Inequalities for relative operator entropies

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INEQUALITIES FOR RELATIVE OPERATOR ENTROPIES

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1. Introduction. We start with some notation (see [2, p. 112]).

As usual, the symbol $M_n(\mathbb{C})$ denotes the $C^*$-algebra of $n \times n$ complex matrices. For matrices $X, Y \in M_n(\mathbb{C})$, we write $Y \leq X$ (resp., $Y < X$) if $X - Y$ is positive semidefinite (resp., positive definite).

A linear map $\Phi : M_n(\mathbb{C}) \to M_k(\mathbb{C})$ is said to be positive if $0 \leq \Phi(X)$ for $0 \leq X \in M_n(\mathbb{C})$. If $0 < \Phi(X)$ for $0 < X \in M_n(\mathbb{C})$ then $\Phi$ is said to be strictly positive.

A real function $f : J \to \mathbb{R}$ defined on interval $J \subset \mathbb{R}$ is called an operator monotone function, if for all Hermitian matrices $A$ and $B$ (of the same order) with spectra in $J$ $A \leq B \implies f(A) \leq f(B)$.

Let $f : J \to \mathbb{R}$ be a continuous function on an interval $J \subset \mathbb{R}$. Let $A$ be an $n \times n$ positive definite matrix and $B$ be an $n \times n$ Hermitian matrix such that the spectrum $\text{Sp}(A^{-1/2}BA^{-1/2}) \subset J$. Then the operator $\sigma_f$ given by

\begin{equation}
A\sigma_fB = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}
\end{equation}
Note that for the functions $p + 1 - p$ and $t^p$, the definition of Eq. (1.1) leads to the arithmetic and geometric operator means (1.2) and (1.3), respectively.

For $A > 0$, $B > 0$ and $p \in [0, 1]$, the $p$-arithmetic mean is defined as follows

$$A \nabla_p B = (1 - p)A + pB.$$  

(1.2)

For $A > 0$, $B > 0$ and $p \in [0, 1]$, the $p$-geometric mean is defined by (see [12, 17])

$$A \sharp_p B = A^{1/2}(A^{-1/2}BA^{-1/2})^pA^{1/2}.$$  

(1.3)

We now give definitions of some operator entropies.

For $A > 0$, $B > 0$, the relative operator entropy is defined by (see [4])

$$S(A, B) = A^{1/2} \log(A^{-1/2}BA^{-1/2})A^{1/2}.$$  

(1.4)

For $A > 0$, $B > 0$ and $p \in \mathbb{R}$, the generalized relative operator entropy is given by (see [14, 18])

$$S_p(A, B) = A^{1/2}(A^{-1/2}BA^{-1/2})^p \log(A^{-1/2}BA^{-1/2})A^{1/2}.$$  

(1.5)

For $A > 0$, $B > 0$ and $0 < p \leq 1$, the Tsallis relative operator entropy is defined as follows (see [18])

$$T_p(A, B) = \frac{A \nabla_p B - A}{p}.$$  

(1.6)

It is not hard to check that (1.4), (1.5) and (1.6) are of the form (1.1) for the functions $\log t$, $t^p \log t$ and $\ln t = e^{t-1}$, respectively.

In recent years there has been a growing interest in the study of entropies and means [5, 6, 7, 8, 9, 16, 19].

**Theorem A.** (Furuichi et al. [17, Theorem 3.6]) For $A > 0$, $B > 0$, $1 \geq p > 0$ and $a > 0$, the following inequality holds:

$$A \nabla_p B - \frac{1}{a}A \nabla_{p-1} B + \frac{1 - a^p}{pa^p} A \leq T_p(A, B) \leq \frac{1}{a}B - \frac{1 - a^p}{pa^p} A \nabla_p B - A.$$  

(1.7)

The next known double inequalities are consequences of (1.7) (see [5, 7, 8, 19]):

$$A - AB^{-1}A \leq T_p(A, B) \leq B - A,$$
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\[ A - AB^{-1}A \leq S(A, B) \leq B - A, \]

and

\[ (1 - \log a)A - \frac{1}{a}AB^{-1}A \leq S(A, B) \leq (\log a - 1)A + \frac{1}{a}B \quad \text{for } a > 0. \]

