x + 2(2r_1r_2 + r_1n_1 + r_2n_2)$.

**Corollary 2.5.** Let $G$ be an $r$-regular graph on $n$ vertices and $K_{p,q}$ a complete bipartite graph. Then

\[
\phi(Q(G \vee K_{p,q}); x) = \frac{\phi(Q(G); x - p - q)\phi(Q(K_{p,q}); x - n)}{(x - 2r - p - q)(x - n - p - q)} \cdot g(x),
\]

where $g(x) = x^3 - 2(n + p + q + r)x^2 + (n^2 + (p + q + 2r)(p + q + 2n))x - 2n(r(p + q) - p - q)^2$.
q + n) + 2pq).

**Corollary 2.6.** Let $K_{m,n}$ and $K_{p,q}$ be two complete bipartite graphs. Then

$$\phi(Q(K_{m,n} \lor K_{p,q}); x) = \frac{\phi(K_{m,n}; x - p - q)\phi(Q(K_{p,q}); x - m - n)}{(x - p - q)(x - m - n)(x - m - n - p - q)^2} h(x),$$

where $h(x) = (x-p-q)(x-m-n)(x-m-n-p-q)^2 - (m+n)(x-p-q) - (m-n)^2) \times ((p+q)(x-m-n) - (p-q)^2).

### 3. A lower bound for the $i$th largest signless Laplacian eigenvalue.

In this section, we will give a lower bound for the $i$th largest signless Laplacian eigenvalue of a graph. Before proceeding, we need to mention two well known results from matrix theory. Recall that $\theta_1(M) \geq \theta_2(M) \geq \ldots \geq \theta_n(M)$ denote the eigenvalues of an $n \times n$ symmetric matrix $M$.

**Lemma 3.1.** (See [14], Cauchy interlacing theorem) Let $H$ be a Hermitian matrix of order $n$. If $H_m$ is a principal submatrix of $H$ of order $m$ with $1 \leq m \leq n$, then for $1 \leq i \leq m$,

$$\theta_{i+n-m}(H) \leq \theta_i(H_m) \leq \theta_i(H).$$

**Lemma 3.2.** (See [14], Weyl's inequality) Let $H = M + P$, where $M$ and $P$ are Hermitian matrices of order $n$. Then for each $i = 1, 2, \ldots, n$,

$$\theta_i(M) + \theta_n(P) \leq \theta_i(H) \leq \theta_i(M) + \theta_i(P).$$

**Theorem 3.3.** Let $G$ be a graph of order $n$ with non-increasing degree sequence $\deg(G) = (d_1, d_2, \ldots, d_n)$. Let $G_i$ be the graph induced by vertices $v_1, v_2, \ldots, v_i$, where $2 \leq i \leq n$. Then

$$\nu_i(G) \geq d_i + \lambda_i(G_i).$$

**Proof.** Index the rows and columns of $Q(G)$ by $v_1, v_2, \ldots, v_n$. Let $M$ be the top-left principal $i \times i$ submatrix of $Q(G)$. By Lemma 3.1, $\nu_i(G) \geq \theta_i(M)$. Note that $M = D_i + (d_i I_i + A(G_i))$, where $D_i$ is the $i \times i$ diagonal matrix with $d_1 - d_i, d_2 - d_i, \ldots, d_{i-1} - d_i, 0$ as diagonal entries. Thus, by Lemma 3.2, $\theta_i(M) \geq \theta_i(D_i) + \theta_i(d_i I_i + A(G_i)) = d_i + \lambda_i(G_i)$. \[\square\]
Let \( P_n \) denotes a path on \( n \) vertices. Note that \( \lambda_2(2P_1) = 0 \) and \( \lambda_2(P_2) = -1 \). By Theorem 3.3, we obtain the following result immediately.

