Orthogonal $*$-$*$-basis of relative symmetry classes of polynomials

Kijti Rodtes
Department of Mathematics, Faculty of Science, Naresuan University, kijtir@nu.ac.th

Follow this and additional works at: http://repository.uwyo.edu/ela
Part of the Algebra Commons

Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.2843
ORTHOGRAPHICAL *-BASIS OF SYMMETRY CLASSES OF POLYNOMIALS*

KIJTI RODTES†

Abstract. In this note, the existence of orthogonal *-basis of the symmetry classes of polynomials is discussed. Analogously to the orthogonal *-basis of symmetry classes of tensors, some criteria for the existence of the basis for finite groups are provided. A condition for the existence of such basis of symmetry classes of polynomials associated to symmetric groups and some irreducible characters is also investigated.

Key words. Symmetry classes of polynomials, Orthogonal *-basis.

AMS subject classifications. 05E05, 15A69.

1. Introduction. One of the classical areas of algebra, the theory of symmetric polynomials is well-known because of its role in branches of algebra, such as Galois Theory, representation theory and algebraic combinatorics. For a review of the theory of symmetric polynomials, one can see the book of Macdonald, [6]. The relative symmetric polynomials as a generalization of symmetric polynomials are introduced by M. Shahryari in [11]. In fact, he used the idea of symmetry classes of tensors to introduce such notions.

One of the most interesting topics about symmetry classes of tensors is the issues of finding a necessary condition for the existence of an orthogonal *-basis for the symmetry classes of tensors associated with a finite group and an irreducible character. Many researchers pay a lot of attention to investigate condition stated above. For example, M.R. Pournaki, [8], gave such a necessary condition for the irreducible constituents of the permutation character of the finite groups in which he extended a result of R.R. Holmes, [2]. Also, M. Shahryari provided an excellent condition for the existence of such basis in [10]. Furthermore, the existence of the special basis for particular groups have been discussed by many authors, see, for example, [3, 4, 13]. Similar questions concerning about the existence of an orthogonal *-basis arise in the context of relative symmetric polynomials as well, see, for example [9, 14, 15]. The general criterion is still an open problem, [11].

†Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand (kijtir@nu.ac.th). Supported by Naresuan University on the project R2558C030.

Received by the editors on December 17, 2014. Accepted for publication on March 29, 2015.

Handling Editor: Tin-Yau Tam.
In this article, we provide some criteria for the existence of the special basis of symmetry classes of polynomials for finite groups and some corresponding permutation characters which are parallel to those of M.R. Pournaki in [8], R.R. Holmes in [2] and M. Shahryari in [10]. We also investigate some condition for the existence of such basis of symmetry classes of polynomials associated to symmetric groups and some irreducible characters, which are similar to the results of Y. Zamani in [12].

2. Notations and background. Let $G$ be a subgroup of the full symmetric group $S_m$ and $\chi$ be an irreducible character of $G$. Let $H_d[x_1,\ldots,x_m]$ be the complex space of homogenous polynomials of degree $d$ with the independent commuting variables $x_1,\ldots,x_m$. Let $\Gamma_{m,d}^+$ be the set of all $m$-tuples of non-negative integers $\alpha = (\alpha_1,\ldots,\alpha_m)$, such that $\sum_{i=1}^m \alpha_i = d$. For any $\alpha \in \Gamma_{m,d}^+$, let $X^\alpha$ be the monomial $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_m^{\alpha_m}$. Then the set $\{X^\alpha | \alpha \in \Gamma_{m,d}^+\}$ is a basis of $H_d[x_1,\ldots,x_m]$. An inner product on $H_d[x_1,\ldots,x_m]$ is defined by

$$\langle X^\alpha, X^\beta \rangle = \delta_{\alpha,\beta}. \quad (2.1)$$

The group $G$, as a subgroup of the full symmetric group $S_m$, acts on $H_d[x_1,\ldots,x_m]$ by (for $\sigma \in G$),

