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FACTORIZATION OF PERMUTATIONS*

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Abstract. The problem of factoring a permutation as a product of special types of transpositions, namely, those transpositions involving two positions with bounded distances, is considered. In particular, the minimum number, δ, such that every permutation can be factored into no more than δ special transpositions is investigated. This study is related to sorting algorithms, Cayley graphs, and genomics.

Key words. Bubble sort, Cayley graph, Permutation, Symmetric group, Genomics.

AMS subject classifications. 20B30.

1. Introduction.

A basic problem in computer science concerns the sorting of a list of elements in a random order to a specific order. For example, the bubble sort algorithm can be used to restore the order of a list of numbers, say, $[i_1, \ldots, i_n]$, an arrangement of the numbers $[1, \ldots, n]$, by swapping adjacent elements to restore the list to its natural (ascending) order.

Mathematically, we identify the list $\sigma = [i_1, \ldots, i_n]$ as a permutation in S_n, the symmetric group of degree n, such that $\sigma(j) = i_j$ for $j = 1, \ldots, n$. Define the number of inversions of the permutation $\sigma = [i_1, \ldots, i_n]$ as the sum of the numbers $\mathrm{inv}(j)$, where $\mathrm{inv}(j)$ is the number of integers smaller than j lying on the right side of j in $[i_1, \ldots, i_n]$. For example, the number of inversions in $[3, 2, 4, 5, 1]$ is $2+1+1+1+0 = 5$.

Applying bubble sort to the permutation $\sigma = [i_1, \ldots, i_n]$ corresponds to restoring σ to the identity permutation $[1, \ldots, n]$ by exchanging two adjacent numbers in each step. It is not hard to see that the minimum number of steps needed is the number of inversions in $\sigma = [i_1, \ldots, i_n]$. In fact, switching two adjacent numbers of a permutation will increase or decrease the number of inversions by one. So, for $\sigma \in S_n$, if in each

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step one swaps two adjacent numbers that are in the wrong order, i.e., so that the left one is larger than the right one, then one will get identity permutation after k steps, where k is the number of inversions of σ. Hence, the worst scenario is when the permutation $[n, n-1, \dots, 1] \in S_n$, which has the maximum number of inversions: $(n-1) + \cdots + 1 = n(n-1)/2$. There are many other efficient sorting algorithms. We refer the readers to [7] for more details.

In this note, we consider the problem of finding the minimum number of steps needed to convert a permutation to the identity permutation if one is allowed to switch the numbers in the ith and jth positions as long as $|i - j| \le m$, for some $m \in \{1, \dots, n-1\}$. Clearly, the bubble sort algorithm is the case when $m = 1$.

Let G_m be the set of transpositions (i, j) in S_n with $|j - i| \le m$. We investigate the minimum number $\delta(n, m)$ such that every permutation can be factored into no more than $\delta(n, m)$ transpositions in G_m. For $(n-1)/2 \le m$, we give a formula for $\delta(n, m)$, and characterize all permutations in S_n requiring $\delta(n, m)$ transpositions in its factorization (see Theorem 2.6). For $1 < m < (n-1)/2$, we obtain an upper bound for $\delta(n, m)$ (see Section 2.2).

Note that the result on bubble sort can be formulated in terms of the Cayley graph of S_n, constructed as follows: Represent every permutation as a vertex, and connect two vertices σ_1 and σ_2 if $\sigma_2 = \tau\sigma_1$ for a permutation $\tau = (i, i+1)$ which exchanges two adjacent numbers at the ith and $(i+1)$th positions for some $i \in \{1, \dots, n-1\}$. Then the bubble sort algorithm amounts to moving a permutation $[i_1, \dots, i_n]$ to $[1, \dots, n]$ in the Cayley graph most efficiently (using the minimum number of steps). Moreover, $n(n-1)/2$ is the maximum distance from $[1, \dots, n]$ to another permutation $[i_1, \dots, i_n]$, which is $[n, \dots, 1]$. One easily shows that the value $n(n-1)/2$ is actually the maximum distance between any two vertices, and is known as the diameter of the Cayley graph. It also indicates that a permutation is a product of no more than $n(n-1)/2$ transpositions of the form $(i, i+1)$.

Our study concerns the Cayley graph of using elements in S_n as vertices so that two vertices σ_1 and σ_2 are connected if $\sigma_2 = \tau\sigma_1$ for a permutation $\tau \in G_m$.

The study of Cayley graphs of S_n has a long history; see [1, 2, 3, 5, 6, 8, 9, 10, 13] and their references. Note that in some of these papers, the authors study the minimum number of transpositions needed in the factorization. It is in essence studying the diameter of the underlying Cayley graphs.

It is interesting to note that the study is related to other topics such as genome rearrangement. In nature, some species have similar genetic make up and differ only in the order of their genes. Finding the shortest rearrangement path between two related bacteria or viruses is useful in drug discovery and vaccine development. The study is

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also useful in the study of mutations. In fact, a slight change of the genetic sequence may have significant effect, and it is more likely to see a change (permutation) of the positions of the nucleotides close to each other in the genetic sequences. That is why there is keen interest in studying such permutations in genetic sequences; for example, see [4, 11, 12] and their references.

2. Main results. Following Section 1, for $m \in \{1, \ldots, n-1\}$, let G_m be the set of transpositions (i, j) in S_n with $|j - i| \leq m$. Then G_m generates S_n, i.e., every permutation in S_n is a product of transpositions in G_m.

We are interested in finding the smallest number of transpositions in G_m needed to convert a given permutation to the identity, and those permutations which require the maximum number of transpositions to do the reduction.

Consider the Cayley graph $\Gamma_{n,m}$ so that the vertices are elements in S_n, and two vertices σ_1 and σ_2 are adjacent if $\sigma_1\sigma_2^{-1} \in G_m$. Denote by $\mathbf{1}$ the identity permutation $[1, \ldots, n]$. We are interested in the shortest path connecting the identity permutation $\mathbf{1}$ to a given permutation σ. The length of this path is denoted by $d(\mathbf{1}, \sigma, m)$. Also, we are interested in the permutation σ^* with a maximum distance to the identity $\mathbf{1}$. Clearly, the maximum distance will be the same as the maximum distance between any two vertices in the Cayley graph, and we will denote this quantity by $\delta(n, m)$, as introduced in Section 1. By the discussion in Section 1, we have $\delta(n, 1) = n(n-1)/2$.

Let $\sigma = (j_1, j_2, \ldots, j_r) \in S_n$ be a cycle of length r, i.e., the permutation $\sigma \in S_n$ such that $\sigma(j_1) = j_2, \ldots, \sigma(j_{r-1}) = j_r, \sigma(j_r) = j_1$ and $\sigma(j) = j$ for other j. Also denote by $\lceil x \rceil$ the smallest integer greater than or equal to x and $\lfloor x \rfloor$ the largest integer less than or equal to x respectively.

The following lemma will be used frequently in our discussion.

Lemma 2.1. *Let $C_1 = (i_1, i_2, \ldots, i_k)$ and $C_2 = (j_1, j_2, \ldots, j_s)$ be two disjoint cycles in S_n. Suppose $\tau_1 = (i_\alpha, i_\beta)$ and $\tau_2 = (i_u, j_v)$ with $1 \leq \alpha < \beta \leq k$, $1 \leq u \leq k$ and $1 \leq v \leq s$. Then*

$$\tau_1 C_1 = (i_\alpha, i_\beta)(i_1, i_2, \ldots, i_k) = (i_1, i_2, \ldots, i_{\alpha-1}, i_\beta, i_{\beta+1}, \ldots, i_k)(i_\alpha, i_{\alpha+1}, \ldots, i_{\beta-1})$$

is the product of two disjoint cycles, and

$$\begin{aligned}\tau_2 C_1 C_2 &= (i_u, j_v)(i_1, i_2, \ldots, i_k)(j_1, j_2, \ldots, j_s)\\ &= (i_1, i_2, \ldots, i_{u-1}, j_v, j_{v+1}, \ldots, j_s, j_1, j_2, \ldots, j_{v-1}, i_u, i_{u+1}, \ldots, i_k)\end{aligned}$$

is a cycle.

The result for $m = 1$ was discussed in Section 1. The other extreme is when

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$m = n-1$, i.e., G_m is the set of all transpositions. We have the following known result, see [1, 9]. Here we give a short proof of it for completeness.