**Theorem B.** (Zou [19, Theorem 2.2]) For \( A > 0, B > 0, 1 \geq p > 0 \) and \( a > 0 \), the following inequality holds:

\[
(1.8) \quad -\left(\log a + \frac{1 - a^p}{p a^p}\right)A + a^{-p}T_p(A, B) \leq S(A, B) \leq T_p(A, B) - \frac{1 - a^p}{p}A^p B - (\log a)A.
\]

It is easily seen that (1.8) implies a result in [7]:

\[ T_p(A, B) \leq S(A, B) \leq T_p(A, B). \]

**Theorem C.** (Furuta [9, Theorem 2.1]) Let \( A \) and \( B \) be \( n \times n \) positive definite matrices such that \( M_1 I \geq A \geq m_1 I > 0 \) and \( M_2 I \geq B \geq m_2 I > 0 \). Put \( m = \frac{m_2}{M_1}, M = \frac{M_2}{m_1}, h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1 \) and \( p \in (0, 1] \). Let \( \Phi \) be normalized positive linear map on \( B(H) = M_n(\mathbb{C}) \). Then the following inequalities hold:

\[ \Phi(T_p(A, B)) \leq T_p(\Phi(A), \Phi(B)) \leq \Phi(T_p(A, B)) + \frac{1 - K(p)}{p} \Phi(A)^p \Phi(B) \]

and

\[ \Phi(T_p(A, B)) \leq T_p(\Phi(A), \Phi(B)) \leq \Phi(T_p(A, B)) + F(p)\Phi(A), \]

where \( K(p) \) is the generalized Kantorovich constant defined by

\[ K(p) = \frac{h^p - h}{(p - 1)(h - 1)} \left(\frac{(p - 1)(h^p - 1)}{p(h^p - h)}\right)^p \]

and

\[ F(p) = \frac{m^p}{p} \left(\frac{h^p - h}{h - 1}\right) \left(1 - K(p)\right)^{\frac{1}{p}} \geq 0. \]

For a positive concave function \( g : J \to \mathbb{R}_+ \) defined on an interval \( J = [m, M] \) with \( m < M \), we define (see [13])

\[ a_g = \frac{g(M) - g(m)}{M - m}, b_g = \frac{M g(m) - m g(M)}{M - m} \quad \text{and} \quad c_g = \min_{t \in J} a_t^{a_t + b_t} g(t). \]
In order to unify our further studies, we introduce the notion of relative $g$-entropy as follows. Let $g: J \rightarrow \mathbb{R}$ be a continuous function defined on an interval $J \subset \mathbb{R}$. For $A > 0$, $B > 0$ with the spectrum of $A^{-1/2}BA^{-1/2}$ in $J$, we define the relative $g$-entropy of $A$ and $B$ as

\begin{equation}
S_g(A, B) = A\sigma_g B = A^{1/2}g(A^{-1/2}BA^{-1/2})A^{1/2}.
\end{equation}

In the present paper, our aim is to provide some further operator inequalities for entropies and means transformed by a strictly positive linear map $\Phi$.

\section{Furuta type inequalities.}

Throughout $f(t, p)$ is a real function of two variables $t \in J$ and $p \in P = (0, p_0]$, $0 < p_0 \leq 1$. We use the notation

\begin{align}
&f_p(t) = f(t, p) \quad \text{for } t \in J \text{ and } p \in P, \\
g_p(t) = g(t, p) = \frac{f(t, p) - f(t, 0)}{p} \quad \text{for } t \in J \text{ and } p \in P.
\end{align}

If there exist the following limits, then we write

\begin{align}
f_0(t) &= f(t, 0) = \lim_{p \to 0^+} f(t, p) \quad \text{for } t \in J, \\
g_0(t) &= g(t, 0) = \lim_{p \to 0^+} g(t, p) \quad \text{for } t \in J.
\end{align}

For example, by substituting $f(t, p) = t^p$ for $t > 0$, $0 < p \leq p_0 = 1$, we get $f_0(t) = 1$, $g(t, p) = \ln p(t)$ and $g_0(t) = \log t$.

\textbf{Lemma 2.1.} Let $f(t, p)$ be a real function of two variables $t \in J$ and $p \in P = (0, p_0]$, $0 < p_0 \leq 1$, with an interval $J \subset (0, \infty)$. Assume $f(t, 0) = 1$, $t \in J$. For $n \times n$ positive definite matrices $A$ and $B$ with spectrum $\operatorname{Sp}(A^{-1/2}BA^{-1/2}) \subset J$, the following identity holds:

\begin{equation}
S_g(A, B) = \frac{S_f(A, B) - A}{p} \quad \text{for } p \in P,
\end{equation}

where $f_p$ and $g_p$ are defined by \eqref{eq:fp}–\eqref{eq:gp}.

