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SPECTRAL PROPERTIES OF FINITE-DIMENSIONAL WAVEGUIDE SYSTEMS∗

NURHAN ÇOLAKOĞLU† AND PETER LANCASTER‡

Abstract. This is a largely expository paper in which a finite dimensional model for gyroscopic/waveguiding systems is studied. Properties of the spectrum that play an important role when computing with such models are studied. The notion of “waveguide-type” is explored in this context. The main theorem provides a form of the central result (due to Abramov) concerning the existence of real spectrum for such systems. The roles of semisimple/defective eigenvalues are discussed, as well as the roles played by eigenvalue “types” (or “Krein signatures”). The theory is illustrated with examples.

Key words. Matrix polynomial, Waveguide, Eigenfunctions.

AMS subject classifications. 15A22, 47A56, 78M10.

1. Introduction. We make some preliminary definitions concerning matrix-valued functions of a complex variable. They are consistent with the usage of references [6]–[8], for example. See also [10]. Let \( L_1, L_2 \in \mathbb{C}^{n \times n} \) and consider monic, quadratic matrix polynomials:

\[
L(\lambda) = I\lambda^2 + L_1\lambda + L_2, \quad \lambda \in \mathbb{C}.
\]

Definition 1.1. (a) \( L \) is said to be selfadjoint if \( L_1 \) and \( L_2 \) are Hermitian. (b) \( L \) is said to be gyroscopic if it is selfadjoint and \( L_1 = iG \), where \( G^T = -G \in \mathbb{R}^{n \times n} \). Thus, for \( \lambda \in \mathbb{C} \),

\[
L(\lambda) = I\lambda^2 + iG\lambda + L_2, \quad G^T = -G \in \mathbb{R}^{n \times n}, \quad L_2^* = L_2 \in \mathbb{C}^{n \times n}.
\]

More generally, the leading coefficient could be a positive definite matrix \( A \in \mathbb{R}^{n \times n} \), but we assume that the usual reduction to the identity has been applied maintaining the symmetries of the definitions (i.e., \( L(\lambda) \rightarrow A^{-1/2}L(\lambda)A^{-1/2} \)). Given the
skew-symmetry of $G$, systems of the form (1.2) may be described as \textit{gyroscopic}, but it is important to note that the coefficient $L_2$ will generally be non-real and indefinite. Indeed, $L_2$ is generally parameter dependent with the form $C - \omega^2 R$, $\omega \in \mathbb{R}$. See Section 2.6.1 of the comprehensive work by Silbergleit and Kopilevich [18], where the problems are first formulated in a Hilbert space context. In that work, and many others, propagating waves are studied - in contrast to evanescent (fading) waves, and this implies a special interest in real eigenvalues for $\mathbb{L}(\lambda)$. Indeed, a major result of the following theory is the fact that a waveguide system always has at least two real eigenvalues (see Theorem 3.1). For a general treatment of waveguides in the context of electromagnetism see [17].

We adapt the earlier treatment of Abramov [1] to our finite-dimensional setting (see also [4] and [11]). In particular, gyroscopic and waveguide systems as defined here (see Definition 2.2 and Proposition 2.3) are obtained when using finite-dimensional approximations (finite-element or finite-difference methods) for continuous systems. Truncation errors involved in such an approximation process are important, but are not our concern in this paper.

The theory is further developed in an infinite-dimensional context in [5] and [11] where special attention is paid to the distribution of the real spectrum (see Figure 4 of [5]) and, physically, to the presence of specific energy-transporting waveforms. It is our objective to study these phenomena in the context of finite dimensional problems - and hence linear algebra - which is generally necessary before computation is possible. This is the context of papers of Chugunova and Pelinovsky [3], Nicolet and Geuzaine [10], and Treyszéde and Laguerre [19], for example. As in the more general theoretical context adopted by Abramov, and by Silbergleit and Kopilevich, stability depends on the presence of real spectrum (see Section 19.1 of [18]). Eigenvalue problems similar to (1.2) appear in the theory of “photonic crystal fibres”. However, in that case, the leading coefficient is singular (see Section 9.4 of Zolla et al. [20].)

Our discussion depends heavily on the four following notions, which are generally useful in the spectral analysis of selfadjoint matrix polynomials.