PROPOSITION 2.2. *Suppose $\sigma \in S_n$. Then $d(\mathbf{1}, \sigma, n-1) = n-r$, where r is the number of cycles in the disjoint cycle representation of σ under the convention that each fixed point is counted as a 1-cycle. Thus, $\delta(n, n-1) = n-1$ is attained at a n-cycle.*

Proof. Writing $\sigma = \tau_1 \cdots \tau_k \in S_n$ for a minimum number of transpositions $\tau_1, \ldots, \tau_k$ is the same as finding the minimum number of transpositions $\tau_1, \ldots, \tau_k$ such that $\tau_k \cdots \tau_1 \sigma = \mathbf{1}$. Using Lemma 2.1, to convert σ to the product of n disjoint cycles, i.e., back to the identity permutation $[1, \ldots, n]$ most efficiently by composing σ with transpositions, the most efficient way is to choose transposition (i, j) in each step such that i and j lie in the same cycle. So, using $n-r$ transpositions $\tau_1, \ldots, \tau_{n-r}$ to convert σ to the identity permutation will be the most efficient scheme. □

2.1. The case when $m \geq (n-1)/2$. Given $\sigma \in S_n$, denote by $K_1(\sigma)$ the set of transpositions in G_{n-1} splitting a cycle of σ into two and by $K_2(\sigma)$ the set of transpositions in G_{n-1} jointing two cycles of σ into one. We call $K_1(\sigma)$ and $K_2(\sigma)$ type one and type two transpositions, respectively. By Lemma 2.1, we have

PROPOSITION 2.3. *Let $\sigma \in S_n$, $\tau_1 \in K_1(\sigma)$ and $\tau_2 \in K_2(\sigma)$. Then*

$$d(\mathbf{1}, \tau_1\sigma, n-1) = d(\mathbf{1}, \sigma, n-1) - 1 \quad \text{and} \quad d(\mathbf{1}, \tau_2\sigma, n-1) = d(\mathbf{1}, \sigma, n-1) + 1.$$

For any cycle $C = (i_1, \ldots, i_p)$ in S_n, we say that C is in the set L_m if for each term i_u in C, there is a term i_v such that $|i_u - i_v| > m$. On the other hand, a cycle $C \notin L_m$ if there is some $1 \leq r \leq p$ such that $|i_t - i_r| \leq m$ for all $t = 1, \ldots, p$.

LEMMA 2.4. *Let n and m be positive integers and $5 \leq n \leq 2m+1$. Suppose $C = (i_1, i_2, \ldots, i_p)$ is a length p cycle in S_n. Then one of the following holds.*

(a) *If $C \in L_m$, then C can be written as a product of $p+1$ transpositions in G_m.*
(b) *If $C \notin L_m$, then C can be written as a product of $p-1$ transpositions in G_m.*

Furthermore, suppose $C_i = (i_1, \ldots, i_p)$ and $C_j = (j_1, \ldots, j_q)$ are two cycles in L_m. If there are $1 \leq r \leq p$ and $1 \leq s \leq q$ such that

$$|i_t - j_s| \leq m \quad \text{for all} \quad t = 1, \ldots, p \quad \text{and} \quad |j_u - i_r| \leq m \quad \text{for all} \quad u = 1, \ldots, q, \tag{2.1}$$

then C_iC_j can be written as a product of $p+q$ transpositions in G_m.

Proof. Suppose $C \in L_m$. Notice that $|i_t - (m+1)| \leq m$ for all t. Then one can write

$$C = (m+1, i_p)\,(m+1, i_{p-1})\,(m+1, i_{p-2}) \cdots (m+1, i_2)\,(m+1, i_1)\,(m+1, i_p).$$

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Then the result (a) holds.

Suppose now $C \notin L_m$. That is, there is some $1 \le r \le p$ such that $|i_t - i_r| \le m$ for all $t = 1, \ldots, p$. Without loss of generality, we may assume that $r = p$. Then C can be written as

$$C = (i_p, i_{p-1})\,(i_p, i_{p-2})\,(i_p, i_{p-3})\,\cdots\,(i_p, i_2)\,(i_p, i_1).$$

Thus, the result (b) holds.

Suppose now C_i and C_j are disjoint cycles and satisfying (2.1). we may assume that $r = p$ and $s = q$ in (2.1). Then C_iC_j can be written as

$$C_iC_j = (j_q, i_p)(j_q, i_{p-1})(j_q, i_{p-2})\cdots(j_q, i_2)(j_q, i_1)(i_p, j_{q-1})(i_p, j_{q-2})\cdots(i_p, j_1)(i_p, j_q).$$

Thus, the result follows. □

Lemma 2.5. *Let n and m be positive integers and $5 \le n \le 2m+1$. Suppose $\sigma \in S_n$ has a disjoint cycle decomposition $C_1 \cdots C_r$ under the convention that each fixed point is counted as a 1-cycle. If s of the cycles C_i belongs to L_m and among them, t disjoint pairs of C_i and C_j satisfy condition (2.1) in Lemma 2.4, then*

$$d(\mathbf{1}, \sigma, m) \le n - r + 2s - 2t.$$

Proof. Suppose $\sigma \in S_n$ has a disjoint cycle decomposition $C_1 \cdots C_r$ such that $C_j \in L_m$ for $j = 1, \ldots, s$, $C_j \notin L_m$ for $j = r-s+1, \ldots, r$. Further, the cycles C_{2k-1} and C_{2k} satisfy condition (2.1) for $k = 1, \ldots, t$.

Assume that C_j has length ℓ_j for $j = 1, \ldots, r$. Then by Lemma 2.4,

$$\begin{aligned} d(\mathbf{1}, \sigma, m) &\le \sum_{j=1}^{2t} \ell_j + \sum_{j=2t+1}^{s} (\ell_j + 1) + \sum_{j=s+1}^{r} (\ell_j - 1) \\ &= \left(\sum_{j=1}^{r} \ell_j \right) + (s - 2t) - (r - s) = n - r + 2s - 2t. \quad \square \end{aligned}$$

Theorem 2.6. *Let n and m be positive integers and $5 \le n \le 2m+1$. Then*

$$\delta(n, m) = n + d - 1 \quad \text{with} \quad d = \left\lfloor \frac{n-m}{2} \right\rfloor.$$

A permutation $\sigma \in S_n$ attains $d(\mathbf{1}, \sigma, m) = n + d - 1$ if and only if one of the following holds.

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(a) $(n-m)$ *is even and* σ *is a product of* $d+1$ *disjoint cycles of the form*

$$(i_1, j_1)(i_2, j_2)\cdots(i_d, j_d)(k_1, \ldots, k_{n-2d}),$$

where $\{i_1, \ldots, i_d\} = \{1, \ldots, d\}$, $\{j_1, \ldots, j_d\} = \{n-d+1, \ldots, n\}$, *and* $\{k_1, \ldots, k_{n-2d}\} = \{d+1, \ldots, n-d\}$.

(b) $(n-m)$ *is odd and* σ *is a product of* $d+1$ *disjoint cycles of the form*

(b.1) $(i_1, j_1)(i_2, j_2)\cdots(i_{d-1}, j_{d-1})(i_d, j_d)(k_1, \ldots, k_{n-2d})$,

where $\{i_1, \ldots, i_d\} \subseteq \{1, \ldots, d+1\}$, $\{j_1, \ldots, j_d\} \subseteq \{n-d, \ldots, n\}$, *and* $\{k_1, \ldots, k_{n-2d}\} \subseteq \{d+1, \ldots, n-d\}$ *such that* $\{d+1, n-d\} \cap \{k_1, \ldots, k_{n-2d}\} \neq \emptyset$.

(b.2) $(i_1, j_1)(i_2, j_2)\cdots(i_{d-1}, j_{d-1})(i_d, j_d, i_{d+1})(k_1, \ldots, k_{n-2d-1})$,

where $\{i_1, \ldots, i_{d+1}\} = \{1, \ldots, d+1\}$, $\{j_1, \ldots, j_d\} = \{n-d+1, \ldots, n\}$, *and* $\{k_1, \ldots, k_{n-2d-1}\} = \{d+2, \ldots, n-d\}$.

(b.3) $(i_1, j_1)(i_2, j_2)\cdots(i_{d-1}, j_{d-1})(i_d, j_d, j_{d+1})(k_1, \ldots, k_{n-2d-1})$,

where $\{i_1, \ldots, i_d\} = \{1, \ldots, d\}$, $\{j_1, \ldots, j_{d+1}\} = \{n-d, \ldots, n\}$, *and* $\{k_1, \ldots, k_{n-2d-1}\} = \{d+1, \ldots, n-d-1\}$.