\textbf{Proof.} By \eqref{eq:entropy} and \eqref{eq:gf}, we establish the equalities

\begin{align}
\frac{S_f(A, B) - A}{p} &= A\sigma_f B - A \\
&= \frac{A^{1/2}f_p(A^{-1/2}BA^{-1/2})A^{1/2} - A^{1/2}IA^{1/2}}{p} \\
&= A^{1/2}\frac{f_p(A^{-1/2}BA^{-1/2}) - I}{A^{1/2}}.
\end{align}
Therefore, from (2.6), we derive that

\[ \mu \]

matrices such that

\[ 0 < m < M \]

This proves (2.5).

Therefore, from (2.6), we derive

\[
\begin{align*}
S_{f_p}(A, B) - A &= A^{1/2} f_p(Z) - U^* U A^{1/2} \\
&= A^{1/2} U^* \text{diag} (f(\mu_1, p), f(\mu_2, p), \ldots, f(\mu_n, p)) U - U^* U A^{1/2} \\
&= A^{1/2} U^* \text{diag} \left( f(\mu_1, p), f(\mu_2, p), \ldots, f(\mu_n, p) \right) U A^{1/2} \\
&= A^{1/2} g_p(U^* \text{diag}(\mu_1, \mu_2, \ldots, \mu_n) U) A^{1/2} \\
&= A^{1/2} g_p(A^{-1/2} BA^{-1/2}) A^{1/2} = A \sigma_p B = S_{g_p}(A, B).
\end{align*}
\]

This proves (2.5). \( \square \)

In the forthcoming theorem, we extend Furuta’s inequality (1.9) from the functions \( t \to t^p \), \( p \in (0, 1] \), to positive operator monotone functions \( t \to f_p(t) \) on \( J = [m, M] \), \( 0 < m < M \).

**Theorem 2.2.** Let \( f(t, p) \) be a real function of two variables \( t \in J = [m, M] \) with \( 0 < m < M \), and \( p \in P = (0, p_0] \) with \( 0 < p_0 \leq 1 \). Let \( f(t, 0) = 1, t \in J \). Assume that \( f_p > 0, p \in P \), is operator monotone on \( J \). Let \( A \) and \( B \) be \( n \times n \) positive definite matrices such that \( mA \leq B \leq MA \).

If \( \Phi : M_n(\mathbb{C}) \to M_k(\mathbb{C}) \) is a strictly positive linear map, then

\[
S_{g_p}(\Phi(A), \Phi(B)) \leq \Phi(S_{g_p}(A, B)) + \frac{1-c_{f_p}}{p} \Phi(A) \sigma_p \Phi(B),
\]

where \( f_p \) and \( g_p \), \( p \in P \), are defined by (2.7) and (2.3), respectively, and \( c_{f_p} = \min_{t \in J} \frac{a_{f_p} t^p + b_{f_p}}{f_p(t)} \) with \( a_{f_p} = \frac{f_p(M) - f_p(m)}{M - m} \) and \( b_{f_p} = \frac{M f_p(m) - m f_p(M)}{M - m} \).

If in addition \( \frac{1-c_{f_p}}{p} \to d \) as \( p \to 0 \), then

\[
S_{g_0}(\Phi(A), \Phi(B)) \leq \Phi(S_{g_0}(A, B)) + d \Phi(A),
\]
where $f_0$ and $g_0$ are defined by (2.3) and (2.4), respectively.

**Proof.** It is not hard to verify that the assertion of [13, Corollary 3.4] can be extended to the case $0 < mA \leq B \leq MA$. In consequence, since $f_p > 0$ is operator monotone on $J$, the following inequality is met (cf. [13, Corollary 3.4]):

$$c f_p \Phi(A) \sigma f_p \Phi(B) \leq \Phi(A \sigma f_p B).$$

(2.9)

In addition, $\Phi(A) \sigma f_p \Phi(B) = \Phi(A)$, because $f_0 \equiv 1$. So, it follows from (2.5) and (2.9) that

$$\Phi(A) \sigma g_p \Phi(B) - \frac{1 - c f_p}{p} \Phi(A) \sigma f_p \Phi(B) = \frac{c f_p \Phi(A) \sigma f_p \Phi(B) - \Phi(A)}{p} \leq \frac{\Phi(A \sigma f_p B) - \Phi(A)}{p} = \Phi \left( \frac{A \sigma f_p B - A}{p} \right) = \Phi(A \sigma g_p B).$$

Therefore, we have

$$\Phi(A) \sigma g_p \Phi(B) \leq \Phi(A \sigma g_p B) + \frac{1 - c f_p}{p} \Phi(A) \sigma f_p \Phi(B).$$

(2.10)

Now, the inequality (2.7) can be deduced from (2.10) via (1.12).