\subsection{1.1. The spectrum.}

The spectrum of $L$ is defined by

$$\sigma(L) = \{ \lambda \in \mathbb{C} : \det \mathbb{L}(\lambda) = 0 \},$$

and members of $\sigma(L)$ are known as \textit{eigenvalues} of $L$. An eigenvalue is said to be semisimple if its algebraic and geometric multiplicities are equal. Because the leading coefficient of $\mathbb{L}$ is invertible the spectrum (the set of all eigenvalues) is bounded. The selfadjoint property of Definition 1.1(a) ensures that $\sigma(L)$ is symmetric with respect to the real axis. In the special case that $L_2 \in \mathbb{R}^{n \times n}$, $\sigma(L)$ has Hamiltonian symmetry,
but this is not generally the case if $L_2 \notin \mathbb{R}^{n \times n}$.

**Definition 1.2.** The system will be said to be unstable if either or both of the following conditions hold: 
(a) There is a real eigenvalue which is not semisimple.
(b) There is at least one non-real eigenvalue (and hence a conjugate pair).

Or, what is equivalent: The system is stable if and only if all eigenvalues are real and semisimple. Note that, when $L_2 \in \mathbb{R}^{n \times n}$ the spectrum has Hamiltonian symmetry and a nonzero real eigenvalue $\mu$ is always accompanied by the real eigenvalue $-\mu$. (In the theory of waveguides real eigenvalues are of special interest and are associated with “running waves”.)

1.2. **The eigencurves.** The eigencurves are defined on $\mathbb{R}$ as
$$\{\mu \in \mathbb{R} : \mu \in \sigma\{L(\lambda_0)\} \text{ for some } \lambda_0 \in \mathbb{R}\}.$$ Notice that, because $L_2(\lambda_0)$ is Hermitian when $\lambda_0 \in \mathbb{R}$ then, for a fixed $\lambda_0$, $L(\lambda_0)$ has $n$ real eigenvalues, $\mu$, (counting multiplicities). Indeed, there are real analytic eigenfunctions $\mu_1(\lambda), \ldots, \mu_n(\lambda)$ defined on $\mathbb{R}$, whose zeros are the eigenvalues of $L(\lambda)$ (see Section 12.4 of [6]). Their graphs generate $n$ smooth eigencurves. Thus, the points at which these curves meet the real axis are the eigenvalues of $L(\lambda)$. Notice that, for every eigencurve, $\mu \to -\infty$ as $\lambda_0 \to -\infty$.

When $L_2^T = L_2 \in \mathbb{R}^{n \times n}$ these curves are symmetric about the origin, but this symmetry is lost if $L_2^* = L_2 \notin \mathbb{R}^{n \times n}$ (compare Figures 3.1 and 3.2 below). See Sections 12.4, 12.5 of [8] and [10] for details, and compare with Sections 14 and 19 of [18].

The eigencurves admit direct analysis of the system without recourse to the technique of linearization in which (1.2) is transformed to a linear eigenvalue problem on $\mathbb{C}^{2n}$.

1.3. **Real eigenvalue types.**

**Definition 1.3.** Let $\lambda_0$ be a real eigenvalue (in the sense of item 1.1 above) and suppose that there are exactly $k$ eigenfunctions $\mu_1(\lambda), \ldots, \mu_k(\lambda)$ which vanish at $\lambda_0$.

Then:
(a) $\lambda_0$ has positive (resp., negative) type if $\mu_j^{(1)}(\lambda_0) > 0$, (resp., $< 0$) for $j = 1, 2, \ldots, k$.
(b) $\lambda_0$ has neutral type if $\mu_j^{(1)}(\lambda_0) = 0$ for $j = 1, 2, \ldots, k$.
(c) If there are eigenfunctions $\mu_r(\lambda), \mu_s(\lambda)$, $r \neq s$, for which $\mu_r(\lambda_0) = \mu_s(\lambda_0) = 0$, $\mu_r^{(1)}(\lambda_0)$ and $\mu_s^{(1)}(\lambda_0)$ are not both zero, and $\mu_r^{(1)}(\lambda_0)\mu_s^{(1)}(\lambda_0) \leq 0$ then $\lambda_0$ is said to have mixed type.
(d) The *sign-characteristic* of the real eigenvalue \( \lambda_0 \) is the set of integers consisting of +1’s, -1’s, or 0’s defined by:

\[
\{ \text{sgn } \mu^{(1)}_j(\lambda_0) \}_{j=1}^k.
\]

(The derivative \( \mu^{(1)}(\lambda_0) \) is associated with the notion of *group velocity* in Chapter 5 of [18]. The positive and negative “types” can be associated with a direction of motion (p. 230 of [9]). See also the discussion of [2].)