$$q^\sigma(x_1,\ldots,x_m) = q(x_{\sigma^{-1}(1)},\ldots,x_{\sigma^{-1}(m)}).$$

It also acts on $\Gamma_{m,d}^+$ by

$$\sigma\alpha = (\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(m)}).$$

Let $\Delta$ be a set of representatives of orbits of $\Gamma_{m,d}^+$ under the action of $G$. Now consider the symmetrizer associated with $G$ and $\chi$

$$T(G,\chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma. \quad (2.2)$$

It is well known that $T(G,\chi)^2 = T(G,\chi)$ and $T(G,\chi)^* = T(G,\chi)$. The image of $H_d[x_1,\ldots,x_m]$ under the map $T(G,\chi)$ is called the symmetry class of polynomials of degree $d$ with respect to $G$ and $\chi$ and it is denoted by $H_d(G;\chi)$.

For any $q \in H_d[x_1,\ldots,x_m]$,

$$q^*_\chi = T(G,\chi)(q)$$

is called a symmetrized polynomial with respect to $G$ and $\chi$. Note that

$$H_d(G;\chi) = \langle X^\alpha^*; \alpha \in \Gamma_{m,d}^+ \rangle.$$
We write $X^{\alpha, \ast}$ instead of $X^{\tilde{\alpha}, \ast}_\chi$ unless it is necessary to avoid confusion.

**Definition 2.1.** An orthogonal *-basis (o-basis, for short) of a subspace $U$ of $H_d(G; \chi)$ is an orthogonal basis of $U$ of the form $\{X^{\alpha_1, \ast}_\chi, X^{\alpha_2, \ast}_\chi, \ldots, X^{\alpha_t, \ast}_\chi\}$ for some $\alpha_i \in \Gamma_{m,d}^\ast$.

Since the set $\{T(G, \chi) : \chi \in \text{Irr}(G)\}$ is a complete set of orthogonal idempotents, where $\text{Irr}(G)$ is the set of irreducible complex characters of $G$, we have the following orthogonal direct sum decomposition (cf. Remark 2.3 in [11])

$$H_d[x_1, \ldots, x_m] = \bigoplus_{\chi \in \text{Irr}(G)} H_d(G; \chi). \quad (2.3)$$

Note that $X^{\alpha, \ast}_\chi$ is a generator of $H_d(G; \chi)$ if $X^{\alpha, \ast}_\chi \neq 0$, which can be checked from $(\chi, 1)_{G_\alpha} \neq 0$, where $(\chi, \phi)_K = \frac{1}{|K|} \sum_{\sigma \in K} \chi(\sigma) \psi(\sigma^{-1})$. Namely, (see, [9, 11]),

$$X^{\alpha, \ast}_\chi \neq 0 \text{ if and only if } (\chi, 1)_{G_\alpha} \neq 0. \quad (2.4)$$

Also, for the induced inner product on $H_d(G; \chi)$, we have (see, [9, 11]).

$$\langle X^{\sigma_1 \alpha, \ast}_\chi, X^{\sigma_2 \beta, \ast}_\chi \rangle = \begin{cases} 0, & \text{if } \alpha \not\in \text{Orb}(\beta); \\ \frac{1}{|G|} \sum_{\sigma \in G_\alpha} \chi(\sigma) \psi(\sigma^{-1}), & \text{if } \alpha = \beta, \end{cases} \quad (2.5)$$

where $\text{Orb}(\beta)$ is the orbit of $\beta$ in $\Gamma_{m,d}^\ast$ under the action of $G$. Then the norm of $X^{\alpha, \ast}_\chi$, with respect to the induced inner product, is given by

$$\|X^{\alpha, \ast}_\chi\|^2 = (\chi(1) \frac{1}{|G_\alpha : G_\alpha|}, \chi(1))_{G_\alpha}. \quad (2.6)$$