By passing to the limit in (2.7) as $p \to 0$, we get $\Phi(A) \sigma f_p \Phi(B) \to \Phi(A) \sigma f_0 \Phi(B)$, $A \sigma g_p B \to A \sigma g_0 B$ and $\Phi(A) \sigma f_p \Phi(B) \to \Phi(A) \sigma f_0 \Phi(B) = \Phi(A)$. Thus, (2.8) leads to (2.7).

This completes the proof of Theorem 2.2.

For $A > 0$, $B > 0$ and $p, q \geq 0$, $p + q \leq 1$, the $(p, q)$-generalized relative operator entropy is defined by

$$S_{p,q}(A, B) = A^{1/2}(A^{-1/2}BA^{-1/2})^p(\log(A^{-1/2}BA^{-1/2}))^q A^{1/2}.\tag{2.11}$$

Notice that for $q = 0$ one has $S_{p,q}(A, B) = A_{p^2}B$, and for $q = 1$ and $p = 0$, $S_{p,q}(A, B) = S(A, B)$.

It is worth emphasizing that the function $J \ni t \to t^q(\log t)^p, p, q \geq 0, p + q \leq 1$, is operator monotone on any interval $J = [m, M], 1 < m < M$ (see [11 Corollary 2.7]).

Below we give an interpretation of statement (2.7) for the $(p, q)$-generalized relative operator entropy.

**Corollary 2.3.** Let $A$ and $B$ be $n \times n$ positive definite matrices such that $mA \leq B \leq MA, 1 < m < M$.
If $\Phi : M_n(\mathbb{C}) \to M_k(\mathbb{C})$ is a strictly positive linear map, then

\begin{equation}
S_{p,q}(\Phi(A), \Phi(B)) \leq \Phi(S_{p,q}(A,B)) + \frac{1-c_{f_{p,q}}}{p} \Phi(A) \sigma_{f_{p,q}} \Phi(B),
\end{equation}

where $p,q \geq 0$, $p+q \leq 1$, and $S_{p,q}$ is the $(p,q)$-generalized relative operator entropy defined by (2.11), and $c_{f_{p,q}}$ is defined by (2.12).

**Proof.** Apply Theorem 2.2 to the functions $f_{p,q}(t) = pt^p/\log t^q + 1$, $f_{0,q}(t) = 1$, and $g_{p,q}(t) = t^p/\log t^q$, $t \in [m,M]$ with fixed $q$ and $p \in [0,p_0]$, $p_0 = 1 - q$. \hfill $\Box$

### 3. Extending Furuichi et al. and Zou’s results

In this section, we develop some results due to Furuichi et al. \[7\] and Zou \[19\]. To do so, we involve star-shaped functions.

Remind that a real nonnegative function $F$ on $[0,p_0]$, $0 < p_0 \leq \infty$, with $F(0) = 0$ is said to be star-shaped if $F(\alpha p) \leq \alpha F(p)$ for $p \in [0,p_0]$ and $0 \leq \alpha \leq 1$.

**Theorem 3.1.** With the definitions (2.7)–(2.8) for a real function $f(t,p)$ of two variables $t \in J \subset (0,\infty)$ with an interval $J$ and $p \in P = [0,1]$, assume that for each $t \in J$ the function $p \to f(t,p) - f(t,0)$, $p \in P$, is positive and star-shaped. Let $\varphi : J \to J$, i.e., $\varphi(t) \in J$ for $t \in J$. Let $A$ and $B$ be $n \times n$ positive definite matrices such that the spectrum $\text{Sp}(A^{-1/2}BA^{-1/2}) \subset J$. Then for any $p \in (0,1]$, the following two inequalities hold:

\begin{align*}
(3.1) & \quad S_{g_p}(A,B) \leq S_{g_1}(A,\varphi(A,B)) - S_{h_p}(A,B), \\
(3.2) & \quad S_{g_0}(A,B) \leq S_{g_1}(A,\varphi(A,B)) - S_{h_0}(A,B),
\end{align*}

where

\begin{align*}
(3.3) & \quad h_p(t) = h(t,p) = g(\varphi(t),p) - g(t,p) \quad \text{for } t \in J, \\
(3.4) & \quad h_0(t) = h(t,0) = g(\varphi(t),0) - g(t,0) \quad \text{for } t \in J.
\end{align*}

