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THE NUMERICAL RADIUS OF A WEIGHTED SHIFT OPERATOR

BATZORIG UNDRAKH†, HIROSHI NAKAZATO‡, ADIYASUREN VANDANJAV§, AND MAO-TING CHIEN¶

Abstract. In this paper, the point spectrum of the real Hermitian part of a weighted shift operator with weight sequence \(a_1, a_2, \ldots, a_n, 1, 1, \ldots\) is investigated and the numerical radius of the weighted shift operator in terms of the weighted shift matrix with weights \(a_1, a_2, \ldots, a_n\) is formulated explicitly.

Key words. Numerical radius, Weighted shift operator, Weighted shift matrix.

AMS subject classifications. 47A12, 15A60.

1. Introduction. Let \(A\) be an operator on a separable Hilbert space \(H\). The numerical range of \(A\) is defined to be the set

\[
W(A) = \{\langle Ax, x \rangle : \|x\| = 1, x \in H\}.
\]

The numerical radius \(w(A)\) is the supremum of the modulus of \(W(A)\). It is a classical result due to Toeplitz and Hausdorff that the numerical range is a convex set. For references on the theory of numerical range, see, for instance, [1, 10, 11, 12]. We consider a weighted shift operator \(A\) with weights \((a_1, a_2, \ldots)\) on the Hilbert space.
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\[ A = A(a_1, a_2, \ldots) = \begin{pmatrix}
0 & 0 & 0 & \cdots & \cdots \\
a_1 & 0 & 0 & \ddots & \ddots \\
0 & a_2 & 0 & \ddots & \ddots \\
0 & 0 & a_3 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}, \]

where \( \{a_n\} \) is a bounded sequence. From the operator-theoretic point of view, the class of weighted shift operators contains a typical non-unitary isometry \( A(1, 1, \ldots) \) (cf. [2]). The real part Hermitian operator \( \Re(A) = (A + A^*)/2 \) of a weighted operator \( A \) is interpreted as the adjacency matrix of weighted \( A_\infty \)-graph (cf. [8]). In addition, weighted shift operators are closely related to numerical analysis and information theory (cf. [11]).

Shields [16] proved that the numerical range \( W(A) \) of a weighted shift operator \( A \) is a circular disk centered at the origin. In this case, the radius of the disk equals its numerical radius \( w(A) \) which is the maximal spectrum of the self-adjoint operator \( \Re(A) \). Ridge [15] computed the radius for a weighted shift operator with periodic weights. Computations of the radii of weighted shift operators with typical weights such as \((r, 1, 1, \ldots), (1, s, 1, 1, \ldots), (r, s, 1, 1, \ldots) \) and \((r, r^2, r^3, \ldots) \) were carried out in [2, 5, 6, 18, 19].

Stout [17] provided a method to obtain the numerical radius of a weighted shift operator \( A(a_1, a_2, \ldots) \) with square summable weights by introducing the analytic function

\[ F_A(z) = \det(I - z\Re(A(a_1, a_2, \ldots))). \]

It is shown in [17] that the analytic function is given by

\[ F_A(z) = 1 + \sum_{k=1}^{\infty} \left(-\frac{1}{4}\right)^k c_k z^{2k}, \]

where

\[ c_k = \sum a_{i_1}^2 a_{i_2}^2 \cdots a_{i_k}^2, \]

the sum being taken over

\[ 1 \leq i_1 < i_2 < \cdots < i_k < \infty, \quad i_2 - i_1 \geq 2, \quad i_3 - i_2 \geq 2, \ldots, \quad i_k - i_{k-1} \geq 2, \]

and the radius \( w(A(a_1, a_2, \ldots)) = 1/\lambda \) where \( \lambda \) is the minimal positive root of \( F_A(z) = 0 \).
In the finite-dimensional case, an \( n \times n \) weighted shift matrix with weights \((a_1, a_2, \ldots, a_{n-1})\) is defined by

\[
A(a_1, a_2, \ldots, a_{n-1}) = 
\begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
a_1 & 0 & 0 & \ddots & \vdots \\
0 & a_2 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & a_{n-1} & 0 & 0
\end{pmatrix}
\]

It is easy to see that a weighted shift matrix or operator \( A \) is unitarily equivalent to its entry-wise modulus operator \(|A|\). Hence we may assume that the weights are nonnegative for the discussion of the numerical range of a weight shift matrix or operator. There have been a number of interesting papers on the properties of the numerical ranges of weighted shift matrices \([3, 6, 7, 9, 16, 17, 20]\). The numerical range of a weighted matrix \( A \) is a closed disk centered at the origin. Various radii of the disk were studied, e.g., \([6, 13, 19]\). In particular, \( w(A(1, 1, \ldots, 1)) = \cos(\pi/(n+1)) \) for the \( n \times n \) shift matrix \( A(1, 1, \ldots, 1) \) (cf. \([13]\)). The numerical radii of the modified shift matrices \( A(1, 1, r, 1, \ldots, 1) \) and \( A(1, \ldots, 1, r, 1, \ldots, 1) \) were computed respectively in \([6]\) and \([19]\).

In this paper, we consider weighted shift operators \( A(a_1, a_2, \ldots, a_{n-1}, 1, 1, \ldots) \) which perturb the canonical shift operator \( A(1, 1, \ldots) \), and investigate the point spectrum of the Hermitian operator \( \Re(A(a_1, a_2, \ldots, a_{n-1}, 1, 1, \ldots)) \) which gives the numerical radius of the operator \( A(a_1, a_2, \ldots, a_{n-1}, 1, 1, \ldots) \). Furthermore, we explicitly formulate the radius \( w(A(a_1, a_2, \ldots, a_{n-1}, 1, 1, \ldots)) \) in terms of the weighted shift matrix \( A(a_1, a_2, \ldots, a_{n-1}) \).

2. Weighted shift matrices. Let \( A(a_1, a_2, \ldots, a_{n-1}) \) be an \( n \times n \) weighted shift matrix with nonnegative weights. The spectral analysis of a real symmetric tridiagonal matrix \( \Re(A) \) is related to numerical analysis (cf. \([4]\)). The Hermitian matrix \( 2I_n - 2\Re(A(1, \ldots, 1)) \) is also discussed as the discrete Laplacian in applied mechanics \([14]\).