According to (2.4), let $\Omega = \{\alpha \in \Gamma_{m,d}^\ast : (\chi, 1)_{G_\alpha} \neq 0\}$. Since

$$H_d[x_1, \ldots, x_m] = \bigoplus_{\alpha \in \Delta} \langle X^{\sigma \alpha, \ast}_\chi : \sigma \in G \rangle,$$

we have the orthogonal direct sum

$$H_d(G; \chi) = \bigoplus_{\alpha \in \Delta} H_d^{\alpha, \ast}(\chi), \quad (2.7)$$

where $\Delta = \Delta \cap \Omega$ and $H_d^{\alpha, \ast}(\chi) = \langle X^{\sigma \alpha, \ast}_\chi : \sigma \in G \rangle$. The dimension of $H_d^{\alpha, \ast}(\chi)$ can be calculated by using Freese’s Theorem (see, e.g. [4, 9])

$$\dim H_d^{\alpha, \ast}(\chi) = \chi(1) (\chi, 1)_{G_\alpha} = \frac{1}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma). \quad (2.8)$$
Orthogonal $\ast$-Basis of Symmetry Classes of Polynomials

As an immediate consequence of (2.7) and (2.8),

$$\dim H_d(G; \chi) = \chi(1) \sum_{\alpha \in \Delta} (\chi, 1)_{G, \ast}. \quad (2.9)$$

In particular, if $\chi$ is linear, then the set $\{X^{\alpha, \ast} : \alpha \in \Delta\}$ is an orthogonal basis of $H_d(G; \chi)$ and $\dim H_d(G; \chi) = |\Delta|$. Thus, the orthogonal $\ast$-basis for $H_d(G; \chi)$ exists for any abelian group $G$.

3. Main criteria. According to the notations in the previous section, $H_d(G; \chi)$ denotes the relative symmetry classes of polynomials of degree $d$ with respect to $G$ and $\chi$. This class is equipped with the induced inner product as in (2.5). Let $\Lambda$ be a set of $m$ elements. Suppose $G$ acts faithfully on $\Lambda$. So, we consider $\{f_\sigma : \sigma \in G\}$ as the group $G$, where $f_\sigma : \Lambda \to \Lambda$ defined by $f_\sigma(\lambda) = \sigma \cdot \lambda$, for all $\lambda \in \Lambda$. Namely, $G$ can be viewed as a subgroup of $S_m$ in this way. We also denote the permutation character of $G$ by $\theta$. It is well known that $\theta(\sigma) = \{|\lambda \in \Lambda | \sigma \cdot \lambda = \lambda|\}$, for each $\sigma \in G$. The similar criterion as in the main theorem of [3] is shown below.

**Theorem 3.1.** Let $G$ be a finite group and let $\Lambda$ be a set of $m$ elements, $m > 1$. Assume that $G$ acts transitively and faithfully on $\Lambda$. Let $\chi$ be an irreducible constituent of permutation character $\theta$ of $G$. If $\chi(1)(\chi, \theta)_G > \frac{m}{2}$, then $H_d(G; \chi)$ does not have an orthogonal $\ast$-basis.

**Proof.** Suppose $H_d(G; \chi)$ has an orthogonal $\ast$-basis. Then, by (2.7), $H^\ast_d(\chi)$ has an $\ast$-basis for each $\alpha \in \Delta$. We now consider $\alpha = (d, \mathbf{0}, \ldots, \mathbf{0}) \in \Gamma_{m,d}$ and choose $\Delta$ to be the set of representatives of orbits of $\Gamma_{m,d}$ under the action of $G$ in which $\alpha \in \Delta$. We can assume without loss of generality that $\Lambda = \{1, 2, \ldots, m\}$ and thus $G_\alpha = G_1$, where $G_\alpha$ refers to the stabilizer subgroup of $\alpha$ (when $G$ acts on $\Gamma_{m,d}$) and $G_1$ refers to the stabilizer subgroup of $1$ (when $G$ acts on $\Lambda$). Since $G$ acts transitively on $\Lambda$, $(1_{G_\alpha})^G = (1_{G_1})^G = \theta$, by Lemma 5.14 of [3]. Hence, by (2.8) and Frobenius reciprocity, we have that

$$\sum_{\sigma \in G_\alpha} \chi(\sigma) = |G_\alpha|(\chi, 1_{G_\alpha})_{G, \ast} = |G_\alpha|(\chi, (1_{G_\alpha})^G)_G = |G_\alpha|(\chi, \theta)_G.$$