**Proof.** The function $[0,1] \ni p \to \frac{f(t,p) - f(t,0)}{p} = g(t,p)$ is nondecreasing \[3\] Lemma 3], i.e.,

$$
0 < p_1 \leq p_2 \leq 1 \quad \text{implies} \quad \frac{f(t,p_1) - f(t,0)}{p_1} \leq \frac{f(t,p_2) - f(t,0)}{p_2}.
$$

Hence,

$$
g(t,0) = \lim_{p_1 \to 0^+} \frac{f(t,p_1) - f(t,0)}{p_1} \leq \frac{f(t,p_2) - f(t,0)}{p_2} \quad \text{for any} \quad 0 < p_2 \leq 1.
$$
Consequently, the following double inequality is valid:

\[(3.5) \quad g(t, 0) \leq g(t, p) \leq g(t, 1) \text{ for any } 0 < p \leq 1.\]

To prove (3.1), we employ the inequality \(g(t, p) \leq g(t, 1)\) for \(t \in J, 0 < p \leq 1\) (see (3.5)). Since \(\varphi(t) \in J\) for \(t \in J\), we obtain

\[g(\varphi(t), p) \leq g(\varphi(t), 1) \text{ for } t \in J,
\]
or, equivalently,

\[g(t, p) \leq g(\varphi(t), 1) - [g(\varphi(t), p) - g(t, p)] \text{ for } t \in J.
\]

So, by (3.3), we find that

\[g(t, p) \leq g(\varphi(t), 1) - h(t, p) \text{ for } t \in J.
\]

In other words, we have

\[(3.6) \quad g_p(t) \leq g_1(\varphi(t)) - h_p(t) \text{ for } t \in J.
\]

By denoting \(Z = A^{-1/2}BA^{-1/2}\) and making use of (3.6), we get

\[g_p(Z) \leq g_1(\varphi(Z)) - h_p(Z).
\]

Hence,

\[A^{1/2}g_p(Z)A^{1/2} \leq A^{1/2}g_1(\varphi(Z))A^{1/2} - A^{1/2}h_p(Z)A^{1/2},
\]

which means

\[(3.7) \quad A\sigma_g B \leq A\sigma_{g_1 \circ \varphi} B - A\sigma h_p B.
\]

However, we can show that

\[(3.8) \quad A\sigma_{g_1 \circ \varphi} B = A\sigma_{g_1} (A\sigma_\varphi B).
\]

Indeed, by using (1.4), we derive

\[A\sigma_{g_1 \circ \varphi} B = A^{1/2}(g_1 \circ \varphi)(A^{-1/2}BA^{-1/2})A^{1/2} = A^{1/2}g_1(\varphi(A^{-1/2}BA^{-1/2}))A^{1/2} = A^{1/2}g_1(A^{-1/2}A^{1/2}\varphi(\varphi(A^{-1/2}BA^{-1/2}))A^{1/2} = A^{1/2}g_1(A^{-1/2}(A\sigma_\varphi B)A^{-1/2})A^{1/2} = A\sigma_{g_1} (A\sigma_\varphi B),
\]

completing the proof of (3.8).
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So, by virtue of (3.7)–(3.8), we infer that
\[ S_{g_p}(A, B) \leq S_{g_1}(A, A\sigma_\varphi B) - S_{h_0}(A, B), \]
which proves (3.1).