**Corollary 3.4.** (See [11 Corollary 3.2]) Let \( G \) be a graph of order \( n \) with non-increasing degree sequence \( \text{deg}(G) = (d_1, d_2, \ldots, d_n) \). Then

(a) \( \nu_2(G) \geq d_2 \), if \( v_1 \) and \( v_2 \) are not adjacent;
(b) \( \nu_2(G) \geq d_2 - 1 \), if \( v_1 \) and \( v_2 \) are adjacent.

Since \( \lambda_3(3P_1) = 0 \), \( \lambda_3(3P_2) = \lambda_3(K_4) = -1 \) and \( \lambda_3(P_3) = -\sqrt{2} \), Theorem 3.3 implies the following result.

**Corollary 3.5.** (See [21 Theorem 3.1]) Let \( G \) be a graph of order \( n \) with non-increasing degree sequence \( \text{deg}(G) = (d_1, d_2, \ldots, d_n) \). Then

(a) \( \nu_3(G) \geq d_3 \), if \( v_1, v_2 \) and \( v_3 \) induce \( 3P_1 \);
(b) \( \nu_3(G) \geq d_3 - 1 \), if \( v_1, v_2 \) and \( v_3 \) induce \( P_2 \cup P_1 \) or \( K_3 \);
(c) \( \nu_3(G) \geq d_3 - \sqrt{2} \), if \( v_1, v_2 \) and \( v_3 \) induce \( P_3 \).

**Corollary 3.6.** Let \( G \) be a graph of order \( n \) with non-increasing degree sequence \( \text{deg}(G) = (d_1, d_2, \ldots, d_n) \). Let \( G_4 \) be the graph induced by vertices \( v_1, \ldots, v_4 \). Then

(a) \( \nu_4(G) \geq d_4 \), if \( G_4 \cong 4P_1 \);
(b) \( \nu_4(G) \geq d_4 - 1 \), if \( G_4 \cong 2P_1 \cup P_2 \), \( 2P_2 \), \( P_1 \cup G_3 \) or \( K_4 \);
(c) \( \nu_4(G) \geq d_4 - \sqrt{2} \), if \( G_4 \cong P_1 \cup P_3 \);
(d) \( \nu_4(G) \geq d_4 + \lambda_{\text{min}}, \) if \( G_4 \cong G_4^1 \), where \( \lambda_{\text{min}} \) is the smallest root of \( \lambda^3 - \lambda^2 - 3\lambda + 1 = 0 \);
(e) \( \nu_4(G) \geq d_4 - \frac{\sqrt{2} - 1}{2} \), if \( G_4 \cong G_4^2 \);
(f) \( \nu_4(G) \geq d_4 - \frac{\sqrt{3} - 2}{6} \), if \( G_4 \cong P_4 \);
(g) \( \nu_4(G) \geq d_4 - \frac{1}{2} \), if \( G_4 \cong K_1,3 \);
(h) \( \nu_4(G) \geq d_4 - 2 \), if \( G_4 \cong C_4 \),

where \( G_4^1 \) and \( G_4^2 \) are shown in Figure 3.1.

**Proof.** The result follows from Theorem 3.3 and Table 3.1.

**Corollary 3.7.** Let \( G \) be a graph of order \( n \) with non-increasing degree sequence \( \text{deg}(G) = (d_1, d_2, \ldots, d_n) \). Then \( \nu_4(G) \geq d_4 - 2 \).

**Proof.** Let \( f(\lambda) = \lambda^3 - \lambda^2 - 3\lambda + 1 \). Note that \( f(-2) = -5 < 0 \), \( f(-1) = 2 > 0 \), \( f(1) = -2 < 0 \) and \( f(3) = 10 > 0 \). This implies that the smallest root of \( f(\lambda) = 0 \) lies in \((-2, -1)\). Then the result follows from Corollary 3.6.

4. Some joins determined by their signless Laplacian spectra. As mentioned in Section 1 that characterizing which graphs are determined by the spectra is very challenging, and that only several joins are proved to be determined by their
signless Laplacian spectra. In this section, we will prove that some new families of joins are determined by the signless Laplacian spectra.