(b.4) $(i_1, j_1)(i_2, j_2)\cdots(i_{d-1}, j_{d-1})(i_d, j_d, i_{d+1}, j_{d+1})(k_1, \ldots, k_{n-2d-2})$,

where $\{i_1, \ldots, i_{d+1}\} = \{1, \ldots, d+1\}$, $\{j_1, \ldots, j_{d+1}\} = \{n-d, \ldots, n\}$, *and* $\{k_1, \ldots, k_{n-2d-2}\} = \{d+2, \ldots, n-d-1\}$ *such that* $\{d+1, n-d\} \cap \{i_d, j_d, i_{d+1}, j_{d+1}\} \neq \emptyset$.

Proof. Suppose σ has a disjoint cycles decomposition $C_1 \cdots C_r$. Assume $C_1, \ldots, C_s$ are disjoint cycles in L_m while $C_{s+1}, \ldots, C_r$ are not in L_m. Notice that $|i - \lceil \frac{n+1}{2} \rceil| \leq m$ for all $i = 1, \ldots, n$. It follows that the cycle containing $\lceil \frac{n+1}{2} \rceil$ is not in L_m and $r > s$. For $j = 1, \ldots, s$, let u_j and v_j be the smallest term and largest term of the cycle C_j, respectively. Since $C_i \in L_m$, we have $|v_i - u_i| > m$,

$$\{u_1, \ldots, u_s\} \subseteq \{1, \ldots, n-m-1\} \quad \text{and} \quad \{v_1, \ldots, v_s\} \subseteq \{m+2, \ldots, n\}.$$

Note that if $|v_j - u_i| \leq m$ then C_i and C_j satisfy (2.1). Moreover, since $n-m-1 < m+2$, we have

$$\min_{1 \leq i \leq s} v_i > \max_{1 \leq i \leq s} u_i.$$

Now assume there are t disjoint pairs of cycles satisfying (2.1). Without loss of generality, we assume that no pair satisfying (2.1) can be found among the cycles $C_1, \ldots, C_{\hat{s}}$ with $\hat{s} = s - 2t$. Then we have $|v_j - u_i| > m$ for all $1 \leq i, j \leq \hat{s}$, and

$$\hat{s} \leq \max_{1 \leq i \leq \hat{s}} u_i \leq \min_{1 \leq j \leq \hat{s}} v_j - (m+1) \leq (n - \hat{s} + 1) - (m+1).$$

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Thus, $2\hat{s} \leq n-m$, and hence, $\hat{s} \leq \left\lfloor \frac{n-m}{2} \right\rfloor = d$. By Lemma 2.5 and the fact that $r > s$,

$$d(\mathbf{1}, \sigma, m) \leq n - r + 2s - 2t = n + \hat{s} - (r-s) \leq n + d - 1. \tag{2.2}$$

Furthermore, equality holds only if $\hat{s} = d$ and $r - s = 1$.

Assume now that σ attains the upper bound. Then $\hat{s} = d$. As any two cycles of $C_1, \ldots, C_{\hat{s}}$ do not satisfy (2.1), we must have

$$\max_{1 \leq i \leq d} u_i \leq d+1 \quad \text{and} \quad \min_{1 \leq j \leq d} v_j \geq n-d. \tag{2.3}$$

Furthermore, at most one of the inequalities is actually an equality if $n-m$ is odd while both two inequalities are strict inequalities if $n-m$ is even. If $t > 0$, the union of the two sets

$$\{u_{d+1}, \ldots, u_{d+2t}\} \cap \{1, \ldots, d\} \quad \text{and} \quad \{v_{d+1}, \ldots, v_{d+2t}\} \cap \{n-d+1, \ldots, n\}$$

contains at most one element. Therefore, there is k such that $u_{d+k} > d$ and $v_{d+k} < n-d+1$. But then

$$v_{d+k} - u_{d+k} < n - d - (d+1) = n - 2d - 1 \leq m,$$

which contradicts that $C_{d+k} \in L_m$. Hence, $t = 0$. Thus, $\hat{s} = s = d$ and σ has a $d+1$ disjoint cycle decomposition $C_1 \cdots C_{d+1}$ with $\{C_1, \ldots, C_d\} \subseteq L_m$ and $C_{d+1} \notin L_m$.

If $n-m$ is even, then $2d = n-m$ and by (2.3) we have

$$\max_{1 \leq i \leq d} u_i = d \quad \text{and} \quad \min_{1 \leq j \leq d} v_j = n-d+1.$$

It follows that

$$\{u_1, \ldots, u_d\} = \{1, \ldots, d\} \quad \text{and} \quad \{v_1, \ldots, v_d\} = \{n-d+1, \ldots, n\}. \tag{2.4}$$

Thus, each C_i has exactly one element in $\{1, \ldots, d\}$ and one element in $\{n-d+1, \ldots, n\}$.

Suppose any of $C_1, \ldots, C_d$ has length greater than 2. Without loss of generality, assume $C_1 = (i_1, \ldots, i_p)$ has length $p > 2$, with $d+1 \leq i_p \leq n-d$. By symmetry, let us first assume that $d+1 \leq i_p \leq m$. For the case for $m < i_p \leq n-d$, one can obtain the same conclusion by a similar argument. Let i_ℓ be the only element of C_1 that lies in $\{n-d+1, \ldots, n\}$. Then one can see that

$$|i_t - i_p| \leq m \quad \text{for} \quad t = 1, \ldots, p,\ t \neq \ell.$$

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Since $C_1 \in L_m$, we have $i_l - i_p > m$, i.e., $i_l > i_p + m \geq d + 1 + m = n - d + 1$. By (2.4), there is another cycle, say C_2, with $v_2 = n - d + 1$ such that $|v_2 - i_p| \leq m$. Let $C_2 = (j_1, \ldots, j_q)$. Then

$$|j_t - i_p| \leq m \quad \text{for} \quad t = 1, \ldots, q.$$

We assume that $j_q = v_2$. Then C_1C_2 can be written as

$$C_1C_2 = (i_p, i_{p-1}) \cdots (i_p, i_{\ell+1})(i_p, j_q)(j_q, i_\ell)(i_p, j_{q-1}) \cdots (i_p, j_1)(i_p, j_q)(i_p, i_{\ell-1}) \cdots (i_p, i_1),$$

which is a product of $p + q$ transpositions in G_m. By Lemma 2.4, $C_3 \cdots C_{d+1}$ can be written as a product of $n - (p + q) + (d - 3)$ transpositions in G_m. Thus, σ is a product of $n+d-3$ transpositions, which contradicts that σ attains the upper bound. Therefore, all C_i have length 2 and the case (a) holds by (2.4).

Now if $n - m$ is odd, then $2d = m - n - 1$. By (2.3), we have either

$$\text{(i)} \ \max_{1\leq i\leq d} u_i = d \quad \text{and} \quad \min_{1\leq j\leq d} v_j \geq n - d$$

or

$$\text{(ii)} \ \max_{1\leq i\leq d} u_i \leq d + 1 \quad \text{and} \quad \min_{1\leq j\leq d} v_j = n - d + 1.$$

Then either

$$\text{(i)} \quad \{u_1, \ldots, u_d\} = \{1, \ldots, d\} \quad \text{and} \quad \{v_1, \ldots, v_d\} \subseteq \{n - d, \ldots, n\} \tag{2.5}$$

or

$$\text{(ii)} \quad \{u_1, \ldots, u_d\} \subseteq \{1, \ldots, d + 1\} \quad \text{and} \quad \{v_1, \ldots, v_d\} = \{n - d + 1, \ldots, n\}. \tag{2.6}$$

Suppose any of C_i, $1 \leq i \leq d$, contains an element in $\{d + 2, \ldots, n - d - 1\}$. Without loss of generality, assume $C_1 = (i_1, \ldots, i_p)$ is the cycle and $d+2 \leq i_p \leq m$. Then there is another length q cycle, say C_2, such that $|v_2 - i_p| \leq m$. Following the same above argument, we conclude that C_1C_2 is a product of $p + q$ transpositions in G_m. By a similar argument as above, one can conclude that this contradicts our assumption. Therefore, the elements of all C_i, $1 \leq i \leq d$, lie in $\{1, \ldots, d + 1\} \cup \{n - d, \ldots, n\}$. Furthermore, at most two cycles have length greater than 2.