Since $\chi$ is an irreducible constituent of permutation character $\theta$ of $G$, $(\chi, \theta)_G \neq 0$ and $\sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0$. Thus $\alpha \in \Delta$. So, $H^\ast_d(\chi)$ has an $\ast$-basis.

By orbit-stabilizer theorem and transitive action of $G$ on $\Lambda$, we have that $m = |\Lambda| = \text{Orb}(1) = [G : G_1] = [G : G_\alpha]$. So, $G = \bigcup_{i=1}^m \sigma_i G_\alpha$, where $\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ is a
system of distinct representatives of left cosets of \( G_\alpha \) in \( G \). Let

\[
\dim H_d^{\alpha,*}(\chi) = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) = \chi(1)(\chi,\theta)_G := t.
\]

We can assume that \( \{ X^{\sigma_1\alpha,*}, X^{\sigma_2\alpha,*}, \ldots, X^{\sigma_t\alpha,*} \} \) is an \( \alpha \)-basis for \( H_d^{\alpha,*}(\chi) \). Define the \( m \times m \) complex matrix \( D = [D_{ij}] \) by \( D_{ij} := \langle X^{\sigma_i\alpha,*}, X^{\sigma_j\alpha,*} \rangle \). Note that \( D \) is idempotent. In fact,

\[
(D^2)_{ij} = \sum_{k=1}^{m} D_{ik} D_{kj} = \sum_{k=1}^{m} \langle X^{\sigma_i\alpha,*}, X^{\sigma_k\alpha,*} \rangle \langle X^{\sigma_k\alpha,*}, X^{\sigma_j\alpha,*} \rangle
\]

\[
= \sum_{k=1}^{m} \left( \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\alpha \sigma_k^{-1}} \chi(\sigma) \right) \left( \frac{\chi(1)}{|G|} \sum_{\tau \in G_\alpha \sigma_k^{-1}} \chi(\tau) \right)
\]

\[
= \frac{\chi(1)^2}{|G|^2} \sum_{k=1}^{m} \sum_{\sigma \in G_\alpha} \sum_{\tau \in G_\alpha} \chi(\sigma \sigma_k^{-1}) \chi(\sigma \tau \sigma_k^{-1})
\]

\[
= \frac{\chi(1)^2}{|G|^2} \sum_{k=1}^{m} \sum_{\lambda \in G_\alpha} \sum_{\mu \in G_\alpha} \chi(\lambda \sigma_k^{-1}) \chi(\sigma_j \mu).
\]

Now, let \( \mu \lambda = \delta \in G_\alpha \). Then \( \mu = \delta \lambda^{-1} \) and we have

\[
(D^2)_{ij} = \chi(1)^2 \sum_{k=1}^{m} \sum_{\lambda \in G_\alpha} \sum_{\sigma \in G_\alpha} \chi(\lambda \sigma_k^{-1}) \chi(\sigma_j \delta \lambda^{-1})
\]

\[
= \chi(1)^2 \sum_{\delta \in G_\alpha} \left( \frac{\chi(1)}{|G|} \sum_{\lambda \in G} \chi(\lambda \sigma_k^{-1}) \chi(\sigma_j \delta \lambda^{-1}) \right)
\]

\[
= \chi(1)^2 \sum_{\delta \in G_\alpha} \left( \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \chi(\sigma_j \delta \sigma_k^{-1} \sigma^{-1}) \right).
\]

By orthogonal relations of irreducible character, we have

\[
(D^2)_{ij} = \chi(1)^2 \sum_{\delta \in G_\alpha} \chi(\sigma_j \delta \sigma_k^{-1}),
\]

which shows that \( D^2 = D \).