The characteristic polynomial

\[
p_n(t) = \det \left( tI_n - \Re(A(a_1, a_2, \ldots, a_{n-1})) \right)
\]

has the recurrence

\[
p_n(t) = t p_{n-1}(t) - \frac{1}{4} a_{n-1}^2 p_{n-2}(t).
\]
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By the formula [17, Lemma 1],

\[ p_n(t) = t^n + \sum_{1 \leq k \leq n/2} \left(-\frac{1}{4}\right)^k S_k(a_1, \ldots, a_{n-1}) t^{n-2k}, \]

where the circularly symmetric functions

\[ S_k(a_1, \ldots, a_{n-1}) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n-1} a_{j_1}^2 a_{j_2}^2 \cdots a_{j_k}^2, \]

over all \(1 \leq j_1 < j_2 < \cdots < j_k \leq n-1\) satisfying \(j_2 - j_1 \geq 2, j_3 - j_2 \geq 2, \ldots, j_k - j_{k-1} \geq 2\). The circularly symmetric function \(S_k(a_1, \ldots, a_{n-1})\) is abbreviated to \(S_k^{(n-1)}\) if there is no confusion. We will use Chebyshev polynomials of the second kind to find \(\alpha = w(A(a_1, a_2, \ldots, a_{n-1}))\). If the weights \(a_1, \ldots, a_{n-1}\) are positive and relatively small, e.g., less than 1, we have \(0 < \alpha < 1\). Then \(0 < \theta_0 = \arccos(\alpha) < \pi\) is the minimal zero of the trigonometric polynomial

\[ 2^n \sin \theta \det \left( \cos \theta I_n - \Re(A(a_1, a_2, \ldots, a_{n-1})) \right). \]

On the other hand, if the weights \(a_1, \ldots, a_{n-1}\) are relatively large, say, greater than \(\sec(\pi/(n + 1))\), then \(\alpha > 1\) and \(\theta_0 = \arccosh(\alpha)\) is the minimal zero of the trigonometric polynomial

\[ 2^n \sinh \theta \det \left( \cosh \theta I_n - \Re(A(a_1, a_2, \ldots, a_{n-1})) \right). \]

Let \(\ell C_k, 0 \leq k \leq \ell\), denote the binomial coefficients with boundary values \(\ell C_0 = \ell C_\ell = 1\). For \(1 \leq k \leq \ell - 1\), \(\ell C_k = \ell!/(k!(\ell - k)!))\). The following lemma is essential to expand the trigonometric polynomial (2.2).

**Lemma 2.1.**

\[ 2^n \sin \theta \cos^n \theta = \sum_{k=0}^{n-1} \ell C_k \sin((n + 1 - 2k)\theta) \]

\[ = \sum_{0 \leq k \leq n/2} (n-1)_{C_k - (n-1)_{C_k-2}} \sin((n + 1 - 2k)\theta), \]

and

\[ 2^n \sinh \theta \cosh^n \theta = \sum_{k=0}^{n-1} \ell C_k \sinh((n + 1 - 2k)\theta) \]

\[ = \sum_{0 \leq k \leq n/2} (n-1)_{C_k - (n-1)_{C_k-2}} \sinh((n + 1 - 2k)\theta). \]
Proof. We compute

\[
\sum_{k=0}^{n-1} n-1 C_k \sin((n+1-2k)\theta) = \Im \left( \sum_{k=0}^{n-1} n-1 C_k e^{i(n+1-2k)\theta} \right) \\
= \Im \left( e^{i(n+1)\theta} \sum_{k=0}^{n-1} n-1 C_k e^{-2ik\theta} \right) \\
= \Im \left( e^{i(n+1)\theta} \left( 1 + e^{-2i\theta} \right)^{n-1} \right) \\
= \Im \left( e^{2i\theta} \left( e^{i\theta} + e^{-i\theta} \right)^{n-1} \right) \\
= 2^{n-1} \cos^{n-1} \theta \Im(e^{2i\theta}) \\
= 2^n \sin \theta \cos \theta,
\]

where \( \Im(z) \) denotes the imaginary part of a complex number. This proves the first equality of (2.3). Next, we prove the second equality of (2.3).

Assume \( n = 2m + 1 \) is odd. Then

\[
\sum_{0 \leq k \leq 2m} 2m C_k \sin((2m + 2 - 2k)\theta) \tag{2.5}
\]

\[
= 2m C_0 \sin((2m + 2)\theta) + 2m C_1 \sin(2m\theta) \\
+ \sum_{2 \leq k \leq m} 2m C_k \sin((2m + 2 - 2k)\theta) \\
+ \sum_{m+2 \leq k \leq 2m} 2m C_k \sin((2m + 2 - 2k)\theta).
\]

Observe that

\[
\sin((2m + 2 - 2k)\theta) = -\sin((2m + 2 - 2(2m + 2 - k))\theta) = -\sin((2m + 2 - 2j)\theta),
\]

where \( j = 2m + 2 - k \). Since \( m + 2 \leq k \leq 2m \), we have \( 2 \leq j \leq m \). Hence,

\[
\sum_{m+2 \leq k \leq 2m} 2m C_k \sin((2m + 2 - 2k)\theta) = \sum_{m+2 \leq k \leq 2m} 2m C_{2m-k} \sin((2m + 2 - 2k)\theta) \\
= -\sum_{2 \leq j \leq m} 2m C_{j-2} \sin((2m + 2 - 2j)\theta). \tag{2.6}
\]

Taking together (2.5) and (2.6), we have

\[
\sum_{0 \leq k \leq 2m} 2m C_k \sin((2m + 2 - 2k)\theta) = \sum_{0 \leq k \leq m} (2mC_k - 2mC_{k-2}) \sin((2m + 2 - 2k)\theta),
\]

where \( 2mC_{-2} = 2mC_{-1} = 0. \)
Assume \( n = 2m \) is even. Similar computations yield that

\[
\begin{align*}
\sum_{0 \leq k \leq 2m-1} 2m-1C_k \sin((2m + 1 - 2k)\theta) &= 2m-1C_0 \sin(2m + 1)\theta + 2m-1C_1 \sin((2m - 1)\theta) \\
&\quad + \sum_{2 \leq k \leq m} 2m-1C_k \sin((2m + 1 - 2k)\theta) \\
&\quad + \sum_{m+1 \leq k \leq 2m-1} 2m-1C_k \sin((2m + 1 - 2k)\theta) \\
&= \sum_{0 \leq k \leq m} (2m-1C_k - 2m-1C_{k-2}) \sin((2m + 1 - 2k)\theta),
\end{align*}
\]

where \( 2m-1C_{-2} = 2m-1C_{-1} = 0 \).