Case I. Suppose all these cycles have length 2. Then by (2.5) and (2.6), the case (b.1) follows.

Case II. Suppose all cycles $C_1, \ldots, C_d$ have length at most 3. Since all these cycles are in L_m, each of them can contain at most one of $d+1$ or $n-d$ but not both. We claim that exactly one of $d+1$ or $n-d$ does not lie in any of cycles. Suppose not, that is, there are two cycles and each of them contains $d + 1$ and $n - d$ respectively.

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If both of two these cycles are of length 2, say $C_1 = (i_1, i_2)$ and $C_2 = (j_1, j_2)$ with $j_1 < i_1 = d + 1 < j_2 = n - d < i_2$. Then $C_1C_2 = (i_1, j_2)(j_1, i_1)(j_2, i_2)(i_1, j_2)$ is a product of 4 transpositions in G_m. By Lemma 2.4, σ is a product of $n + d - 3$ transpositions in G_m, a contradiction. Now if one of these two cycles has length 3 while another has length 2, say $C_1 = (i_1, i_2)$ with $i_1 = d + 1$ and $i_2 \geq n - d + 1$ and $C_2 = (j_1, j_2, j_3)$ with $j_2 = n - d$. Then by (2.6), either $j_1 \leq d < j_2 < j_3$ or $j_3 \leq d < j_2 < j_1$. Then

$$C_1C_2 = \begin{cases} (j_2, j_3)(i_1, j_2)(j_1, i_1)(j_2, i_2)(i_1, j_2) & \text{if } j_1 < j_2 < j_3, \\ (i_1, j_2)(j_3, i_1)(j_2, i_2)(i_1, j_2)(j_2, i_1) & \text{if } j_3 < j_2 < j_1. \end{cases}$$

In both cases, C_1C_2 can be written as a product of 5 transpositions in G_m, and hence, by Lemma 2.4, σ is a product of $n+d-3$ transpositions in G_m, a contradiction. Similar argument can show that it is impossible to have a length 2 cycle containing $n - d$ while another length 3 cycle containing $d + 1$. Finally if there are two length 3 cycles containing $d + 1$ and $n - d$ respectively, say $C_1 = (i_1, i_2, i_3)$ and $C_2 = (j_1, j_2, j_3)$ with $i_2 = d + 1$ and $j_2 = n - d$. By (2.5) and (2.6), we may further assume that $\{u_1, v_1\} = \{i_1, i_3\}$ and $\{u_2, v_2\} = \{j_1, j_3\}$. Then

$$C_1C_2 = \begin{cases} (i_1, i_2)(i_2, j_2)(i_2, j_1)(i_3, j_3)(i_2, j_2)(j_2, j_3) & \text{if } i_1 < i_2 < i_3 \text{ and } j_1 < j_2 < j_3, \\ (i_1, i_2)(j_1, j_2)(i_2, j_2)(i_2, j_3)(i_3, j_2)(i_2, j_2) & \text{if } i_1 < i_2 < i_3 \text{ and } j_3 < j_2 < j_1, \\ (i_2, j_3)(i_2, j_1)(i_1, j_2)(i_2, j_2)(i_2, i_3)(j_2, j_3) & \text{if } i_3 < i_2 < i_1 \text{ and } j_1 < j_2 < j_3, \\ (j_1, j_2)(i_2, j_2)(i_1, j_2)(i_3, j_3)(i_2, j_2)(i_3, j_3) & \text{if } i_3 < i_2 < i_1 \text{ and } j_3 < j_2 < j_1. \end{cases}$$

In all cases, C_1C_2 is a product of 6 transpositions in G_m, and hence, σ is a product of $n + d - 3$ transpositions in G_m, which contradicts our assumption. Therefore, we conclude that one and only one of $d + 1$ and $n - d$ does not lie in any of $C_1, \ldots, C_d$. Hence, exactly one of the cycles has length 3 and all other cycles have length 2. Then (b.2) holds if one of the cycles contains $d + 1$ and (b.3) holds otherwise.

Case III. Suppose exactly one cycle has length 4. Then all other cycles have length 2. By (2.5) and (2.6), the length 4 cycle must contain at least one of $d + 1$ and $n - d$. Suppose the cycle $C_1 = (i_1, i_2, i_3, i_4)$ contains only one of them, say $n - d$. Let $i_1 = n - d$. Then there is another length 2 cycle, say $C_2 = (j_1, j_2)$ with $j_1 = d + 1$ and $j_2 \geq n - d + 1$. If $|i_4 - i_3| \leq m$, then $(i_3, i_4)C_1 = (i_1, i_2, i_3)$ is a length 3 cycle containing $n - d$. Then $(i_3, i_4)C_1$ and C_2 satisfy the condition (2.1). By Lemma 2.4, $(i_3, i_4)C_1C_2$ can be written as a product of 5 transpositions in G_m and so C_1C_2 is a product of 6 transpositions. Applying Lemma 2.4 again one can conclude that σ is a product of $n + d - 3$ transpositions in G_m, a contradiction. Therefore, $|i_4 - i_3| > m$. Similarly, one can show that the other two absolute values $|i_3 - i_2|$ and $|i_2 - i_1|$ are strictly greater than m. Then one must have $i_2, j_4 \in \{1, \ldots, d\}$ and $i_3 \in \{n - d + 1, \ldots, n\}$. Thus, (b.4) follows. By the similar argument, the result holds

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if C_1 contains only $d+1$ but not $n-d$. Finally, suppose C_1 contains both $d+1$ and $n-d$. If $\{i_1, i_2\} = \{d+1, n-d\}$, then $(i_3, i_4)C_1 = (i_1, i_2, i_3)$ contains both $d+1$ and $n-d$ and so it is not in L_m. By Lemma 2.4, it is a product of 2 transpositions in G_m and so C_1 is the product of 3 transpositions in G_m. Thus, σ is a product of $n+d-2$ transpositions in G_m, a contradiction. So $\{i_1, i_2\}$ contain at most one of $d+1$ and $n-d$. The same observation holds for $\{i_2, i_3\}$, $\{i_3, i_4\}$ and $\{i_1, i_4\}$. It follows that either $\{i_1, i_3\} = \{d+1, n-d\}$ or $\{i_2, i_4\} = \{d+1, n-d\}$. Thus, (b.4) holds.

It remains to show that all the permutations mentioned in (a) and (b) attain the upper bound. Suppose $n-m$ is even and σ has the required form in (a). Let $C_t = (i_t, j_t)$ for $t = 1, \ldots, d$ and $C_{d+1} = (k_1, \ldots, k_{n-2d})$. Suppose $d(\mathbf{1}, \sigma, m) = s$ and $\tau_1, \ldots, \tau_s$ are transpositions in G_m such that

$$\tau_1 \cdots \tau_s \sigma = 1.$$

Assume that $\tau_v \in K_2(\tau_{v+1} \cdots \tau_s \sigma)$ for $v = r_1, \ldots, r_q$ with $1 \le r_1 < \cdots < r_q \le s$ and $\tau_v \in K_1(\tau_{v+1} \cdots \tau_s \sigma)$ for other v's. Since $C_1, \ldots, C_d \notin G_m$ and

$$\min_{1 \le t \le d} j_t - \max_{1 \le t \le d} i_t = (n-d+1) - d > m,$$

one needs at least one term in $\{d+1 \ldots, n-d\}$ and d distinct type two transpositions to move elements in $C_1, \ldots, C_d$ back to their natural positions. Thus, we have $q \ge d$. Notice that

$$d(\mathbf{1}, \tau_{r_1} \cdots \tau_s \sigma, m) = r_1 - 1.$$

On the other hand, by Lemma 2.1, the number of disjoint cycles in $\tau_{r_1} \cdots \tau_s \sigma$ is

$$d + 1 + (s - r_1 + 1 - q) - q = d - 2q + s - r_1 + 2,$$

which implies

$$d(\mathbf{1}, \tau_{r_1} \cdots \tau_s \sigma, n-1) = n - (d - 2q + s - r_1 + 2).$$

It follows that

$$\begin{aligned} r_1 - 1 &= d(\mathbf{1}, \tau_{r_1} \cdots \tau_s \sigma, m) \\ &\ge d(\mathbf{1}, \tau_{r_1} \cdots \tau_s \sigma, n-1) \\ &= n - (d - 2q + s - r_1 + 2) \\ &\ge n - (-d + s - r_1 + 2) \end{aligned}$$

which ensures $d(\mathbf{1}, \sigma, m) = s \ge n + d - 1$.