We note that \( m = \theta(1) = \sum_{\chi \in \Theta} \chi(1)(\chi,\theta) \), where \( \Theta \) is the set of all irreducible constituents of the permutation character \( \theta \). Since \( m > 1, |\Theta| > 1 \) and hence \( m > t \). We can now write \( D \) in the form

\[
\begin{bmatrix}
D_1 & D_2 \\
D_3 & D_4
\end{bmatrix},
\]
where $D_1, D_2, D_3$ and $D_4$ are matrices of sizes $t \times t, t \times (m-t), (m-t) \times t$ and $(m-t) \times (m-t)$ respectively. On the matrix $D_1$, we have, by (2.3), that, for $1 \leq i, j \leq t,$

$$
(D_1)_{ij} = \langle X^{\sigma_i,\alpha,\ast}, X^{\sigma_j,\alpha,\ast} \rangle = \begin{cases} 
\chi(1) \sum_{\sigma \in G} \chi(\sigma), & \text{if } i = j \\
0, & \text{if } i \neq j 
\end{cases} = \begin{cases} 
\frac{t}{m}, & \text{if } i = j \\
0, & \text{if } i \neq j 
\end{cases} = \left( \frac{t}{m} I_t \right)_{ij},
$$

where $I_t$ is the $t \times t$ identity matrix. So, $D = \left[ \left( \frac{t}{m} I_t \right) D_1 D_2 D_3 D_4 \right]$. Now, using $D^2 = D$, we get

$$
D_2 D_3 = \left( \frac{t}{m} - \frac{t^2}{m^2} \right) I_t.
$$

Since $t < m$, $\left( \frac{t}{m} - \frac{t^2}{m^2} \right) \neq 0$ and hence $D_2 D_3$ is invertible. This means that if $H_{d,\ast}(\chi)$ has an $o$-basis, then

$$
t = \text{rank } D_2 D_3 \leq \min \{ \text{rank } D_2, \text{rank } D_3 \} \leq \min \{ t, m-t \} \leq m-t.
$$

Therefore, if $\chi(1)(\chi, \theta)_G = t > \frac{m}{2}$, then $H_{d}(G; \chi)$ does not have an orthogonal $*$-basis, by (2.7).

We also obtain a similar results of Holmes in [2].

**Corollary 3.2.** (cf. [2, 5]) Let $G$ be a 2-transitive subgroup of $S_m$, $m > 2$. Let $\chi = \theta - 1_G$, where $\theta$ is the permutation character of $G$. Then $H_{d}(G; \chi)$ does not have an orthogonal $*$-basis.

**Proof.** Note that $G$ has a canonical transitive an faithful action on the set $\Lambda = \{1, 2, \ldots, m\}$, given by $\sigma \cdot i := \sigma(i)$ for each $\sigma \in G \leq S_m$ and $i \in \Lambda$. Since $G$ acts 2-transitively on $\Lambda$ with permutation character $\theta$, by Corollary 5.17 in [5], $\chi = \theta - 1_G$ is an irreducible constituent of $\theta$. We compute that

$$
\chi(1)(\chi, \theta)_G = \chi(1)(\theta - 1_G, \theta)_G = \chi(1)[(\theta, \theta)_G - (\theta, 1_G)_G] = \chi(1)[2 - 1] = m - 1.
$$

Since $m > 2$, $m - 1 > \frac{m}{2}$ and hence $\chi(1)(\chi, \theta)_G > \frac{m}{2}$. Thus, by Theorem 3.1, the result follows.