As to the formula (2.4), we compute that

\[
\begin{align*}
\sum_{0 \leq k \leq n-1} n-1C_k \sinh((n + 1 - 2k)\theta) &= \frac{1}{2} \left( \sum_{0 \leq k \leq n-1} n-1C_k \left( e^{(n+1-2k)\theta} - e^{-(n-1+2k)\theta} \right) \right) \\
&= \frac{1}{2} (e^{2\theta} - e^{-2\theta})(e^{\theta} + e^{-\theta})^{n-1} \\
&= \frac{1}{2} (e^{\theta} - e^{-\theta})(e^{\theta} + e^{-\theta})^{n-1} \\
&= 2^n \left( \frac{e^{\theta} - e^{-\theta}}{2} \right) \left( \frac{e^{\theta} + e^{-\theta}}{2} \right)^n \\
&= 2^n \sinh \theta \cosh^n \theta.
\end{align*}
\]

The second equality of (2.4) can be derived in a similar way.

Applying Lemma 2.1, we expand the determinant in (2.2), which generalizes results of \[6\], Theorem 2.1 and \[19\], Theorem 2.1.

**Theorem 2.2.** Let \( n \geq 2 \). Then

\[
2^n \sin \theta \det \left( \cos \theta I_n - R(A(a_1, a_2, \ldots, a_{n-1})) \right) = \sum_{0 \leq k \leq [n/2]} h_k^{(n-1)} \sin((n + 1 - 2k)\theta)
\]

and

\[
2^n \sinh \theta \det \left( \cosh \theta I_n - R(A(a_1, a_2, \ldots, a_{n-1})) \right) = \sum_{0 \leq k \leq [n/2]} h_k^{(n-1)} \sinh((n + 1 - 2k)\theta),
\]

where

\[
h_k^{(n-1)} = (n-1C_k - n-1C_{k-2}) \\
+ \left( \sum_{\ell=1}^{k-1} (-1)^\ell (n-1-2\ell C_k-\ell - n-1-2\ell C_{k-\ell-2})S_{\ell}^{(n-1)} \right) + (-1)^k S_k^{(n-1)},
\]
The second assertion can be proved in a similar way.

**Proof.** We compute that

\[
2^n \sin \theta \det \left( \cos \theta I_n - \mathcal{R}(A(a_1, a_2, \ldots, a_{n-1})) \right)
= 2^n \sin \theta \left( \cos^n \theta + \sum_{1 \leq k \leq \lfloor n/2 \rfloor} \left( \frac{1}{4} \right)^k S_{k}^{(n-1)} \cos^{n-2k} \theta \right)
= 2^n \sin \theta \cos^n \theta + \sum_{1 \leq k \leq \lfloor n/2 \rfloor} (-1)^k \cos^{n-2k} \theta S_{k}^{(n-1)}
\]

(by (2.1))

\[
= \sum_{0 \leq \ell \leq \lfloor n/2 \rfloor} (n-1)C_{\ell} - n-1C_{\ell-2} \sin((n + 1 - 2\ell)\theta)
+ \sum_{1 \leq k \leq \lfloor n/2 \rfloor} (-1)^k S_{k}^{(n-1)} \times \left( \sum_{0 \leq \ell \leq \lfloor (n-2k)/2 \rfloor} (n-2k-1)C_{\ell} - n-2k-1C_{\ell-2} \sin((n - 2k + 1 - 2\ell)\theta) \right)
\]

(by Lerm \[2.1]\)

\[
= \sin((n + 1)\theta) + n-1C_{1} \sin((n - 1)\theta) + (n-1)C_{2} - n-1C_{0} \sin((n - 3)\theta) + \cdots
- S_{1}^{(n-1)} \left( \sin((n - 3)\theta) + n-3C_{1} \sin((n - 5)\theta) \right)
+ (n-3)C_{2} - n-3C_{0} \sin((n - 5)\theta) + \cdots
+ S_{2}^{(n-1)} \left( \sin((n - 5)\theta) + n-5C_{1} \sin((n - 7)\theta) \right)
+ (n-5)C_{2} - n-5C_{0} \sin((n - 7)\theta) + \cdots
\]

\[
\vdots
\]

\[
= \sum_{0 \leq k \leq \lfloor n/2 \rfloor} \left( (n-1)C_{k} - n-1C_{k-2} \right) + S_{1}^{(n-1)}(n-3)C_{k-1} - (n-3)C_{k-3}
+ S_{2}^{(n-1)}(n-5)C_{k-2} - (n-5)C_{k-4} + \cdots \sin((n + 1 - 2k)\theta)
\]

\[
= \sum_{0 \leq k \leq \lfloor n/2 \rfloor} h_{k}^{(n-1)} \sin((n + 1 - 2k)\theta).
\]

The second assertion can be proved in a similar way. \[\square\]

### 3. Weighted shift operators.

The numerical radius of a weighted shift operator has attracted much attention because of its importance and complexity. Research papers on this subject include the Hilbert-Schmidt class of weighted shift operators, i.e., with square summable weights (cf. [17]), and the modified canonical shift operator such as \((\alpha, 1, 1, \ldots), (1, \alpha, 1, 1, \ldots)\) and \((\alpha, \beta, 1, 1, \ldots)\) (cf. [2, 6, 19]).
In this section, we consider weighted shift operators $A(a_1, a_2, \ldots)$ acting on a complex Hilbert space $\ell^2(\mathbb{N})$ identified with the Hardy space $H^2$ satisfying

$$\lim_{\ell \to \infty} a_\ell = 1, \sum_{\ell=1}^\infty |a_\ell - 1| < \infty \text{ and } \prod_{j=1}^\ell a_j \to \beta \text{ as } \ell \to \infty$$

for some $0 < \beta < \infty$. A typical weighted shift operator of this class is $A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)$ that perturbs the canonical shift operator $A = A(1, 1, \ldots)$.

**Theorem 3.1.** Let $A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)$ be a weighted shift operator with positive weights. Then $w(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)) > 1$ if and only if $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$ has an eigenvalue greater than 1.