When $n-m$ is odd and σ has the required form in (b.1), (b.2) or (b.3), one can use the same above argument to deduce that σ attains the upper bound.

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Now suppose σ has the required form in (b.4). Denote by $C_t = (i_t, j_t)$ for $t = 1, \ldots, d-1$, $C_d = (i_d, j_d, i_{d+1}, j_{d+1})$ and $C_{d+1} = (k_1, \ldots, k_{n-2d-2})$. Again, suppose $d(\mathbf{1}, \sigma, m) = s$ and $\tau_1, \ldots, \tau_s$ are transpositions in G_m such that

$$\tau_1 \cdots \tau_s \sigma = 1. \tag{2.7}$$

Note that $\{i_1, \ldots, i_{d+1}\} = \{1, \ldots, d+1\}$, $\{j_1, \ldots, j_{d+1}\} = \{n-d, \ldots, n\}$. There are at least $d-1$ type two transpositions in $\tau \equiv \{\tau_1, \ldots, \tau_s\}$. If there are d type two transpositions in τ, then the same argument for (a) works for (b.4).

Next we assume there are exactly $d-1$ type two transpositions in τ. Then $C_1, \ldots, C_d$ can be reordered as $C_{p_1}, \ldots, C_{p_v}, C_{p_{v+1}}, \ldots, C_{p_d}$ with $v \geq 1$ such that there is no transposition $(u, w) \in \tau$ with

$$u \in V(C_{p_1}, \ldots, C_{p_v}), w \in V(C_{p_{v+1}}, \ldots, C_{p_d}, C_{d+1}),$$

where $V(C_{p_1}, \ldots, C_{p_v})$ denotes the set of elements in cycles $C_{p_1}, \ldots, C_{p_v}$. Since

$$V(C_{p_1}, \ldots, C_{p_v}) \cap V(C_{p_{v+1}}, \ldots, C_{p_d}, C_{d+1}) = \emptyset,$$

by (2.7), we can assume

$$\tau_1 \cdots \tau_u C_{p_1} \cdots C_{p_v} = \mathbf{1}$$

and

$$\tau_{u+1} \cdots \tau_s C_{p_{v+1}} \cdots C_{p_d} C_{d+1} = \mathbf{1} \tag{2.8}$$

for some u. Notice that to move elements in $V(C_{p_1}, \ldots, C_{p_v})$ back to their natural positions, we must use the transposition $(d+1, n-d)$. Since

$$\{d+1, n-d\} \cap \{i_d, j_d, i_{d+1}, j_{d+1}\} \neq \emptyset,$$

we have $C_d \in \{C_{p_1}, \ldots, C_{p_v}\}$ and $V(C_{p_1}, \ldots, C_{p_v})$ contains $2(v+1)$ elements. Moreover, among the transpositions $\tau_1, \ldots, \tau_u$, there are at least $v+1$ transpositions with form $(d+1, n-d)$, v transpositions with form $(i, d+1)$, $1 \leq i \leq d$, and v transpositions with form $(j, n-d)$, $n-d+1 \leq j \leq n$. Hence,

$$u \geq 3v + 1. \tag{2.9}$$

Suppose there are q type two transpositions in $\tau_{u+1}, \ldots, \tau_s$. By (2.8), $q \geq d - v$. Assume the first type two transpositions among them is τ_r. Denote by $\alpha = \tau_r \cdots \tau_s C_{p_{v+1}} \cdots C_{p_d} C_{d+1}$. Then

$$d(\mathbf{1}, \alpha, m) = r - 1 - u.$$

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On the other hand, the number of disjoint cycles in α is

$$d + 1 - v + (s - r + 1 - q) - q = d - 2q - r + s - v + 2.$$

Therefore,

$$d(\mathbf{1}, \alpha, n-1) = n - 2v - 2 - (d - 2q - r + s - v + 2).$$

It follows that

$$r - 1 - u = d(\mathbf{1}, \alpha, m) \geq d(\mathbf{1}, \alpha, n-1) = n - 2v - 2 - (d - 2q - r + s - v + 2).$$

Hence,

$$s - u \geq n + 2q - d - v - 3 \geq n + d - 3v - 3.$$

By (2.9), we have

$$d(\mathbf{1}, \sigma, m) = s \geq n + d - 2.$$

On the other hand, by the former arguments, σ can be written as a product of $n+d-1$ transpositions from G_m. By even and odd permutation rule, we have

$$d(\mathbf{1}, \sigma, m) \geq n + d - 1. \quad \square$$

2.2. The case when $1 < m < (n-1)/2$. Since $m \geq 2$, we need only to discuss the case when $n \geq 6$. We are not able to determine the exact value of $\delta(n, m)$ for these cases. Nevertheless, we have the following upper bounds.

Proposition 2.7. *Let n, m be integers with $n \geq 6$ and $1 < m < \frac{n-1}{2}$. Then*

$$\delta(n, m) \leq \left\lceil \frac{n-1}{m} \right\rceil + \delta(n-1, m).$$

Proof. Let $\sigma \in S_n$ with $\sigma(i) = n$. Suppose $n - i = qm + r$ with $0 < r \leq m$. Take the transpositions

$$(i, i+m), (i+m, i+2m), \ldots, (i + (q-1)m, i + km), (i + qm, n).$$

Thus, we move n to the last position by $q + 1 = \lceil \frac{n-i}{m} \rceil$ transpositions and get a new permutation $\sigma_1 = [j_1, \ldots, j_{n-1}, n]$. Note that $\tau \equiv [j_1, \ldots, j_{n-1}]$ is a permutation in S_{n-1}. We have

$$\begin{aligned} d(\mathbf{1}, \sigma, m) &\leq \left\lceil \frac{n-i}{m} \right\rceil + d(\mathbf{1}, \sigma_1, m) = \left\lceil \frac{n-i}{m} \right\rceil + d(\mathbf{1}, \tau, m) \\ &\leq \left\lceil \frac{n-i}{m} \right\rceil + \delta(n-1, m) \leq \left\lceil \frac{n-1}{m} \right\rceil + \delta(n-1, m). \quad \square \end{aligned}$$

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Proposition 2.8. *Let n, m be integers with $n \geq 6$ and $1 < m < (n-1)/2$. Then*

$$\delta(n,m) \leq 2\left\lceil \frac{n-1}{m} \right\rceil - 1 + \delta(n-2,m).$$

Proof. Let $\sigma \in S_n$ with $\sigma(i) = n$ and $\sigma(j) = 1$. It suffices to verify that we can move 1 and n back to their positions in $2\left\lceil \frac{n-1}{m} \right\rceil - 1$ steps and get a new permutation $[1, i_2, \ldots, i_{n-1}, n]$. Since $[i_2, \ldots, i_{n-1}]$ is a permutation in S_{n-2}, we can fix it in at most $\delta(n-2,m)$ steps. Therefore

$$\delta(n,m) = \max_{\sigma \in S_n} d(\mathbf{1}, \sigma, m) \leq 2\left\lceil \frac{n-1}{m} \right\rceil - 1 + \delta(n-2,m).$$

Suppose $n-1 = mq+r$ with q, r be integers and $0 < r \leq m$. Since $m < (n-1)/2$, we have $q \geq 2$. Let s and t be positive integers such that

$$(s-1)m + 1 \leq i < sm + 1 \quad \text{and} \quad tm + 1 < j \leq (t+1)m + 1.$$

Notice that $|t-s| \leq q-1$. Suppose $s \geq t$. Then we can use the following transpositions to move 1 and n back to the first position and the last position, respectively

$$(i, sm+1),\ (sm+1, (s+1)m+1),\ \ldots,\ ((q-1)m+1, qm+1),\ (qm+1, n),$$
$$(j, tm+1),\ (tm+1, (t-1)m+1),\ \ldots,\ (2m+1, m+1),\ (m+1, 1).$$

The number of these transpositions is

$$(q-s+2) + (t+1) = q + 3 + t - s \leq q + 3 \leq 2q + 1 = 2\left\lceil \frac{n-1}{m} \right\rceil - 1.$$

Suppose $s < t$. Then we can use the following transpositions to move 1 and n back to the first position and the last position, respectively

$$(j, tm{+}1),\ (i, sm{+}1),\ (sm{+}1, (s{+}1)m{+}1),\ \ldots,\ ((q{-}1)m{+}1, qm{+}1),\ (qm{+}1, n),$$
$$((t-1)m+1, (t-2)m+1),\ \ldots,\ (2m+1, m+1),\ (m+1, 1).$$

The number of these transpositions is

$$(q-s+3) + (t-1) = q + 2 + t - s \leq q + 2 + (q-1) = 2q + 1 = 2\left\lceil \frac{n-1}{m} \right\rceil - 1. \quad \square$$

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2.3. Results on $\delta(n,m)$ for small n. By the results in the previous sections and numerical computation, we have the following table for $\delta(n,m)$.