### 1.4. The numerical range.

The numerical range of \( L \) is defined by

\[
NR(L) := \{ \lambda \in \mathbb{C} : \ x^* L(\lambda)x = 0 \text{ for some nonzero } x \in \mathbb{C}^n \}.
\]

See Section 10.6 of [6] for the following facts:

1. \( \sigma(L) \subset NR(L) \).
2. Every real frontier point of \( NR(L) \) is an eigenvalue.
3. There are no real eigenvalues if and only if there are no real numbers in \( NR(L) \).
4. \( NR(L) \) is bounded.

(Notice the significance of item 3 in the context of waveguide systems where, as noted above, stable systems have only real eigenvalues.)

### 2. Systems of waveguide-type.

We first summarize the approach of Abramov [1] in the context of (1.2), i.e., finite-dimensional gyroscopic systems. Since we are particularly interested in real eigenvalues, an important role is played by the real-valued “discriminant functional” associated with the scalar quadratic equation,

\[
x^* L(\lambda)x = 0,
\]

of (1.3). Thus, for any \( x \in \mathbb{C}^n \),

\[
d(x) := (iGx,x)^2 - 4(x,x)(L_2x,x) \in \mathbb{R},
\]

is defined to be the *discriminant functional*. Then \( \{ x \in \mathbb{C}^n : \ d(x) \geq 0 \} \) contains all eigenvectors of the real spectrum of \( L \), and \( \{ x \in \mathbb{C}^n : \ d(x) < 0 \} \) contains all eigenvectors of the non-real spectrum of \( L \).

The set of \( x \) for which \( d(x) = 0 \) may be described as a “pointed cone”, \( K \subset \mathbb{C}^n \). The cone has vertex at the origin. Clearly, if \( d(x) = 0 \) then \( d(\alpha x) = 0 \) for all \( \alpha \in \mathbb{C} \). In general, this cone provides a “boundary” in \( \mathbb{C}^n \) separating a zone containing two distinct real roots for \( (L(\lambda)x,x) = x^* L(\lambda)x = 0 \) (when \( d(x) > 0 \), the “interior”) from a zone containing a non-real conjugate pair of roots (when \( d(x) < 0 \), the “exterior”).

Following Abramov we name\(^1\) the open and closed interiors of \( K \) as follows:

\[
G := \{ x \in \mathbb{C}^n : \ d(x) > 0 \} \quad \text{and} \quad G' := \{ x \in \mathbb{C}^n : x \neq 0, \ d(x) \geq 0 \}.
\]

---

\(^1\) There seems to be a misprint at this point in [1].
We say that a gyroscopic system (1.2) is **strongly stable** if 
\[ d(x) > 0 \] for all nonzero \( x \in \mathbb{C}^n \) (i.e., if \( \mathcal{G} = \mathcal{G}' = \mathbb{C}^n \setminus \{0\} \)).

Thus, if the system is strongly stable then \( \mathcal{G} = \mathbb{C}^n \setminus \{0\} \). In particular, 
\[ \sigma(L) \subset NR(L) \subset \mathbb{R}, \]
all eigenvalues are real and semisimple (as we shall see in Proposition 1.2), and the corresponding eigenvectors are inside \( K \). So the system is stable in the sense of Definition 1.2.

If the system is **not** strongly stable there is at least one \( x \neq 0 \) such that 
\[ d(x) \leq 0. \]
Consequently, there could be either a multiple real eigenvalue (when \( x \neq 0 \) and 
\[ d(x) = 0 \] so that \( x \) is on the cone \( K \)), or a conjugate pair of non-real eigenvalues (when \( d(x) < 0 \) so that \( x \) is outside \( K \)), or both.

It is important to note that in the special case \( L^* = L \in \mathbb{R}^{n \times n} \), there is Hamiltonian symmetry of the spectrum, and this implies that nonzero real eigenvalues occur in positive/negative pairs. Similarly for pairs and quadruples of non-real eigenvalues.