**Proof.** Since $W(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$ is a circular disk centered at the origin, it follows that $w(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)) = w(\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))).$ The sufficiency is trivial. Assume $w(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)) > 1$. For the Hermitian operator $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$,

$$\|\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))\| = w(\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))).$$

Then, by [19, Lemma 3.1], $w(\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)))$ and thus $w(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$ is an eigenvalue of $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$. $\square$

The following result characterizes the existence of an eigenvalue $\alpha \geq 0$ of the Hermitian operator $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$, and it turns out the eigenvalue must be greater than 1.

**Theorem 3.2.** Let $n \geq 1$ and $A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)$ be a weighted shift operator with positive weights. A value $\alpha \geq 0$ is an eigenvalue of $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$ if and only if there is a nonzero formal power series

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^{(n)}(0)}{n!}z^n + \frac{f^{(n+1)}(0)}{(n+1)!}z^{n+1} + \cdots$$

with $f(0) \neq 0$ belonging to the Hardy space $H^2$ which satisfies the following recurrence relations

\[
\begin{align*}
f'(0) & = \frac{2\alpha}{a_1} f'(0), \\
f^{(k)}(0) & = \frac{2\alpha}{a_k} f^{(k-1)}(0) - \frac{a_{k-1}}{a_k} f^{(k-2)}(0) \\
\end{align*}
\]

for $k = 2, \ldots, n$, and

\[
\begin{align*}
& f^{(n+1)}(0) \quad (n+1)! = 2\alpha \frac{f^{(n)}(0)}{n!} - a_n \frac{f^{(n-1)}(0)}{(n-1)!}, \\
& f^{(m)}(0) \quad m! = 2\alpha \frac{f^{(m-1)}(0)}{(m-1)!} - \frac{f^{(m-2)}(0)}{(m-2)!}
\end{align*}
\]
for $m = n + 2, n + 3, \ldots$. In this case, the eigenvalue $\alpha > 1$ and

$$\frac{f^{(n+1)}(0)}{(n+1)!} - (\alpha - \sqrt{\alpha^2 - 1}) \frac{f^{(n)}(0)}{n!} = 0.$$ (3.4)

**Proof.** Assume that $\alpha \geq 0$. Clearly, $\alpha$ is an eigenvalue of the Hermitian operator $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$ if and only if there is a corresponding nonzero eigenfunction

$$f(z) = f(0) + f'(0)z + \frac{f^{(2)}(0)}{2!}z^2 + \frac{f^{(3)}(0)}{3!}z^3 + \cdots \in H^2$$

satisfying $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))f = \alpha f$ which is equivalent to the recurrence relations (3.1), (3.2) and (3.3).

From the recurrence relation (3.3), the coefficients of the eigenfunction satisfy the recurrence

$$\frac{f^{(k)}(0)}{k!} = 2\alpha \frac{f^{(k-1)}(0)}{(k-1)!} - \frac{f^{(k-2)}(0)}{(k-2)!}$$ (3.5)

for $k = n + 2, n + 3, \ldots$. If $\alpha = 0$, the recurrence relation (3.5) implies that

$$\frac{f^{(n+2p)}(0)}{(n+2p)!} = (-1)^p \frac{f^{(n)}(0)}{n!}$$

and

$$\frac{f^{(n+1+2p)}(0)}{(n+1+2p)!} = (-1)^p \frac{f^{(n+1)}(0)}{(n+1)!},$$

$p = 1, 2, \ldots$. Since $f \in H^2$, it follows that

$$\frac{f^{(n+1)}(0)}{(n+1)!} = \frac{f^{(n)}(0)}{n!} = 0,$$

and this derives the function $f = 0$ from (3.1).

If $0 < \alpha < 1$ and is expressed as $\alpha = \cos \theta$ for some $0 < \theta < \pi/2$, then the difference equation (3.5) implies that

$$\frac{f^{(n+p)}(0)}{(n+p)!} = \gamma e^{i\theta} + \delta e^{-i\theta},$$

$p = 0, 1, 2, \ldots$, for some constants $\gamma$ and $\delta$. Again, for $f \in H^2$, we have $\gamma = 0$ and $\delta = 0$ as well.

If $\alpha = 1$, then by (3.5),

$$\frac{f^{(n+p+2)}(0)}{(n+p+2)!} - \frac{f^{(n+p+1)}(0)}{(n+p+1)!} = \frac{f^{(n+p+1)}(0)}{(n+p+1)!} - \frac{f^{(n+p)}(0)}{(n+p)!}.$$
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$p = 0, 1, 2, \ldots$, and hence

$$\frac{f^{(n+p)}(0)}{(n+p)!} = \frac{f^{(n)}(0)}{n!} + p\eta,$$

$p = 0, 1, 2, \ldots$, for some constant $\eta$. By the same fact that $f \in H^2$, we have $\eta = 0$ and

$$\frac{f^{(n+1)}(0)}{(n+1)!} = \frac{f^{(n)}(0)}{n!} = 0,$$

and thus $f = 0$. This proves the eigenvalue $\alpha > 1$.

The characteristic equation of the difference equation (3.5) is

$$r^2 - 2\alpha r + 1 = 0.$$

The general solution of the difference equation becomes

$$\frac{f^{(n+p)}(0)}{(n+p)!} = \mu_1(\alpha + \sqrt{\alpha^2 - 1})^p + \mu_2(\alpha - \sqrt{\alpha^2 - 1})^p,$$

$p = 0, 1, 2, \ldots$, for some constants $\mu_1$ and $\mu_2$. Notice that $\alpha + \sqrt{\alpha^2 - 1} > 1$. Hence, $\mu_1 = 0$, and the initial condition implies that $\mu_2 = \frac{f^{(n)}(0)}{n!}$, and this concludes (3.3).

In the following, we explicitly formulate a characteristic equation for the point spectrum of the operator $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$ in terms of the weighted shift matrix $A(a_1, a_2, \ldots, a_n)$.

**Theorem 3.3.** Let $n \geq 1$ and $A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)$ be a weighted shift operator with positive weights, and let $f(z)$ be a nonzero formal power series:

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^{(n)}(0)}{n!}z^n + \frac{f^{(n+1)}(0)}{(n+1)!}z^{n+1} + \cdots.$$

Assume $\alpha > 1$. Then $\alpha$ is an eigenvalue of $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$ with eigenfunction $f(z) \in H^2$ if and only if the coefficients of $f(z)$ satisfy condition (3.1), and the value $z = \alpha - \sqrt{\alpha^2 - 1}$ is a zero of the polynomial

$$F_n(z) = Q_{n-1}(z) - a_n^2z^2Q_{n-2}(z),$$

where

$$Q_\ell(z) = \det \left( (z^2 + 1) I_{\ell+1} - 2z \Re(A(a_1, \ldots, a_{\ell})) \right),$$

$\ell = 1, 2, \ldots$ and $Q_0(z) = z^2 + 1, Q_{-1}(z) = 1$. 