$n\backslash m$	1	2	3	4	5	6	7	8	9	10	11
2	1										
3	3	2									
4	6	4	3								
5	10	5	5	4							
6	15	[7]	6	6	5						
7	21	[10]	8	7	7	6					
8	28	[14]	[10]	9	8	8	7				
9	36	[16]	[11]	10	10	9	9	8			
10	45	[19]	[14]	[12]	11	11	10	10	9		
11	55	[23]	[16]	[14]	13	12	12	11	11	10	
12	66	29*	20*	17*	16*	14	13	13	12	12	11

Table 1

Here, the number marked with "$*$" are upper bounds for $\delta(n,m)$. Note that the upper bounds of the numbers in the table are obtained by Propositions 2.7 and 2.8. For example, by Proposition 2.7, $\delta(12,2) \le 29$, $\delta(12,3) \le 20$; by Propositions 2.7 and also 2.8, $\delta(12,4) \le 16$.

The values in square bracket $[\cdot]$ were computed by a Java program written by the third author. The program uses breadth first search to generate permutations from $[1,\ldots,n] \in S_n$ using elements in G_m, and identifies the minimum number of steps needed to generate all permutations in S_n, and also the permutations require the maximum number of steps. One can download the program source code from the link "https://github.com/sharonli/permutation". When the command "Permutation" is executed, one will be asked to input n and m. One will also be asked whether the program should show all the permutations generated in each step. If one says no, only the permutations generated in the final step will be displayed.

Because of memory limitations, this program can handle the problem up to $n = 11$. It is easy to modify the program to determine the diameter of Cayley graphs with vertices connected by other sets of permutations. Some numerical results obtained by the program are shown in the Appendix of the paper.

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Appendix: Some numerical results.

1. The same 4 permutations attain $\delta(7,2) = 10$ and $\delta(7,3) = 8$.
   [6, 7, 4, 5, 2, 3, 1], [6, 7, 4, 5, 3, 1, 2], [6, 7, 5, 3, 4, 1, 2], [7, 5, 6, 3, 4, 1, 2].

2. The same unique permutation attains $\delta(8,2) = 14$ and $\delta(8,3) = 10$, namely, [8, 7, 6, 5, 4, 3, 2, 1].

3. There is a unique permutation attaining $\delta(9,2) = 16$, namely, [9, 8, 7, 6, 5, 4, 3, 2, 1].

4. There are 770 permutations attaining $\delta(9,3) = 11$.

5. There are 39 permutations attaining $\delta(10,2) = 19$.
   [10, 9, 8, 7, 6, 5, 4, 3, 2, 1], [9, 10, 8, 7, 5, 6, 3, 4, 1, 2], [10, 9, 8, 6, 7, 4, 5, 3, 2, 1],
   [10, 9, 8, 7, 6, 4, 5, 2, 3, 1], [9, 10, 8, 7, 6, 5, 3, 4, 2, 1], [10, 9, 8, 7, 6, 5, 2, 4, 3, 1],
   [9, 10, 7, 8, 5, 6, 3, 4, 2, 1], [10, 9, 7, 8, 5, 6, 4, 3, 2, 1], [10, 9, 8, 7, 4, 6, 5, 3, 2, 1],
   [10, 8, 9, 6, 7, 4, 5, 2, 3, 1], [10, 9, 6, 8, 7, 5, 4, 3, 2, 1], [10, 9, 8, 7, 5, 4, 6, 3, 2, 1],
   [10, 9, 7, 8, 5, 6, 3, 4, 1, 2], [10, 9, 8, 7, 6, 5, 4, 1, 3, 2], [10, 9, 8, 6, 7, 5, 4, 2, 3, 1],
   [10, 8, 9, 7, 6, 4, 5, 3, 2, 1], [10, 9, 8, 7, 5, 6, 3, 4, 2, 1], [9, 10, 7, 8, 6, 5, 4, 3, 2, 1],
   [9, 10, 8, 7, 6, 5, 4, 3, 1, 2], [10, 8, 7, 9, 6, 5, 4, 3, 2, 1], [10, 9, 8, 7, 6, 3, 5, 4, 2, 1],
   [9, 8, 10, 7, 6, 5, 4, 3, 2, 1], [9, 10, 7, 8, 5, 6, 4, 3, 1, 2], [10, 9, 8, 7, 6, 5, 4, 2, 1, 3],
   [8, 10, 9, 7, 6, 5, 4, 3, 2, 1], [10, 9, 8, 7, 6, 5, 3, 4, 1, 2], [10, 9, 8, 7, 5, 6, 4, 3, 1, 2],
   [10, 9, 7, 8, 6, 5, 3, 4, 2, 1], [10, 8, 9, 7, 6, 5, 4, 2, 3, 1], [10, 9, 7, 6, 8, 5, 4, 3, 2, 1],
   [10, 8, 9, 6, 7, 5, 4, 3, 2, 1], [10, 7, 9, 8, 6, 5, 4, 3, 2, 1], [10, 9, 7, 8, 6, 5, 4, 3, 1, 2],
   [10, 9, 8, 5, 7, 6, 4, 3, 2, 1], [10, 9, 8, 7, 6, 5, 3, 2, 4, 1], [10, 9, 8, 6, 5, 7, 4, 3, 2, 1],
   [9, 10, 7, 8, 6, 5, 3, 4, 1, 2], [9, 10, 8, 7, 5, 6, 4, 3, 2, 1], [10, 9, 8, 7, 6, 4, 3, 5, 2, 1].

6. There are 8 permutations attaining $\delta(10,3) = 14$.
   [9, 10, 7, 8, 6, 5, 4, 3, 1, 2], [10, 9, 7, 8, 6, 5, 4, 3, 2, 1], [10, 9, 8, 7, 6, 5, 3, 4, 2, 1],
   [9, 10, 8, 7, 6, 5, 3, 4, 1, 2], [10, 9, 7, 8, 6, 4, 5, 1, 2, 3], [8, 9, 10, 5, 6, 7, 3, 4, 2, 1],
   [8, 9, 10, 6, 7, 5, 3, 4, 2, 1], [10, 9, 7, 8, 4, 5, 6, 1, 2, 3].

7. There are 38 permutations attaining $\delta(10,4) = 12$.
   [8, 9, 10, 6, 7, 5, 4, 1, 2, 3], [8, 9, 10, 7, 4, 5, 6, 1, 2, 3], [10, 7, 8, 9, 6, 5, 4, 3, 2, 1],
   [9, 8, 10, 6, 7, 5, 4, 2, 1, 3], [8, 10, 9, 7, 4, 5, 6, 1, 3, 2], [10, 8, 9, 5, 7, 4, 6, 2, 3, 1],
   [8, 10, 9, 6, 4, 7, 5, 1, 3, 2], [10, 9, 8, 7, 6, 4, 5, 3, 2, 1], [8, 9, 10, 6, 4, 7, 5, 1, 2, 3],
   [8, 10, 9, 5, 7, 4, 6, 1, 3, 2], [8, 9, 10, 5, 7, 4, 6, 1, 2, 3], [8, 10, 9, 7, 6, 4, 5, 1, 3, 2],
   [10, 8, 9, 6, 7, 5, 4, 2, 3, 1], [10, 8, 9, 7, 4, 5, 6, 2, 3, 1], [10, 8, 9, 6, 4, 7, 5, 2, 3, 1],
   [9, 8, 10, 5, 6, 7, 4, 2, 1, 3], [8, 10, 9, 5, 6, 7, 4, 1, 3, 2], [9, 10, 8, 6, 4, 7, 5, 3, 1, 2],
   [9, 10, 8, 6, 7, 5, 4, 3, 1, 2], [9, 8, 10, 7, 6, 4, 5, 2, 1, 3], [10, 9, 8, 5, 6, 7, 4, 3, 2, 1],
   [10, 9, 8, 5, 7, 4, 6, 3, 2, 1], [10, 9, 8, 7, 6, 5, 2, 3, 4, 1], [9, 8, 10, 6, 4, 7, 5, 2, 1, 3],

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[8, 9, 10, 7, 6, 4, 5, 1, 2, 3], [8, 9, 10, 5, 6, 7, 4, 1, 2, 3], [9, 8, 10, 5, 7, 4, 6, 2, 1, 3],
[9, 10, 8, 7, 4, 5, 6, 3, 1, 2], [10, 8, 9, 7, 6, 4, 5, 2, 3, 1], [9, 10, 8, 5, 6, 7, 4, 3, 1, 2],
[9, 10, 8, 7, 6, 4, 5, 3, 1, 2], [8, 10, 9, 6, 7, 5, 4, 1, 3, 2], [9, 8, 10, 7, 4, 5, 6, 2, 1, 3],
[10, 8, 9, 5, 6, 7, 4, 2, 3, 1], [9, 10, 8, 5, 7, 4, 6, 3, 1, 2], [10, 9, 8, 7, 4, 5, 6, 3, 2, 1],
[10, 9, 8, 6, 4, 7, 5, 3, 2, 1], [10, 9, 8, 6, 7, 5, 4, 3, 2, 1].