We shall show (3.2). According to the inequality \( g(t, 0) \leq g(t, p) \) for \( t \in J, 0 < p \leq 1 \) (see (3.5)), we get
\[ g(\varphi(t), 0) \leq g(\varphi(t), p) \quad \text{for} \quad t \in J, \]
because \( \varphi(t) \in J \) for \( t \in J \). So, by (3.4), we have
\[ g(t, 0) \leq g(t, p) - [g(t, p) - g(\varphi(t), p)] - h_0(t) \quad \text{for} \quad t \in J, \]
which means
\[ g_0(t) \leq g_p(t) - [g_p(t) - g_p(\varphi(t))] - h_0(t) \quad \text{for} \quad t \in J. \]

With the notation \( Z = A^{-1/2}BA^{-1/2} \), inequality (3.9) gives
\[ g_0(Z) \leq g_p(Z) - [g_p(Z) - g_p(\varphi(Z))] - h_0(Z). \]
Next, by pre- and post-multiplying by \( A^{1/2} \) we obtain
\[ A^{1/2}g_0(Z)A^{1/2} \leq A^{1/2}g_p(Z)A^{1/2} - [A^{1/2}g_p(Z)A^{1/2} - A^{1/2}g_p(\varphi(Z))A^{1/2}] - A^{1/2}h_0(Z)A^{1/2}. \]
This amounts to
\[ A\sigma_{g_p}B \leq A\sigma_{g_p}B - [A\sigma_{g_p}B - A\sigma_{g_p}\circ\varphi B] - A\sigma_{h_0}B. \]

Similarly as in (3.8), we have
\[ A\sigma_{g_p}\circ\varphi B = A\sigma_{g_p}(A\sigma_\varphi B). \]

Therefore, (3.10)–(3.11) lead to
\[ S_{g_p}(A, B) \leq S_{g_p}(A, B) - [S_{g_p}(A, B) - S_{g_p}(A, A\sigma_\varphi B)] - S_{h_0}(A, B), \]
completing the proof of (3.2). \( \square \)

Remark 3.2. According to [3, Theorem 5], Theorem 3.1 remains valid if the star-shapedness of the function \( p \to f(t, p) - f(t, 0) \), is replaced by convexity or convexity on the average.

Remark 3.3. (i). It is not hard to verify that Theorem 3.1 Eq. (3.1), reduces to Theorem A, with the following specification
\[ f_p(t) = t^p, \quad g_p(t) = \ln_p t = \frac{t^p - 1}{p}, \quad g_1(t) = t - 1, \quad \varphi(t) = \frac{t}{a}, \quad a > 0. \]
(ii). Likewise, Theorem 3.1, Eq. (3.2), becomes Theorem B, whenever
\[ f_p(t) = t^p, \quad g_0(t) = \log t, \quad g_p(t) = \frac{t^p - 1}{p}, \quad \varphi(t) = at, \quad a > 0. \]

In the next corollary, we provide analogs of Theorem A and Theorem B for the
generalized relative operator entropy defined by (1.5).

Corollary 3.4. Let \( A \) and \( B \) be \( n \times n \) positive definite matrices such that the
spectrum \( \text{Sp} (A^{-1/2}BA^{-1/2}) \subset (1, \infty) \). Then for any \( p \in P = (0, 1] \) and \( a \geq 1 \), the
following two inequalities hold:
\[
\begin{align*}
(3.12) & \quad S_p(A, B) \leq a^{1-p}(\log a)B + a^{1-p}S_1(A, B) - (\log a)A_{ap}B, \\
(3.13) & \quad a^{-p}S(A, B) + (a^{-p}\log a)A - (\log a)A_{ap}B \leq S_p(A, B),
\end{align*}
\]
where \( S_p \) is the generalized relative operator entropy defined by (1.5), and \( S \) is the
relative operator entropy defined by (1.4).

Proof. We apply Theorem 3.1 to the functions \( f(t, p) = pt^p \log t, \ f(t, 0) = 0, \ g_p(t) = g(t, p) = t^p \log t, \ g_0(t) = g(t, 0) = \log t, \) and \( \varphi(t) = at, \ a \geq 1, \) for \( t \in J = (1, \infty) \) and \( p \in (0, 1] \). So, it is easily seen that \( S_q(A, B) = aB. \)

Next, we shall show the identity
\[
(3.14) \quad S_q(A, aB) = (a^q \log a)A_{aq}B + a^q S_q(A, B) \quad \text{for} \quad q \in (0, 1].
\]