Proof. Assume that $\alpha$ is an eigenvalue of the Hermitian operator $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$, and its corresponding nonzero eigenfunction is given by

$$f(z) = f(0) + f'(0)z + \frac{f^{(2)}(0)}{2}z^2 + \frac{f^{(3)}(0)}{3!}z^3 + \cdots \in H^2.$$ 

The equation $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))f = \alpha f$ leads to the relation

$$(z^2 - 2\alpha z + 1)f(z) = f(0) - (a_1 - 1)z^2 f(0) - \left( \sum_{k=1}^{n-1} ((a_{k+1} - 1)z^{k+2} + (a_k - 1)z^{k}) \frac{f^{(k)}(0)}{k!} \right) - (a_n - 1) \frac{f^{(n)}(0)}{n!} z^n.$$ 

We define the above polynomial by

$$H_n(z) = f(0) - (a_1 - 1)z^2 f(0) - \left( \sum_{k=1}^{n-1} ((a_{k+1} - 1)z^{k+2} + (a_k - 1)z^{k}) \frac{f^{(k)}(0)}{k!} \right) - (a_n - 1) \frac{f^{(n)}(0)}{n!} z^n,$$

and

$$F_n(z) = \frac{a_1 a_2 \cdots a_n}{f(0)} H_n(z).$$

Since $z = \alpha - \sqrt{\alpha^2 - 1}$ is a root of $z^2 - 2\alpha z + 1 = 0$, it follows that $F_n(\alpha - \sqrt{\alpha^2 - 1}) = H_n(\alpha - \sqrt{\alpha^2 - 1}) = 0$. By the relation of the sequence $f^{(k)}(0)/k!$ in (3.1), we have that

$$a_1 a_2 \cdots a_k \frac{f^{(k)}(0)}{k!} z^k = f(0) z^k \left( (2\alpha)^k - (2\alpha)^{k-2} S_{k-1}^1 + (2\alpha)^{k-4} S_{k-1}^2 + \cdots \right)$$

$$= f(0) z^k \left( (2\alpha)^k + \sum_{1 \leq \ell \leq k/2} (-1)^\ell (2\alpha)^{k-2\ell} S_{k-1}^\ell \right)$$

$$= f(0) \left( (z^2 + 1)^k + \sum_{1 \leq \ell \leq k/2} (-1)^\ell z^{2\ell} (z^2 + 1)^{k-2\ell} S_{k-1}^\ell \right).$$
For \( k \) (3.8) and hence

Hence, \( c \)

Substituting (3.7) into (3.6), we obtain that

\[
F_n(z) = a_1 a_2 \cdots a_n + (a_1 a_2 \cdots a_n - a_1^2 a_2 a_3 \cdots a_n) z^2 \\
+ \sum_{k=1}^{n-1} (a_{k+1} a_{k+2} \cdots a_n - a_{k+1}^2 a_{k+2} \cdots a_n) z^2 \\
\times \left( (z^2 + 1)^k + \sum_{1 \leq \ell \leq k/2} (-1)^\ell z^{2\ell} (z^2 + 1)^{k-2\ell} S^{(k-1)}_\ell \right) \\
+ \sum_{k=1}^{n-1} (a_{k+1} a_{k+2} \cdots a_n - a_k a_{k+1} \cdots a_n) \\
\times \left( (z^2 + 1)^k + \sum_{1 \leq \ell \leq k/2} (-1)^\ell z^{2\ell} (z^2 + 1)^{k-2\ell} S^{(k-1)}_\ell \right) \\
+ (1 - a_n) \left( (z^2 + 1)^n + \sum_{1 \leq \ell \leq n/2} (-1)^\ell z^{2\ell} (z^2 + 1)^{n-2\ell} S^{(n-1)}_\ell \right).
\]

The polynomial \( F_n(z) \) with corresponding coefficients \( c_k^{(n)} \) is written as

(3.8) \[
F_n(z) = c_0^{(n)} + c_1^{(n)} z^2 + c_2^{(n)} z^4 + c_3^{(n)} z^6 + \cdots + c_n^{(n)} z^{2n}.
\]

We compute the coefficients \( c_k^{(n)} \). For \( k = 0 \),

\[
c_0^{(n)} = a_1 a_2 \cdots a_n + \sum_{k=1}^{n-1} (a_{k+1} a_{k+2} \cdots a_n - a_k a_{k+1} \cdots a_n) + (1 - a_n) \\
= 1 - a_n + (a_n - a_n a_{n-1} a_n) + (a_{n-1} a_n - a_{n-2} a_n a_{n-1} a_n) + \cdots \\
+ (a_{2} a_3 \cdots a_n - a_{1} a_2 \cdots a_n) + a_1 a_2 \cdots a_n \\
= 1.
\]

For \( k = n \), and \( n = 1 \), by using \( a_1 z f'(0) = (z^2 + 1)f(0) \), we obtain that

\[
H_1(z) = f(0) - (a_1 - 1)z f(0) - (a_1 - 1)z f'(0), \\
F_1(z) = 1 + (1 - a_1)z^2,
\]

and hence \( c_1^{(1)} = 1 - a_1^2 \). For \( n \geq 2 \),

\[
a_1 a_2 \cdots a_n \frac{1}{f(0)} (1 - a_n) \left( z^n \frac{f(n)(0)}{n!} + z^{n+1} \frac{f(n-1)(0)}{(n-1)!} \right) \\
= (1 - a_n) \left( z^n + a_n z^{2n-2} + \gamma z^{2n-2} + \cdots \right) \\
= (1 - a_n^2) z^n + (1 - a_n) \gamma z^{2n-2} + \cdots .
\]

Hence, \( c_n^{(n)} = 1 - a_n^2 \).
For $1 \leq k \leq n - 1$, we deduce from the definition of $H_n(z)$ in (3.6) that

$$H_n(z) = H_{n-1}(z) - (a_n - 1) \left( z^{n+1} \frac{f^{(n-1)}(0)}{(n-1)!} + z^n \frac{f^{(n)}(0)}{n!} \right).$$