Recalling (2.2), the scalar quadratic formula, and Definition 1.3 we define the real-valued functionals \( p_{\pm}(x) \) on \( \mathcal{G}' \) by
\[
2p_{\pm}(x) = \frac{-iGx, x \pm \sqrt{d(x)}}{(x, x)}.
\]
(This is consistent with the notation of \([1]\).) If \( x_0 \) is an eigenvector associated with eigenvalue \( \lambda \), then \( \mathbb{L}(\lambda)x_0 = 0, x_0 \neq 0 \), so that \( x_0^*\mathbb{L}(\lambda)x_0 = 0 \), and at least one of \( p_{\pm}(x_0) \) is an eigenvalue. Furthermore, \( p_{\pm}(x_0) \in \mathbb{R} \) if \( d(x_0) \geq 0 \), i.e., if \( x_0 \in \mathcal{G}' \), the zone in which there are two (possibly coincident) real zeros for \( x^*\mathbb{L}(\lambda)x \).

Then the bounds, \( k'_- \leq k_- \leq k_+ \leq k'_+ \) are defined by
\[
k'_- := \inf_{\mathcal{G}'} p_-, \quad k_- := \inf_{\mathcal{G}} p_-, \quad k_+ := \sup_{\mathcal{G}} p_+, \quad k'_+ := \sup_{\mathcal{G}'} p_+.
\]
Following Abramov, we define systems \( \mathbb{L}(\lambda) \) of waveguide-type in terms of five hypotheses concerning these four bounds. The first hypothesis is ensured by assuming that \( \mathbb{L} \) is monic. The next two are guaranteed simply because we pose the problem on finite-dimensional space. The fourth and fifth are:
\[
\mathcal{G} \neq \emptyset \quad \text{and} \quad \mathcal{G} \neq \mathbb{C}^n \setminus \{0\},
\]
\[
-\infty < k'_- \quad \text{and} \quad k'_+ < \infty.
\]

\[ ^2 \text{As there is no damping in the system, we avoid the term “overdamped”, cf. Chapter 13 of \([8]\).} \]
The first statement of (2.5) is necessary for the existence of some real spectrum.

**Definition 2.2.** A (finite-dimensional) gyroscopic system

\[
L(\lambda) = \lambda I + iG\lambda + L_2, \quad \lambda \in \mathbb{C},
\]

(with \(G^T = -G \in \mathbb{R}^{n \times n}\) and \(L_2^* = L_2 \in \mathbb{C}^{n \times n}\)) has waveguide-type if conditions (2.5) and (2.6) are satisfied.

For our problems, posed on finite-dimensional space, the conditions (2.6) are automatically satisfied. Conditions (2.5) admit the existence of a real eigenvalue, and ensure that the system is not strongly stable (in the sense of Definition 2.1).

Thus, a system (1.2) of waveguide-type is certainly not strongly stable. In terms of the discriminant functional we have:

**Proposition 2.3.** A gyroscopic system (2.7) has waveguide-type if and only if:

1. There is a nonzero \(x \in \mathbb{C}^n\) such that \(d(x) > 0\), and
2. There is a nonzero \(y \in \mathbb{C}^n\) such that \(d(y) \leq 0\), i.e., the system is not strongly stable.

In particular, if there is an eigenpair \(\lambda_1, x\) with \(\lambda_1 \in \mathbb{R}\), \(d(x) > 0\), and an eigenpair \(\lambda_2, y\) with \(\lambda_2 \notin \mathbb{R}\), \(d(y) < 0\), then the eigenvectors \(x\) and \(y\) satisfy Proposition 2.3. Thus, in general, it can be said that “waveguide-type” is primarily concerned with systems having both real and non-real eigenvalues. Nevertheless, we shall see in Examples 3.3 and 5.1 that there are systems with entirely real spectrum and waveguide-type.

Observe that, in condition (a), \(x\) may or may not be an eigenvector. Example 3.5 (below) is a case in which condition (a) is not satisfied at any eigenvector but there is a nonzero \(x\) for which \(d(x) = 0\).

Also, if \(d(y) \leq 0\) at an eigenvector \(y\), then item (b) implies that there is either a multiple real eigenvalue \(\lambda_2\) (when \(d(y) = 0\)), or there is a non-real eigenvalue (when \(d(y) < 0\)). However, this inequality may well be satisfied at a \(y\) which is not an eigenvector. The beauty of the criterion of Proposition 2.3 lies in the fact that it does not require the calculation of spectral properties. Our concern is with the spectral properties implied by the criterion. Note the following simple corollary of the proposition:

If \(L_2 < 0\) then \(L(\lambda)\) of (1.2) is not of waveguide-type. This is simply because \(L_2 < 0\) implies \(d(x) > 0\) for all \(x \neq 0\). In other words, \(L_2 < 0\) implies strong stability.