**Example 3.1.** (cf. [5]) Let $G = \Lambda = A_4$ be the alternating group of degree 4. We know that $G$ acts transitively and faithfully on $\Lambda$ by left multiplication. Then we can view $G$ as a subgroup of $S_12$. Note that $G$ has an irreducible character, $\chi$, of degree 3 and the permutation character $\theta$ of $G$ is regular. Thus $\chi$ is an irreducible constituent of $\theta$ of multiplicity 3. Hence, $\chi(1)(\chi, \theta)_G = 9 > \frac{12}{2} = \frac{12}{2}$ and then $H_{d}(A_4; \chi)$ does
not have an orthogonal $*$-basis, by Theorem 3.1. In this example, however, the action $G$ on $\Lambda$ is not 2-transitive.

By using the same technique as in the proof of Theorem 3.1, we also obtain an analogous criterion of Shahryari in [10].

**Theorem 3.3.** Let $G$ be a permutation group of degree $m$ and $\chi$ be a non-linear irreducible character of $G$. If there is $\alpha \in \Gamma_{m,d}^+$ such that

$$\frac{\sqrt{2}}{2} < \| X^{\alpha,*}_\chi \| < 1,$$

then $H_d(G;\chi)$ does not have an orthogonal $*$-basis.

**Proof.** Let $\alpha \in \Gamma_{m,d}^+$. Suppose the orbit of $\alpha$ under the action of $G$ is $\text{Orb}(\alpha) = \{\sigma_1 \alpha, \sigma_2 \alpha, \ldots, \sigma_r \alpha\}$. Then, by orbit-stabilizer theorem, $r = [G : G_\alpha]$ and $G = \bigcup_{i=1}^r \sigma_i G_\alpha$ is a partition. Now, we construct $r \times r$ matrix $D = [D_{ij}]$ by $D_{ij} := \langle X^{\sigma_i \alpha,*}, X^{\sigma_j \alpha,*} \rangle$ which is idempotent as before. Next, suppose $\chi$ is a non-linear irreducible character of $G$ and $\alpha \in \Delta$ and assume also that $\{X^{\sigma_1 \alpha,*}, X^{\sigma_2 \alpha,*}, \ldots, X^{\sigma_t \alpha,*}\}$ is an $o$-basis for $H_d^{\alpha,*}(\chi)$ in which $t < r$, where $t = \dim H_d^{\alpha,*}(\chi)$. So, the matrix $D$ has the block partition form

$$D = \begin{bmatrix} (\frac{1}{t}) I_t & D_2 \\ D_3 & D_4 \end{bmatrix},$$

where $D_2, D_3$ and $D_4$ are matrices of sizes $t \times (r-t), (r-t) \times t$ and $(r-t) \times (r-t)$ respectively. By the same arguments as in the proof of Theorem 3.1, we reach to the conclusion that $t \leq r-t$ or $t \leq \frac{r}{2}$. Thus if $t < r$ and $t > \frac{r}{2}$, then $H_d^{\alpha,*}(\chi)$ does not have $o$-basis. Substituting $r = [G : G_\alpha]$, $t = \chi(1)(\chi,1)_{G_\alpha}$ in the inequality $\frac{r}{2} < t < r$ and using (2.6) and (2.7), the result follows. $\blacksquare$

4. **Symmetric groups.** It is well known that there is a standard one-to-one correspondence between the complex irreducible characters of the symmetric group $S_m$ and the partitions of $m$. Here, a partition $\pi$ of $m$ of length $t$, denoted by $\pi \vdash m$, means an unordered collection of $t$ positive integers that sum to $m$. In this article, we use the same symbol to denote an irreducible character of $S_m$ and the partition of $m$ corresponding to it. Typically, we represent the partition by a sequence $\pi = [\pi_1, \pi_2, \ldots, \pi_t]$ in which $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_t > 0$. A partition $\pi = [\pi_1, \pi_2, \ldots, \pi_t]$ is usually represented by a collection of $m$ boxes arranged in $t$ rows such that the number of boxes of row $i$ is equal to $\pi_i$, for $i = 1, 2, \ldots, t$. This collection is called the **Young diagram** associated with $\pi$ and denoted by $[\pi]$. The **Young subgroup** corresponding to $\pi \vdash m$ is the internal direct product