Then

$$F_n(z) = \frac{a_1 a_2 \cdots a_n}{f(0)} \left( H_{n-1}(z) - (a_n - 1) \left( z^{n+1} \frac{f^{(n-1)}(0)}{(n-1)!} + z^n \frac{f^{(n)}(0)}{n!} \right) \right)$$

$$= a_n F_{n-1}(z) + (a_n - a_n^2) \left( z^2(2z + 1)^{n-1} \right) + \sum_{1 \leq \ell \leq (n-1)/2} (-1)^\ell z^{2\ell}(2z + 1) \frac{n-1-2\ell}{n} S_{n-1}(z)$$

$$+ (1 - a_n) \left( z^2(2z + 1)^n \right) + \sum_{1 \leq \ell \leq n/2} (-1)^\ell z^{2\ell}(2z + 1) \frac{n-2\ell}{n} S_{n-1}(z).$$

Comparing the coefficients of (3.8) and (3.9), we have the following recurrence relation for $c_k^{(n)}$, $1 \leq k \leq n - 1$:

$$0 = -c_k^{(n)} + a_n c_k^{(n-1)} + (a_n - a_n^2) \sum_{1 \leq \ell \leq k-1, 2\ell \leq n-1} (-1)^\ell n-1-2\ell C_{k-1-\ell} S_{n-1}^{(n-2)}$$

$$+ (1 - a_n) \sum_{1 \leq \ell \leq k, 2\ell \leq n} (-1)^\ell n-2\ell C_{k-\ell} S_{n-1}^{(n-1)}$$

$$= -c_k^{(n)} + B_k^{(n)} - a_n (-c_k^{(n-1)} + A_k^{(n-1)}),$$

where

$$A_k^{(n)} = n C_k - n-1 C_{k-1} a_n^2 - n-2 C_{k-1} \sum_{j=1}^{n-1} a_j^2$$

$$+ \sum_{\ell=2}^{k-1} (-1)^\ell \left[ n-2\ell+1 C_{k-\ell} S_{n-1}^{(n-2)} a_n^2 + n-2\ell C_{k-\ell} S_{n-1}^{(n-1)} \right] + (-1)^k S_k^{(n)}$$

and

$$B_k^{(n)} = n C_k - n-1 C_{k-1} a_n^2 - a_n^2 \sum_{1 \leq \ell \leq k-1, 2\ell \leq n-2} (-1)^\ell n-1-2\ell C_{k-1-\ell} S_{n-1}^{(n-2)}$$

$$+ \sum_{1 \leq \ell \leq k, 2\ell \leq n} (-1)^\ell n-2\ell C_{k-\ell} S_{n-1}^{(n-1)}.$$
We decompose the coefficient \( c_k^{(n)} = B_k^{(n)} \) for \( k \geq 2 \) into two parts. By letting
\[
c_k^{(n)} = n C_k + \sum_{1 \leq \ell \leq n/2} (-1)^{\ell} n_{-2\ell} C_{k-\ell} S_{\ell}^{(n)}, \quad k = 0, 1, \ldots, n
\]
and
\[
d_k^{(n)} = n_{-2} C_{k-2} + \sum_{1 \leq \ell \leq (n-2)/2} (-1)^{\ell} n_{-2-2\ell} C_{k-2-\ell} S_{\ell}^{(n-2)}, \quad k = 2, 3, \ldots, n.
\]
We compute that
\[
\tilde{c}_k^{(n)} - B_k^{(n)} = n C_k + \sum_{\ell=1}^{k-1} (-1)^{\ell} n_{-2\ell} C_{k-\ell} S_{\ell}^{(n)} - n C_k - \sum_{\ell=1}^{k-1} n_{-2\ell} C_{k-\ell} S_{\ell}^{(n-1)}
\]
\[
+ n_{-1} C_{k-1} a_n^2 + a_n^2 \sum_{\ell=1}^{k-1} (-1)^{\ell} n_{-1-2\ell} C_{k-1-\ell} S_{\ell}^{(n-2)}
\]
\[
= a_n^2 \sum_{\ell=1}^{k} (-1)^{\ell} n_{-2\ell} C_{k-\ell} S_{\ell-1}^{(n-2)} + n_{-1} C_{k-1} a_n^2
\]
\[
+ a_n^2 \sum_{\ell=1}^{k-1} (-1)^{\ell} n_{-1-2\ell} C_{k-1-\ell} S_{\ell}^{(n-2)}
\]
\[
= a_n^2 d_k^{(n)}.
\]
This shows that
\[
c_k^{(n)} = \tilde{c}_k^{(n)} - a_n^2 d_k^{(n)},
\]
and thus
\[
c_k^{(n)} z^{2k} = \tilde{c}_k^{(n)} z^{2k} - a_n^2 z^4 d_k z^{2k-4}
\]
with \( c_0^{(n)} = 1, \quad c_1^{(n)} = -S_1^{(n)} + n \). Therefore,
\[
F_n(z) = G_n(z) - a_n^2 z^4 L_n(z),
\]
where
\[
G_n(z) = \tilde{c}_0^{(n)} + \tilde{c}_1^{(n)} z^2 + \tilde{c}_2^{(n)} z^4 + \tilde{c}_3^{(n)} z^6 + \cdots + \tilde{c}_n^{(n)} z^{2n},
\]
and
\[
L_n(z) = d_2^{(n)} z^2 + \cdots + d_n^{(n)} z^{2n-4}.
\]
Rearranging the terms of $G_n(z)$, we have that

$$G_n(z) = (1 + nC_1z^2 + nC_2z^4 + \cdots + nC_nz^{2n})$$

$$-S_1^{(n)}z^2(n-2C_0 + n-2C_1z^2 + \cdots + n-2C_nz^{2n-2}) + \cdots$$

$$= \sum_{k=0}^{n} nC_kz^{2k} + \sum_{1 \leq \ell \leq [(n+1)/2]} (-1)^{\ell} S_\ell^{(n)} z^{2\ell} \sum_{j=0}^{n-2\ell} n-2\ell C_j z^{2j}$$

$$= (z^2 + 1)^n + \sum_{1 \leq \ell \leq [(n+1)/2]} (-1)^{\ell} S_\ell^{(n)} z^{2\ell} (z^2 + 1)^{n-2\ell}.$$ 