8. There are 19 permutations attaining $\delta(11, 2) = 23$.
[11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1], [11, 10, 9, 8, 7, 6, 5, 2, 4, 3, 1], [11, 10, 9, 7, 6, 8, 5, 4, 3, 2, 1],
[11, 10, 9, 8, 7, 6, 5, 4, 2, 1, 3], [11, 10, 7, 9, 8, 6, 5, 4, 3, 2, 1], [9, 11, 10, 8, 7, 6, 5, 4, 3, 2, 1],
[11, 10, 9, 8, 7, 6, 5, 3, 2, 4, 1], [11, 10, 9, 6, 8, 7, 5, 4, 3, 2, 1], [11, 8, 10, 9, 7, 6, 5, 4, 3, 2, 1],
[11, 9, 8, 10, 7, 6, 5, 4, 3, 2, 1], [11, 10, 9, 8, 7, 6, 5, 4, 1, 3, 2], [11, 10, 8, 7, 9, 6, 5, 4, 3, 2, 1],
[11, 10, 9, 8, 7, 4, 6, 5, 3, 2, 1], [11, 10, 9, 8, 7, 5, 4, 6, 3, 2, 1], [10, 9, 11, 8, 7, 6, 5, 4, 3, 2, 1],
[11, 10, 9, 8, 6, 5, 7, 4, 3, 2, 1], [11, 10, 9, 8, 7, 6, 3, 5, 4, 2, 1], [11, 10, 9, 8, 5, 7, 6, 4, 3, 2, 1],
[11, 10, 9, 8, 7, 6, 4, 3, 5, 2, 1].

9. There are 170 permutations attaining $\delta(11, 3) = 16$.
[8, 10, 9, 11, 6, 7, 5, 4, 2, 3, 1], [10, 11, 8, 9, 6, 7, 5, 4, 3, 2, 1], [11, 10, 9, 5, 6, 7, 8, 4, 2, 3, 1],
[11, 9, 7, 8, 6, 10, 2, 4, 5, 3, 1], [11, 7, 10, 8, 9, 6, 5, 4, 2, 3, 1], [11, 9, 10, 8, 6, 7, 3, 5, 4, 1, 2],
[9, 11, 10, 8, 4, 5, 6, 7, 2, 3, 1], [11, 9, 10, 8, 7, 5, 6, 2, 3, 4, 1], [11, 10, 9, 8, 7, 6, 5, 4, 2, 3, 1],
[11, 10, 9, 8, 6, 5, 7, 4, 2, 3, 1], [11, 8, 9, 10, 6, 7, 5, 3, 4, 2, 1], [11, 9, 10, 7, 8, 5, 6, 4, 3, 2, 1],
[11, 9, 10, 8, 7, 5, 6, 4, 2, 3, 1], [11, 9, 10, 8, 7, 6, 4, 5, 2, 3, 1], [8, 9, 10, 11, 7, 5, 6, 1, 2, 3, 4],
[11, 9, 8, 10, 6, 7, 5, 2, 4, 3, 1], [11, 10, 9, 8, 6, 7, 2, 4, 5, 3, 1], [11, 9, 10, 8, 7, 6, 5, 2, 4, 3, 1],
[11, 9, 10, 8, 7, 5, 6, 1, 3, 2, 4], [11, 9, 10, 8, 6, 5, 7, 2, 4, 3, 1], [11, 10, 9, 8, 7, 5, 6, 3, 4, 1, 2],
[11, 9, 8, 10, 6, 7, 5, 4, 3, 2, 1], [11, 10, 8, 9, 7, 5, 6, 3, 4, 2, 1], [11, 9, 10, 8, 6, 7, 2, 4, 3, 5, 1],
[11, 10, 8, 9, 6, 7, 4, 5, 3, 2, 1], [11, 9, 10, 5, 6, 7, 8, 4, 3, 2, 1], [11, 10, 5, 8, 9, 7, 6, 4, 3, 2, 1],
[11, 8, 10, 9, 7, 5, 6, 3, 2, 4, 1], [11, 9, 7, 8, 10, 2, 6, 4, 5, 3, 1], [10, 11, 7, 8, 9, 5, 6, 3, 4, 1, 2],
[11, 9, 10, 8, 7, 6, 5, 4, 3, 2, 1], [11, 10, 7, 8, 6, 5, 9, 4, 3, 2, 1], [9, 10, 11, 8, 6, 7, 5, 4, 2, 3, 1],
[11, 9, 10, 8, 7, 5, 6, 3, 2, 1, 4], [11, 9, 10, 8, 6, 5, 7, 4, 3, 2, 1], [11, 8, 7, 9, 10, 5, 6, 4, 2, 3, 1],
[11, 10, 9, 8, 3, 5, 6, 7, 4, 2, 1], [11, 9, 10, 7, 6, 5, 8, 4, 2, 3, 1], [11, 10, 7, 8, 9, 5, 6, 4, 2, 3, 1],
[11, 10, 9, 8, 6, 7, 5, 1, 2, 3, 4], [11, 9, 10, 8, 7, 2, 6, 3, 5, 4, 1], [11, 6, 8, 10, 9, 7, 5, 4, 2, 3, 1],
[11, 9, 7, 8, 10, 5, 6, 3, 2, 4, 1], [11, 9, 10, 8, 6, 7, 4, 3, 5, 1, 2], [11, 9, 10, 8, 6, 7, 5, 2, 3, 4, 1],
[11, 10, 5, 8, 6, 7, 9, 3, 4, 2, 1], [11, 7, 10, 8, 9, 5, 6, 2, 4, 3, 1], [11, 7, 9, 8, 10, 5, 6, 4, 2, 3, 1],
[11, 9, 10, 8, 6, 5, 4, 7, 2, 3, 1], [11, 10, 9, 8, 5, 7, 6, 4, 2, 3, 1], [11, 10, 6, 8, 9, 7, 5, 3, 4, 2, 1],
[9, 11, 10, 8, 7, 5, 6, 4, 1, 3, 2], [11, 10, 9, 8, 7, 5, 6, 2, 4, 3, 1], [11, 8, 9, 10, 7, 5, 6, 4, 2, 3, 1],
[11, 10, 9, 8, 6, 7, 3, 4, 2, 5, 1], [10, 11, 9, 8, 6, 7, 5, 4, 2, 3, 1], [11, 9, 10, 8, 6, 7, 5, 4, 2, 3, 1],
[8, 9, 10, 11, 6, 7, 5, 1, 2, 3, 4], [11, 7, 10, 8, 9, 5, 6, 4, 3, 2, 1], [11, 10, 9, 8, 7, 5, 6, 4, 3, 2, 1],
[11, 10, 7, 8, 9, 6, 5, 4, 3, 2, 1], [11, 10, 8, 9, 7, 3, 6, 4, 5, 2, 1], [10, 11, 7, 9, 8, 5, 6, 4, 2, 3, 1],
[11, 9, 10, 8, 7, 5, 6, 4, 1, 2, 3], [11, 10, 8, 9, 4, 5, 6, 7, 3, 2, 1], [11, 9, 10, 8, 5, 7, 6, 2, 4, 3, 1],
[11, 10, 8, 9, 6, 7, 5, 3, 4, 2, 1], [11, 9, 10, 8, 6, 7, 3, 2, 4, 5, 1], [11, 8, 10, 9, 6, 7, 5, 3, 2, 4, 1],
[11, 10, 8, 9, 6, 7, 3, 4, 5, 2, 1], [8, 9, 10, 11, 7, 5, 6, 4, 3, 2, 1], [11, 10, 9, 8, 6, 5, 3, 4, 7, 2, 1],
[11, 10, 8, 5, 6, 7, 9, 4, 3, 2, 1], [11, 9, 10, 8, 5, 7, 6, 4, 3, 2, 1], [11, 9, 8, 10, 6, 7, 5, 4, 2, 1, 3],