Indeed, we have
\[
S_q(A, aB) = A^{1/2}g_q(A^{-1/2}aB)A^{-1/2}A^{1/2} = A^{1/2}g_q(aA^{-1/2}BA^{-1/2})A^{1/2}.
\]
By denoting \( Z = A^{-1/2}BA^{-1/2} \), we write \( Z = U^*(\text{diag} \mu_i)U \) with unitary \( U \) and the
eigenvalues \( \mu_i, \ i = 1, \ldots, n, \) of \( Z. \) Hence,
\[
g_q(aZ) = g_q(U^*(\text{diag} \mu_i)U) = U^*(\text{diag} g_q(\mu_i))U \\
= U^*(\text{diag} ((a\mu_i)^q \log(a\mu_i)))U \\
= U^*(\text{diag} (a^q\mu_1^q \log a + a^q\mu_1^q \log \mu_1))U \\
= U^*(\text{diag} ((a^q \log a)\mu_1^q))U + U^*(\text{diag} (a^q \mu_1^q \log \mu_1))U \\
= (a^q \log a)U^*(\text{diag} \mu_1^q)U + a^qU^*(\text{diag} (\mu_1^q \log \mu_1))U \\
= (a^q \log a)Z^q + a^q g_q(Z).
\]
By pre- and post-multiplying by \( A^{1/2} \) we obtain
\[
S_q(A, aB) = S_{g_q}(A, aB) = A^{1/2}g_q(aZ)A^{1/2} \\
= (a^q \log a)A^{1/2}Z^q A^{1/2} + a^q A^{1/2}g_q(Z)A^{1/2} \\
= (a^q \log a)A_{aq}B + a^q S_q(A, B),
\]
completing the proof of (3.14).

(i) In order to prove (3.12), we derive
\[
  h_p(t) = h(t, p) = g(\varphi(t), p) - g(t, p) \\
  = (at)^p \log(at) - t^p \log t \\
  = (a^p \log a)t^p + (a^p - 1)t^p \log t \\ 
  \text{for } t \in J.
\]

For this reason,
\[
  S_{h_p}(A, B) = (a^p \log a)A\sharp_p B + (a^p - 1)S_p(A, B).
\]

By employing (3.14) with \(q = 1\), we establish
\[
  S_1(A, aB) = (a \log a)B + aS_1(A, B).
\]

(ii) We shall show (3.13). By virtue of (3.2) we get
\[
  S(A, B) \leq S_p(A, aB) - S_{h_0}(A, B).
\]

Putting \(q = p\) into (3.14) yields
\[
  S_p(A, aB) = (a^p \log a)A\sharp_p B + a^p S_p(A, B).
\]

It is obvious that
\[
  h_0(t) = h(t, 0) = g(at, 0) - g(t, 0) = \log(at) - \log t = \log a \\ 
  \text{for } t \in J.
\]

Hence, \(S_{h_0}(A, B) = (\log a)A\).

It now follows from (3.16) that
\[
  S(A, B) \leq a^p S_p(A, B) + (a^p \log a)A\sharp_p B - (\log a)A.
\]

Thus we obtain (3.13), as desired. \(\square\)

We are now in a position to show a complement to Furuta type inequality (2.7).

**Theorem 3.5.** With the definitions (2.1)–(2.4) for a real function \(f(t, p)\) of two variables \(t \in J = (0, \infty)\) and \(p \in P = (0, 1]\), assume that for each \(t \in J\) the function
\[
p \rightarrow f(t, p) - f(t, 0), \ p \in P, \ \text{is positive and star-shaped. Let } \varphi : J \to J \ \text{be such that } \varphi(t) = at \in J, \ a > 0, \ \text{for } t \in J. \ \text{Suppose that } g_1(t) = at + \beta, \ \alpha > 0, \ \text{is an affine function, and that } g_2 \ \text{is an concave function with its chord function } t \to a_{g_2} t + b_{g_2}, \ t \in J, \ a_{g_2} > 0 \ \text{(see (3.11)). Let } A \text{ and } B \ \text{be } n \times n \ \text{positive definite matrices.}
\]

If \( \Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) is a strictly positive linear map, then for any \( p \in P \),

\[
(3.17) \quad S_{g_p}(\Phi(A), \Phi(B)) \leq \frac{\alpha a}{a_{g_2}} \Phi(S_{g_2}(A, B)) - S_{h_p}(\Phi(A), \Phi(B)) + \left( \beta - \frac{b_{g_2}}{a_{g_2}} \right) \Phi(A),
\]

where \( h_p(t) = h(t, p) = g(at, p) - g(t, p) \) for \( t \in J \).