By equation (2.1), the characteristic polynomial of $2\Re(A_1, \ldots, a_n))$ is formulated as

$$\det \left( xI_{n+1} - 2y\Re(A_1, a_2, \ldots, a_n) \right) = \sum_{0 \leq \ell \leq (n+1)/2} (-1)^{\ell} S_\ell^{(n)} x^{n+1-2\ell} y^{2\ell}.$$ 

Then, we obtain that

$$(3.10) \quad (z^2 + 1)G_n(z) = \det \left( (z^2 + 1)I_{n+1} - 2z\Re(A_1, \ldots, a_n) \right) = Q_n(z).$$

Applying the Laplace expansion on the $(n+1)$-th row of the determinant (3.10), we expand

$$\det \left( (z^2 + 1)I_{n+1} - 2z\Re(A_1, \ldots, a_n) \right)$$

$$= (z^2 + 1) \det \left( (z^2 + 1)I_n - 2z\Re(A_1, \ldots, a_{n-1}) \right)$$

$$- a_n^2 z^2 \det \left( (z^2 + 1)I_{n-1} - 2z\Re(A_1, \ldots, a_{n-2}) \right).$$

Similarly, we can prove that

$$(z^2 + 1)L_n(z) = \det \left( (z^2 + 1)I_{n-1} - 2z\Re(A_1, \ldots, a_{n-2}) \right).$$

Then, we have that

$$(z^2 + 1)F_n(z) = (z^2 + 1)G_n(z) - a_n^2 z^4(z^2 + 1)L_n(z)$$

$$= (z^2 + 1) \det \left( (z^2 + 1)I_n - 2z\Re(A_1, \ldots, a_{n-1}) \right)$$

$$- a_n^2 z^2(z^2 + 1) \det \left( (z^2 + 1)I_{n-1} - 2z\Re(A_1, \ldots, a_{n-2}) \right).$$

This implies that

$$F_n(z) = \det \left( (z^2 + 1)I_n - 2z\Re(A_1, \ldots, a_{n-1}) \right)$$

$$- a_n^2 z^2 \det \left( (z^2 + 1)I_{n-1} - 2z\Re(A_1, \ldots, a_{n-2}) \right)$$

$$= Q_{n-1}(z) - a_n^2 z^2 Q_{n-2}(z).$$
To prove the converse part, we first compute that

\[ F_n(z) = Q_{n-1}(z) - a_n^2 z^2 Q_{n-2}(z) \]
\[ = \left( (z^2 + 1)Q_{n-1}(z) - a_n^2 z^2 Q_{n-2}(z) \right) - z^2 Q_{n-1}(z) \]
\[ = Q_n(z) - z^2 Q_{n-1}(z) \]
\[ = \left( (z^2 + 1)^{n+1} + \sum_{1 \leq \ell \leq [(n+1)/2]} (-1)^\ell [z^{2\ell}(z^2 + 1)^{n+1-2\ell} S_{\ell}^{(n)}] \right) \]
\[ - z^2 \left( (z^2 + 1)^n + \sum_{1 \leq \ell \leq [n/2]} (-1)^\ell [z^{2\ell}(z^2 + 1)^{n-2\ell} S_{\ell}^{(n-1)}] \right) \quad \text{(by (3.10))} \]
\[ = z^{n+1} \left( (2\alpha)^{n+1} + \sum_{1 \leq \ell \leq [(n+1)/2]} (-1)^\ell (2\alpha)^{n+1-2\ell} S_{\ell}^{(n)} \right) \]
\[ - z^{n+2} \left( (2\alpha)^n + \sum_{1 \leq \ell \leq [n/2]} (-1)^\ell (2\alpha)^{n-2\ell} S_{\ell}^{(n-1)} \right) \]
\[ = \left( \prod_{j=1}^n a_j \right)^z \left( \frac{f^{(n+1)}(0)}{(n+1)!} - z \frac{f^{(n)}(0)}{n!} \right). \]

Hence, if \( F_n(\alpha - \sqrt{\alpha^2 - 1}) = 0 \), we have

\[ \frac{f^{(n+1)}(0)}{(n+1)!} - (\alpha - \sqrt{\alpha^2 - 1}) \frac{f^{(n)}(0)}{n!} = 0. \]

This equation guarantees that the coefficients of the eigenfunction determined by the difference equation (3.10) satisfy the relation

\[ \frac{f^{(n+p)}(0)}{(n+p)!} = \mu_1 (\alpha + \sqrt{\alpha^2 - 1})^p + \mu_2 (\alpha - \sqrt{\alpha^2 - 1})^p, \]

\( p = 0, 1, 2, \ldots \), for some constants \( \mu_1 \) and \( \mu_2 \), and the constant \( \mu_1 \) necessarily vanishes.

This asserts that the function \( f(z) \) belongs to the Hardy space \( H^2 \). \( \square \)

We consider a special weighted shift operator \( A(a_1, 1, 1, \ldots) \). If \( a_1 \) is large enough, e.g., \( a_1 > \sqrt{2} \), then \( \alpha = \| \mathbb{R}(A(a_1, 1, 1, \ldots)) \| > 1 \) is an eigenvalue of \( \mathbb{R}(A(a_1, 1, 1, \ldots)) \). By Theorem 3.3, \( F_1(z) = z^2 + 1 - a_1^2 z^2 \). The positive root of \( F_1(z) = 0 \) is \( z = 1/\sqrt{a_1^2 - 1} \), and thus,

\[ w(A(a_1, 1, 1, \ldots)) = \alpha = \frac{1}{2} \left( z + \frac{1}{z} \right) = \frac{1}{2} \left( \sqrt{a_1^2 - 1} + \frac{1}{\sqrt{a_1^2 - 1}} \right). \]

This formula is also obtained in [2] [20]. Similarly, for the weighted shift operator \( A(1, a_2, 1, 1, \ldots) \), we have

\[ F_2(z) = Q_1(z) - a_2^2 z^2 Q_0(z) = (z^2 + 1)^2 - z^2 - a_2^2 z^2 (z^2 + 1). \]
The zeros of $F_2(z) = 0$ determine the numerical radius $w(A(1, a_2, 1, \ldots))$ (cf. [12, 20]).