$$S_\pi = S_{\pi_1} \times S_{\pi_2} \times \cdots \times S_{\pi_t}.$$
We write $1_{S_{\pi}} = 1_{\pi}$ for the principle character of $S_{\pi}$. Note that $1_{S_{\pi}}^{S_m}$ is a character of $S_m$, so there must exist integers $K_{\mu,\pi}$ such that

$$1_{S_{\pi}}^{S_m} = \sum_{\mu \vdash m} K_{\mu,\pi} \mu.$$ 

The numbers $K_{\mu,\pi} = (1_{S_{\pi}}^{S_m}, \mu)_{S_m}$ are called Kostka coefficients. By Corollary 4.54 in [7], the Kostka coefficient $K_{\pi,\pi} = 1$ for all $\pi \vdash m$.

For each ordered pair $(i,j)$, $1 \leq i \leq t$, $1 \leq j \leq \pi_i$, there is corresponding a box, $B_{ij}$, in Young diagram $[\pi]$. Each $B_{ij}$ determines a unique hook in $[\pi]$ consisting of $B_{ij}$ itself, all the boxes in row $i$ of $[\pi]$ to the right of $B_{ij}$ and all boxes in column $j$ of $[\pi]$ below $B_{ij}$. The hook length,

$$h_{ij} := (\pi_i - i) + (\pi'_j - j) + 1,$$

where $\pi'_j := \{k \in \{1, 2, \ldots, t\} | \pi_k \geq j\}$ (a $j$ part of conjugate partition of $\pi$), is the number of boxes in the hook determined by $B_{ij}$. By the Frame-Robinson-Thrall Hook Length Formula (see, e.g., Theorem 4.60 in [7]), if $\pi$ is a partition of $m$, then the degree of the irreducible character of $S_m$ corresponding to $\pi = [\pi_1, \pi_2, \ldots, \pi_t]$ is

$$\pi(1) = \frac{m!}{\prod_{i=1}^{\pi_1} \prod_{j=1}^{\pi_i} h_{ij}}. \tag{4.1}$$

As a consequence of Theorem 3.3, we have an analogous result of Y. Zamani in [12].

**Theorem 4.1.** Let $\pi$ be an irreducible character of $S_m$ of the cycle type;

I. $\pi = [m - l, l]$, $d \equiv 0 \mod l$, $d \neq 0$ such $m \geq 3l$, or

II. $\pi = [m - l, l - 1, 1]$, $d \equiv r \mod l$, $0 < r < l$, $l > 2$, $d \neq r$ such $m > 3l + \frac{4}{l-2}$.

Then $H_d(S_m; \pi)$ does not have an orthogonal $*$-basis.

**Proof.** For the form I, we set $\alpha = (0, 0, \ldots, 0, k, k, \ldots, k)$, where $k = \frac{d}{l}$. Then $\alpha \in \Gamma_{m,d}^+$. Under the action of $S_m$ on $\Gamma_{m,d}^+$, we choose a system $\Delta$ of representatives such that $\alpha \in \Delta$. Since $d \neq 0$, $k \neq 0$ and