**Proof.** As in the proof of Theorem 3.1 (see (3.5)), we have \( g_p(t) \leq g_1(t) \) for \( t \in J \), and

\[
g_p(t) \leq g_1(at) - h_p(t) = \alpha at + \beta - h_p(t) \quad \text{for } t \in J.
\]

Since \( g_2 \) is concave with its chord function \( t \to a_{g_2} t + b_{g_2}, \ t \in J, \ a_{g_2} > 0 \) (see (3.11)), we get

\[
a_{g_2} t + b_{g_2} \leq g_2(t) \quad \text{for } t \in J.
\]

By using the last two inequalities, we obtain

\[
g_p(t) + h_p(t) - \beta \leq \alpha \varphi(t) = \alpha at = \frac{\alpha a}{a_{g_2}} a_{g_2} t \leq \frac{\alpha a}{a_{g_2}}(g_2(t) - b_{g_2}) \quad \text{for } t \in J.
\]

In consequence, for \( Z = A^{-1/2}BA^{-1/2} \) and \( W = C^{-1/2}DC^{-1/2} \) with \( C = \Phi(A) \) and \( D = \Phi(B) \), we find that

\[
(3.18) \quad g_p(W) + h_p(W) - \beta I \leq \alpha a W;
\]

\[
\alpha a Z \leq \frac{\alpha a}{a_{g_2}} (g_2(Z) - b_{g_2} I).
\]

Thus, we obtain

\[
C^{1/2} g_p(W) C^{1/2} + C^{1/2} h_p(W) C^{1/2} - \beta C \leq \alpha a C^{1/2} W C^{1/2},
\]

\[
\alpha a A^{1/2} Z A^{1/2} \leq \frac{\alpha a}{a_{g_2}} \left( A^{1/2} g_2(Z) A^{1/2} - b_{g_2} A \right).
\]

That is,

\[
(3.19) \quad C \sigma_{g_p} D + C \sigma_{h_p} D - \beta C \leq \alpha a D,
\]

\[
\alpha a B \leq \frac{\alpha a}{a_{g_2}} (A \sigma_{g_2} B - b_{g_2} A).
\]

Hence,

\[
(3.20) \quad \alpha a \Phi(B) \leq \frac{\alpha a}{a_{g_2}} (\Phi(A \sigma_{g_2} B) - b_{g_2} \Phi(A)).
\]
But (3.19) can be rewritten as

\[(3.21) \quad \Phi(A)\sigma_{g_2}\Phi(B) + \Phi(A)\sigma_{h_2}\Phi(B) - \beta\Phi(A) \leq \alpha a \Phi(B).\]

Now, by combining (3.21) and (3.20), we establish

\[(3.22) \quad \Phi(A)\sigma_{g_2}\Phi(B) + \Phi(A)\sigma_{h_2}\Phi(B) - \beta\Phi(A) \leq \frac{\alpha a}{a_{g_2}}(\Phi(A)\sigma_{g_2}B - b_{g_2}\Phi(A)).\]

So, we infer that

\[(3.23) \quad \Phi(A)\sigma_{g_2}\Phi(B) \leq \frac{\alpha a}{a_{g_2}}(\Phi(A)\sigma_{g_2}B - \Phi(A)\sigma_{h_2}\Phi(B) + \left(\beta - \frac{b_{g_2}}{a_{g_2}}\right)\Phi(A)),\]

which is equivalent to (3.17).

**Corollary 3.6.** With the assumptions of Theorem 3.5, if in addition \(g_2 = g_1\) then (3.17) reduces to

\[(3.22) \quad S_{g_2}(\Phi(A), \Phi(B)) \leq a\Phi(S_{g_1}(A, B)) - S_{h_2}(\Phi(A), \Phi(B)) + \beta (1 - a) \Phi(A).\]

If additionally \(\Phi\) is the identity, then (3.22) yields

\[(3.23) \quad S_{g_2}(A, B) \leq aS_{g_1}(A, B) - S_{h_2}(A, B) + \beta (1 - a) A.\]

**Remark 3.7.** By letting \(\alpha = 1, \beta = -1\) and

\[f_p(t) = t^p, \quad g_p(t) = \ln t = \frac{t^p - 1}{p}, \quad \varphi(t) = \frac{t}{a}, \quad h_p(t) = t^p \ln \frac{1}{a}, 1 \geq a > 0,\]

inequality (3.23) becomes the result (1.7) due to Furuichi et al. (see Theorem A).

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Reference List