The polynomial $F_n(z)$ associated with the weighted shift operator $A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)$ in Theorem 3.3 is also denoted by $F_n(z; a_1, \ldots, a_n)$ if it is necessary to emphasize the first $n$ weights. In consequence of Theorem 3.3 we have the following corollaries.

**Corollary 3.4.** Let $1 \leq m < n$, and $F_n(z)$ be the polynomial defined in Theorem 3.3. Then

$$F_n(z; a_1, \ldots, a_m, 1, \ldots, 1) = F_n(z; a_1, \ldots, a_m).$$

*Proof.* By mathematical induction on $n - m$, it is sufficient to prove the case $n = m + 1$. We have that

$$F_n(z; a_1, \ldots, a_m, 1) = Q_m(z) - z^2Q_{n-1}(z) = (z^2 + 1)Q_{n-1}(z) - a_m^2z^2Q_{n-2}(z) - z^2Q_{n-1}(z) = Q_{n-1}(z) - a_m^2Q_{n-2}(z) = F_m(z; a_1, \ldots, a_m).$$

**Corollary 3.5.** Let $F_n(z)$ be the polynomial defined in Theorem 3.3. Then, the recurrence equation

$$(a_{n+2}^2 - 1)(a_{n+1}^2 - 1)F_{n+3}(z) = (a_m^2 + 1)Q_{n-1}(z) = (z^2 + 1)Q_{n-1}(z) - a_m^2z^2Q_{n-2}(z) - a_m^2z^2Q_{n-1}(z) = (z^2 - a_m^2z^2 + 1)Q_{n-1}(z) - a_m^2z^2Q_{n-2}(z) = (1 - a_m^2z^2)Q_{n-1}(z) + F_n(z).$$

*Proof.* Firstly, we prove the equation

$$(3.11) \quad F_{n+1}(z) - F_n(z) = (1 - a_m^2z^2)Q_{n-1}(z)$$

for $n = 0, 1, 2, \ldots$ Equation (3.11) holds clearly for $n = 0$ with $Q_{-1}(z) = 1$. We assume that equation (3.11) is true for indices less than $n$. Then

$$F_{n+1}(z) = Q_n(z) - a_m^2z^2Q_{n-1}(z) = (z^2 + 1)Q_{n-1}(z) - a_m^2z^2Q_{n-2}(z) - a_m^2z^2Q_{n-1}(z) = (z^2 - a_m^2z^2 + 1)Q_{n-1}(z) - a_m^2z^2Q_{n-2}(z) = (1 - a_m^2z^2)Q_{n-1}(z) + F_n(z).$$
Applying equation (3.11), we compute that
\[
(1 - a_{n+2}^2)(1 - a_{n+1}^2)(F_{n+3}(z) - F_{n+2}(z)) = (1 - a_{n+2}^2)(1 - a_{n+1}^2)(1 - a_{n+3}^2)z^2Q_{n+1}(z)
\]
\[
= (1 - a_{n+2}^2)(1 - a_{n+1}^2)(1 - a_{n+3}^2)z^2\left((z^2 + 1)Q_n(z) - a_{n+1}^2 z^2 Q_{n-1}(z)\right)
\]
\[
= (1 - a_{n+3}^2)(1 - a_{n+1}^2)(z^2 + 1)(F_{n+2}(z) - F_{n+1}(z)) - (1 - a_{n+2}^2)(1 - a_{n+3}^2)a_{n+1}^2 z^2(F_{n+1}(z) - F_n(z)),
\]
which implies the desired recurrence equation. □

The coefficients $c_k^{(n)}$ of the polynomial $F_n(z)$ in Theorem 3.3 and the coefficients $h_k^{(n-1)}$ in Theorem 2.2 are closely related. Comparing the coefficients $c_k^{(n)}$ and $h_k^{(n)}$, we derive the following relation.

**Theorem 3.6.** Let $c_k^{(n)}$ be the coefficients of the polynomial $F_n(z)$ in (3.8).

(I) Suppose $n + 1 = 2m \geq 4$ is an even integer. Then
\[
2^{n+1} \sin \theta \det \left( \cos \theta I_{n+1} - \Re(A(a_1, a_2, \ldots, a_n)) \right)
\]
\[
= \sin((n + 2)\theta) + c_1^{(n)} \sin(n \theta) + (c_2^{(n)} - c_1^{(n)}) \sin((n - 2)\theta) + \cdots + (c_{m}^{(n)} - c_{m+1}^{(n)}) \sin \theta.
\]

(II) Suppose $n + 1 = 2m + 1 \geq 3$ is an odd integer. Then
\[
2^{n+1} \sin \theta \det \left( \cos \theta I_{n+1} - \Re(A(a_1, a_2, \ldots, a_n)) \right)
\]
\[
= \sin((n + 2)\theta) + c_1^{(n)} \sin(n \theta) + (c_2^{(n)} - c_1^{(n)}) \sin((n - 2)\theta) + \cdots + (c_{m}^{(n)} - c_{m+2}^{(n)}) \sin(2 \theta).
\]

In either case,
\[
2^{n+1} \sin \theta \det \left( \cos \theta I_{n+1} - \Re(A(a_1, a_2, \ldots, a_n)) \right) = \sum_{\ell=0}^{n} c_{\ell}^{(n)} \sin((n + 2 - 2\ell)\theta).
\]

**Example 3.7.** Consider the $4 \times 4$ weighted shift matrix $A(2, 1, 3)$. Then, by Theorem 2.2
\[
2^4 \sinh \theta \det \left( \cos \theta I_4 - \Re(A(2, 1, 3)) \right) = \sinh(5\theta) - 11 \sinh(3\theta) + 24 \sinh(\theta).
\]

From Theorem 3.3
\[
F_3(z) = 1 - 11z^2 + 16z^4 - 8z^6,
\]
which yields
\[ h^{(3)}_1 = c^{(3)}_1 = -11, \quad h^{(3)}_2 = c^{(3)}_2 - c^{(3)}_3 = 24. \]

The eigenvalues spectra, greater than 1, of the Hermitian operator \( \Re(A(2,1,3,1,1,\ldots)) \) lie in the set
\[ \left\{ \frac{1}{2}(z + z^{-1}) : 0 < z < 1, \ 1 - 11z^2 + 16z^4 - 8z^6 = 0 \right\}. \]

The numerical value of this set is \( \{1.69504\} \) for \( z = 0.3264004 \).

REFERENCES

The Numerical Radius of a Weighted Shift Operator