**Lemma 4.1.** (See [11, Theorem 4.1]) Let $G$ be connected graph of order $n$. Then $\nu_n(G) < d_n(G)$.

Let $G$ be a graph with $n$ vertices. Define $T_k = \sum_{i=1}^{n} \nu_i(G)^k$ to be the $k$th $Q$-

**Table 3.1**

<table>
<thead>
<tr>
<th>graph</th>
<th>characteristic polynomial</th>
<th>$\lambda_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4P_1$</td>
<td>$\lambda^4$</td>
<td>0</td>
</tr>
<tr>
<td>$2P_1 \cup P_2$</td>
<td>$\lambda^2(\lambda - 1)(\lambda + 1)$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$2P_2$</td>
<td>$(\lambda - 1)^2(\lambda + 1)^2$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$P_1 \cup P_3$</td>
<td>$\lambda^2(\lambda - 2)$</td>
<td>$-\sqrt{2}$</td>
</tr>
<tr>
<td>$P_2 \cup C_3$</td>
<td>$\lambda(\lambda - 2)(\lambda + 1)^2$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$P_4$</td>
<td>$\lambda^2(\lambda^2 - \lambda - 1)(\lambda^2 + \lambda - 1)$</td>
<td>$-1 - \sqrt{2}$</td>
</tr>
<tr>
<td>$K_{1,3}$</td>
<td>$\lambda^4(\lambda^2 - 3)$</td>
<td>$-\sqrt{3}$</td>
</tr>
<tr>
<td>$G^1_4$</td>
<td>$(\lambda + 1)(\lambda^3 - \lambda^2 - 3\lambda + 1)$</td>
<td>the smallest root of $\lambda^3 - \lambda^2 - 3\lambda + 1 = 0$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>$\lambda^2(\lambda - 2)(\lambda + 2)$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$G^2_4$</td>
<td>$\lambda(\lambda + 1)(\lambda^2 - \lambda - 4)$</td>
<td>$1 - \sqrt{13}$</td>
</tr>
<tr>
<td>$K_4$</td>
<td>$(\lambda - 3)(\lambda + 1)^3$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

*The characteristic polynomials and the smallest eigenvalues of graphs in Figure 3.1.*

**Fig. 3.1.** Graphs with 4 vertices.
The spectral moment of $G, k = 0, 1, 2, \ldots$

**Lemma 4.2.** (See [6, Corollary 4.3]) Let $G$ be a graph with $n$ vertices, $m$ edges and $n_3(G)$ triangles. Then

$$T_0 = n, \quad T_1 = \sum_{i=1}^{n} d_i = 2m, \quad T_2 = 2m + \sum_{i=1}^{n} d_i^2, \quad T_3 = 6n_3(G) + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

**Lemma 4.3.** Let $G$ be an $r$-regular graph on $n$ vertices and $G$ is determined by its signless Laplacian spectrum. Let $H$ be a graph $Q$-cospectral with $G \lor K_m$. If $d_1(H) = d_2(H) = \cdots = d_m(H) = n + m - 1$, then $H \cong G \lor K_m$.

**Proof.** The following equations follow from Lemma 4.2

\begin{align}
\sum_{i=1}^{m+n} d_i(H) &= m(n + m - 1) + n(r + m), \\
\sum_{i=1}^{m+n} d_i^2(H) &= m(n + m - 1)^2 + n(r + m)^2.
\end{align}

Plugging $d_1(H) = d_2(H) = \cdots = d_m(H) = n + m - 1$ into equations (4.1) and (4.2), we have

$$\sum_{i=m+1}^{m+n} d_i(H) = n(r + m), \quad \sum_{i=m+1}^{m+n} d_i^2(H) = n(r + m)^2.$$