$$(S_m)_\alpha \cong S_{m-l} \times S_l = S_{\pi},$$

where $(S_m)_\alpha$ is the stabilizer subgroup of $\alpha$ and $S_{\pi}$ is the Young subgroup corre-
sponding to $\pi \vdash m$. Hence, by Frobenius Reciprocity Theorem,
\[
\frac{1}{|S_m\alpha|} \sum_{\sigma \in (S_m\alpha)} \pi(\sigma) = (\pi, 1_{(S_m\alpha)})_{(S_m\alpha)}
\]
\[
= (\pi, 1_{\pi})_{S_m}
\]
\[
= (\pi, 1_{\pi})_{S_m}
\]
\[
= \left(\pi, \sum_{\mu \vdash m} K_{\pi, \pi\mu}\right)_{S_m}
\]
\[
= \sum_{\mu \vdash m} K_{\pi, \pi\mu} (\pi, \mu)_{S_m}
\]
\[
= K_{\pi, \pi} = 1 \neq 0.
\]
This yields $\alpha \in \Delta$ and, moreover, by (2.8), that
\[
\dim H_{d}^{\alpha, \ast}(\pi) = \frac{\pi(1)}{|S_m\alpha|} \sum_{\sigma \in (S_m\alpha)} \chi(\sigma) = \pi(1).
\]
Now, we compute the product of the hook lengths of $[\pi]$ which we get
\[
\prod_{i=1}^{2} \prod_{j=1}^{\pi_i} h_{ij} = (m - l + 1)(m - l) \cdots (m - 2l + 2)(m - 2l) \cdots (2)(1)(l - 1) \cdots (2)(1)
\]
\[
= \frac{(m - l + 1)!l!}{(m - 2l + 1)!}.
\]
Hence, by (4.1),
\[
\dim H_{d}^{\alpha, \ast}(\pi) = \pi(1) = \frac{(m - 2l + 1)m!}{(m - l + 1)!l!}.
\]
Now, using (2.6), we have
\[
\|X^{\alpha, \ast}\|^2 = \frac{\dim H_{d}^{\alpha, \ast}(\pi)}{|S_m : (S_m\alpha)|} = \frac{m - 2l + 1}{m - l + 1}.
\]
Hence, $\frac{1}{2} < \|X^{\alpha, \ast}\|^2 < 1$ if and only if $m \geq 3l$. Thus, the result for the first form follows from Theorem 3.3.

For the form II, $\pi = [m - l, l - 1, 1]$, we set $\alpha = (0, 0, \ldots, 0, k, k, \ldots, k, k + r)$, where $k = \frac{d - r}{l}$. Then $\alpha \in \Gamma_{m,d}^+$. Under the action of $S_m$ on $\Gamma_{m,d}^+$, we choose a system $\Delta$ of representatives such that $\alpha \in \Delta$. Since $d \neq r \neq 0$, $k \neq 0$ and $k + r \neq k$ and hence
\[
(S_m\alpha) \cong S_{m-1} \times S_{l-1} \times S_1 = S_{\pi}.
\]
By the same arguments as the first form, we conclude that \( \dim H^\alpha_d(\pi) = \pi(1) \). For the products of the hook lengths, we compute that

\[
\prod_{i=1}^{m} \prod_{j=1}^{l} h_{ij} = (m - l + 2)(m - l) \cdots (m - 2l + 3)(m - 2l + 1) \cdots (1)l(l - 2)(l - 3) \cdots (1)
\]

\[
= \frac{(m - l + 2)!l!}{(m - l + 1)(m - 2l + 2)(l - 1)}.
\]

Then

\[
\dim H^\alpha_d(\pi) = \pi(1) = \frac{(m - l + 1)(m - 2l + 2)(l - 1)m!}{(m - l + 2)!l!}.
\]

Now, using (2.6) again, we have

\[
\| X^{\alpha,*} \|^2 = \frac{\dim H^\alpha_d(\pi)}{|S_m : (S_m)_\alpha|} = \frac{(m - 2l + 2)(l - 1)}{(m - l + 2)(l)}.
\]

It is now easy to show that \( \frac{1}{2} < \| X^{\alpha,*} \|^2 < 1 \) if and only if \( m > 3l + \frac{4}{1-l} \), because \( l > 2 \). The result for the second form follows from Theorem 3.3.

**Acknowledgment.** The author is grateful to the anonymous referees and Professor Tin-Yau Tam for their recommendations, and would like to thank Naresuan University for financial support on the project R2558C030.

**REFERENCES**