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[10, 9, 11, 8, 7, 5, 6, 4, 2, 1, 3], [11, 10, 9, 8, 6, 7, 4, 5, 2, 3, 1], [11, 10, 9, 7, 8, 5, 6, 3, 4, 2, 1], [11, 8, 10, 9, 7, 6, 5, 4, 2, 3, 1], [11, 9, 10, 5, 6, 7, 8, 4, 2, 1, 3], [11, 8, 10, 9, 6, 5, 7, 4, 2, 3, 1], [10, 9, 11, 8, 4, 5, 6, 7, 2, 3, 1], [11, 10, 9, 8, 6, 7, 5, 2, 4, 3, 1], [9, 11, 10, 8, 6, 7, 5, 4, 1, 3, 2], [11, 8, 9, 10, 6, 7, 5, 4, 2, 3, 1], [11, 9, 10, 8, 4, 6, 5, 7, 2, 3, 1], [11, 10, 9, 8, 6, 7, 2, 3, 4, 5, 1], [11, 9, 7, 8, 10, 6, 5, 4, 2, 3, 1], [11, 10, 9, 5, 6, 7, 8, 3, 4, 2, 1], [11, 8, 10, 9, 6, 7, 2, 4, 5, 3, 1], [11, 6, 10, 8, 9, 7, 3, 4, 2, 5, 1], [11, 10, 9, 8, 7, 6, 5, 3, 4, 2, 1], [11, 10, 9, 8, 6, 7, 5, 4, 3, 2, 1], [11, 10, 8, 9, 7, 5, 6, 2, 3, 4, 1], [10, 9, 11, 8, 7, 5, 6, 3, 2, 4, 1], [11, 8, 10, 9, 6, 7, 5, 4, 1, 3, 2], [11, 10, 9, 8, 6, 7, 3, 5, 4, 2, 1], [11, 10, 9, 8, 6, 5, 7, 3, 4, 2, 1], [11, 10, 9, 8, 7, 6, 3, 4, 5, 2, 1], [11, 10, 8, 9, 7, 5, 3, 4, 6, 2, 1], [11, 9, 10, 8, 7, 6, 5, 3, 2, 4, 1], [11, 9, 10, 8, 6, 7, 4, 5, 3, 2, 1], [11, 9, 10, 8, 7, 5, 6, 3, 4, 2, 1], [11, 9, 10, 5, 8, 7, 6, 4, 2, 3, 1], [11, 9, 8, 10, 7, 6, 5, 4, 2, 3, 1], [11, 9, 10, 8, 6, 5, 7, 3, 2, 4, 1], [11, 10, 8, 9, 7, 5, 6, 4, 2, 3, 1], [11, 10, 7, 9, 8, 5, 6, 4, 3, 2, 1], [11, 9, 8, 10, 6, 5, 7, 4, 2, 3, 1], [11, 9, 10, 5, 7, 6, 8, 4, 2, 3, 1], [11, 10, 7, 8, 6, 9, 5, 3, 4, 2, 1], [9, 11, 10, 8, 7, 5, 6, 2, 4, 3, 1], [10, 11, 8, 7, 9, 5, 6, 4, 2, 3, 1], [11, 10, 9, 8, 3, 7, 6, 4, 5, 2, 1], [11, 10, 9, 8, 4, 5, 6, 7, 2, 3, 1], [11, 10, 9, 8, 4, 5, 6, 3, 7, 2, 1], [10, 9, 11, 8, 6, 7, 5, 4, 2, 1, 3], [11, 10, 8, 9, 3, 5, 6, 4, 7, 2, 1], [11, 9, 10, 8, 6, 7, 2, 3, 5, 4, 1], [11, 8, 10, 9, 5, 7, 6, 4, 2, 3, 1], [11, 9, 8, 10, 6, 7, 2, 4, 5, 3, 1], [11, 10, 8, 9, 6, 7, 2, 4, 3, 5, 1], [11, 8, 10, 9, 6, 7, 3, 4, 2, 5, 1], [11, 10, 5, 9, 6, 7, 8, 4, 3, 2, 1], [9, 10, 11, 8, 6, 7, 5, 3, 4, 2, 1], [11, 7, 10, 8, 9, 5, 6, 3, 2, 4, 1], [11, 10, 8, 9, 7, 6, 5, 4, 3, 2, 1], [11, 9, 10, 8, 7, 6, 2, 4, 5, 3, 1], [11, 9, 10, 8, 4, 7, 6, 5, 2, 3, 1], [11, 10, 9, 8, 7, 5, 6, 3, 2, 4, 1], [11, 10, 8, 9, 6, 5, 7, 4, 3, 2, 1], [11, 10, 7, 8, 9, 5, 6, 3, 4, 2, 1], [11, 8, 10, 9, 7, 5, 6, 4, 3, 2, 1], [11, 7, 8, 10, 9, 5, 6, 4, 2, 3, 1], [11, 9, 10, 5, 6, 7, 8, 4, 1, 3, 2], [11, 7, 10, 8, 9, 5, 3, 4, 2, 6, 1], [11, 10, 9, 8, 6, 7, 3, 1, 5, 2, 4], [11, 9, 10, 7, 8, 6, 5, 4, 2, 3, 1], [11, 9, 10, 8, 7, 5, 6, 4, 3, 1, 2], [8, 11, 10, 9, 6, 7, 5, 4, 2, 3, 1], [11, 9, 7, 8, 10, 5, 6, 2, 4, 3, 1], [11, 7, 9, 8, 10, 5, 6, 3, 4, 2, 1], [11, 10, 9, 8, 6, 7, 4, 3, 5, 2, 1], [11, 10, 9, 8, 5, 7, 6, 3, 4, 2, 1], [11, 9, 10, 8, 4, 5, 6, 7, 3, 2, 1], [11, 8, 7, 9, 6, 10, 5, 4, 2, 3, 1], [10, 9, 8, 11, 6, 7, 5, 4, 2, 3, 1], [11, 9, 7, 8, 10, 5, 6, 4, 3, 2, 1], [10, 11, 8, 9, 6, 7, 3, 4, 5, 1, 2], [10, 11, 9, 8, 6, 7, 3, 4, 5, 2, 1], [11, 9, 10, 8, 6, 7, 5, 3, 4, 2, 1], [11, 9, 10, 8, 7, 5, 6, 1, 4, 3, 2], [11, 9, 10, 8, 5, 7, 6, 3, 2, 4, 1], [11, 9, 10, 8, 6, 7, 3, 4, 5, 2, 1], [11, 10, 8, 9, 6, 7, 5, 4, 2, 3, 1], [8, 10, 7, 11, 9, 5, 6, 4, 3, 2, 1], [11, 9, 8, 10, 7, 5, 6, 2, 4, 3, 1], [11, 9, 8, 10, 5, 7, 6, 4, 2, 3, 1], [11, 9, 10, 8, 7, 5, 3, 2, 4, 6, 1], [11, 9, 8, 10, 6, 7, 3, 4, 2, 5, 1], [11, 9, 10, 8, 7, 6, 3, 4, 2, 5, 1], [11, 10, 8, 9, 7, 5, 6, 4, 1, 2, 3], [11, 10, 7, 8, 9, 5, 6, 4, 3, 1, 2], [11, 9, 8, 10, 7, 5, 6, 4, 3, 2, 1], [11, 10, 9, 7, 8, 5, 6, 4, 2, 3, 1], [11, 10, 8, 7, 9, 5, 6, 4, 3, 2, 1], [11, 7, 8, 9, 10, 5, 6, 4, 3, 2, 1], [11, 10, 9, 8, 6, 7, 5, 3, 2, 4, 1], [11, 10, 8, 9, 5, 7, 6, 4, 3, 2, 1], [11, 8, 10, 9, 6, 7, 5, 4, 3, 2, 1].

10. There are 2 permutations attaining $\delta(11, 4) = 14$.
[10, 11, 8, 9, 7, 5, 6, 3, 4, 1, 2], [10, 11, 8, 9, 6, 7, 5, 3, 4, 1, 2].