Then

$$\sum_{i=m+1}^{m+n} (d_i(H) - r - m)^2 = \sum_{i=m+1}^{m+n} d_i(H)^2 - 2(r + m) \sum_{i=m+1}^{m+n} d_i(H) + \sum_{i=m+1}^{m+n} (r + m)^2 = 0.$$

This implies that $d_{m+1}(H) = d_{m+2}(H) = \cdots = d_{m+n}(H) = r + m$. Then $H \cong G_1 \lor K_m$, where $G_1$ is an $r$-regular graph. Corollary 2.4 implies that $G$ and $G_1$ are $Q$-cospectral graph. Therefore, $H \cong G \lor K_m$ comes from the assumption that $G$ is determined by its signless Laplacian spectrum.

**Remark 4.4.** It is known [9, Proposition 3] that a regular graph $G$ is determined by its signless Laplacian spectrum if and only if $G$ is determined by its adjacency spectrum. So, the assumption that $G$ is determined by its signless Laplacian spectrum in Lemma 4.3 can be replaced by that $G$ is determined by its adjacency spectrum.

**Theorem 4.5.** Let $G$ be an $(n - 2)$-regular graph on $n$ vertices. Then $G \lor K_m$ is determined by its signless Laplacian spectrum.
Proof. Let \( H \) be a graph \( Q \)-cospectral with \( G \lor K_m \). Then Lemma 4.2 implies that

\[
\sum_{i=1}^{m+n} d_i(H) = m(m+n-1) + n(m+n-2) = (m+n)^2 - (m+n) - n.
\] (4.3)

Note that \( d_i(H) \leq m+n-1 \) for \( i = 1, 2, \ldots, m+n \), since \( H \) has \( m+n \) vertices. Consider the following cases.

If \( d_m(H) = m+n-1 \), then \( d_1(H) = d_2(H) = \cdots = d_{m+n}(H) = m+n-1 \). Note that \( G \) is determined by its signless Laplacian spectrum \cite{8}. By Lemma 4.3, \( H \cong G \lor K_m \).

If \( d_m(H) \leq m+n-2 \), then

\[
\sum_{i=1}^{m+n} d_i(H) \leq (m-1)(m+n-1) + (n+1)(m+n-2) = (m+n)^2 - (m+n) - (n+1),
\]

a contradiction to equation (4.3).

Remark 4.6. The special cases of Theorem 4.5 when \( m = 1 \) and \( m = 2 \) were considered in \cite{2} and \cite{20}, respectively.

Note that \( \phi(Q(K_n); x) = x^n \) and \( \phi(Q(K_m); x) = (x-2m+2)(x-m+2)^{m-1} \). The following result follows from Corollary 2.4 by letting \( G_1 = K_n \) and \( G_2 = K_m \).

Lemma 4.7. The signless Laplacian spectrum of \( K_n \lor K_m \) is

\[
\frac{(3m+n-2) \pm \sqrt{(m+n-2)^2 + 4mn}}{2}, m^{n-1}, (n+m-2)^{m-1},
\]

where an exponent indicates the multiplicity of the corresponding signless Laplacian eigenvalue.

Theorem 4.8. \( K_n \lor K_2 \) is determined by its signless Laplacian spectrum except for \( n = 3 \).

Proof. If \( n = 1 \), then \( K_1 \lor K_2 \cong K_3 \), which is determined by its signless Laplacian spectrum. In the following, we consider the cases of \( n \geq 2 \). Lemma 4.7 implies that the signless Laplacian spectrum of \( K_n \lor K_2 \) is

\[
\frac{(n+4) \pm \sqrt{n^2 + 8n}}{2}, n, 2^{n-1}.
\]

Let \( H \) be \( Q \)-cospectral with \( K_n \lor K_2 \). Lemma 4.1 and Corollary 3.5 imply respectively that

\[
d_{n+2}(H) > \frac{(n+4) - \sqrt{n^2 + 8n}}{2} > 0, \quad d_3(H) \leq 2 + \sqrt{2}.
\]
Then \(3 \geq d_3(H) \geq d_{n+2}(H) \geq 1\). Denote by \(x_i\), for \(i = 1, 2, 3\), the number of vertices with degree \(i\) in \(V(H) \setminus \{v_1(H), v_2(H)\}\), where \(v_i(H)\) denotes the vertex with degree \(d_i(H)\) for \(i = 1, 2, 3\). Consider the following cases.

If \(d_3(H) \leq 2\), then Lemma 4.2 implies that \(x_1 + x_2 = n\) and \(x_1 + 2x_2 + d_2(H) + d_1(H) = 2(n + 1) + 2n\). Then \(x_2 = n + 2(n + 1) - d_2(H) - d_1(H) \leq n\). This implies that \(d_1(H) + d_2(H) \geq 2(n + 1)\). Note that \(d_2(H) \leq d_1(H) \leq n + 1\), since \(H\) has \(n + 2\) vertices. Thus, we have \(d_1(H) = d_2(H) = n + 1\). By Lemma 4.3, \(H \cong K_n \lor K_2\).

If \(d_3(H) = 3\), then by Lemma 4.2, we have

\[
\begin{align*}
&x_1 + x_2 + x_3 = n, \\
&x_1 + 2x_2 + 3x_3 + d_2(H) + d_1(H) = 2(n + 1) + 2n, \\
&x_1 + 4x_2 + 9x_3 + d_2(H)^2 + d_1(H)^2 = 2(n + 1)^2 + 4n.
\end{align*}
\]

By solving the above equations, we have

\[x_1 + x_3 = (n + 1 - d_1(H))(n + d_1(H) - 3) + (n + 1 - d_2(H))(n + d_2(H) - 3)\]

If \(n \geq d_1(H) \geq d_2(H) \geq d_3(H) = 3\), then \(n \geq x_1 + x_3 \geq 2n + d_1(H) + d_2(H) - 6 > n\), a contradiction.

If \(d_1(H) = n + 1\) and \(n \geq d_2(H) \geq d_3(H) = 3\), then \(n \geq x_1 + x_3 = (n - 1)^2 - (d_2(H) - 2)^2 \geq 2n - 3\). This implies that \(n = 3\). Then \(d_1(H) = 4\), \(d_2(H) = 3\) and \(x_2 = 0\). Plugging them back into \((4.4)\), we have \(x_1 = 1\) and \(x_3 = 2\). So, \(H\) has \(5\) vertices with \(\text{deg}(H) = (4, 3, 3, 3, 1)\). There is a unique possibility for \(H\) as shown in Figure 4.1. By Maple, \(K_3 \lor K_2\) and \(H_1\) have the same signless Laplacian spectrum: \(3, (7 + \sqrt{33})/2, 2^2\).

If \(d_1(H) = n + 1\) and \(d_2(H) = n + 1\), then by solving \((4.4)\), we have \(x_3 = 0\), a contradiction to our assumption of \(d_3(H) = 3\).

Remark 4.9. The Q-cospectral mate \(K_3 \lor K_2\) and \(H_1\) given in Theorem 4.8 was indeed presented in [2] Theorem 3.6, where the authors showed an infinite family of Q-cospectral mates. We also presented this result in Example 2.3.
Theorem [4.8] proved that $K_n \vee K_2$ is determined by its signless Laplacian spectrum except for $n = 3$. One may naturally ask whether $K_n \vee K_m$ is determined by its signless Laplacian spectrum for $m \geq 3$. From our experiences, the crucial point to prove this is to determine the degree sequence of the $Q$-cospectral graph with $K_n \vee K_m$ for $m \geq 3$.

This will be more difficult than what we did in Theorem [4.8].

**Question 4.10.** Prove or disprove that $K_n \vee K_m$ is determined by its signless Laplacian spectrum for $m \geq 3$.

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**REFERENCES**