---

3It can be shown by example that \(d(y) = 0\) may occur whether the associated real eigenvalue is semisimple or not.
3. Spectra of systems of waveguide-type. We first observe that, for systems of waveguide-type, the real spectrum is not empty. To see this observe that, in Proposition 2.3(a), the condition \( d(x) > 0 \) for some \( x \) ensures that \( x \) determines a real point in the numerical range of \( \mathbb{L} \) (see (1.3)). Thus, we immediately have \( NR(\mathbb{L}) \neq \phi \). But then, since \( NR(\mathbb{L}) \) is bounded it must have real frontier points, and it follows from Theorem 10.15 of [6] that there must be real spectrum.

A major result of this subject is more explicit and asserts that, for systems of waveguide-type, \( k_- \) and \( k_+ \) of (2.4) are, in fact, real eigenvalues. Thus, it is necessary for systems of waveguide-type that at least one eigenvalue be real. Indeed, all eigenvalues may be real\(^4\). We take advantage of the definitions introduced in Subsections 1.1-1.4 to state a variant of Abramov’s theorem and to expand on the role of the eigenfunctions - which are readily visualized. Proof of the theorem is postponed to an Appendix.

**Theorem 3.1.** If a gyroscopic system \( \mathbb{L}(\lambda) \) of (2.7) has waveguide-type then \( k_- \) and \( k_+ \) of (2.4) are real eigenvalues. Furthermore:

(a) There is an eigenfunction \( \mu_\tau(\lambda) \) with \( \mu_\tau(k_-) = 0 \) and \( \mu^{(1)}_\tau(k_-) \leq 0 \), and the sign-characteristic for \( k_- \) contains no +1’s.

(b) There is an eigenfunction \( \mu_\sigma(\lambda) \) with \( \mu_\sigma(k_+) = 0 \) and \( \mu^{(1)}_\sigma(k_+) \geq 0 \) and the sign-characteristic for \( k_+ \) contains no -1’s.

Thus, waveguide-type implies the existence of two (possibly coincident) real eigenvalues. Proposition 2.3 is not difficult to apply, but it may be useful to provide relatively simple sufficient conditions on the spectrum ensuring waveguide-type. We have:

**Proposition 3.2.** If \( \mathbb{L}(\lambda) \) of (2.7) has the following properties:

(a) there is a real eigenvalue \( \lambda_1 \in \mathbb{R} \),

(b) there is at least one linear elementary divisor associated with \( \lambda_1 \),

(c) there exists an eigenvalue \( \lambda_2 \in \mathbb{C} \setminus \mathbb{R} \),

then the system has waveguide-type.

**Proof.** The fact that there exists an eigenvalue \( \lambda_1 \in \mathbb{R} \) with a linear elementary divisor means that \( \mathbb{L}(\lambda) \) has an eigencurve \( \mu(\lambda) \) defined on \( \mathbb{R} \) with a simple zero at \( \lambda_1 \) (see Section 1.2). Furthermore, because \( \lambda_1 \) is a simple zero of \( \mu(\lambda) \), there is an eigenvector \( x \) of \( \mathbb{L}(\lambda_1) \) for which

\[
\mu^{(1)}(\lambda_1) = x^* \mathbb{L}^{(1)}(\lambda_1)x \neq 0,
\]

\(^4\)Since a direct sum of waveguide systems is again of waveguide-type, we can easily construct waveguide systems with prescribed spectrum. Example 4.3 is of this kind.
(see Theorem 12.5.2 of [8], for example). With this choice of $x$, and because $\lambda_1 \in \mathbb{R}$, we satisfy condition (a) of Proposition 2.3.

Condition (b) of that proposition follows from our hypothesis (c). Thus, it follows from Proposition 2.3 that the system has waveguide-type.

The first example seems elementary, but is instructive.

**Example 3.3.** Let $G = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix}$, $g \geq 0$, and $L_2 = I_2$. It is found that the spectrum of $L(\lambda)$ is real for $g \geq 2$ and, otherwise, there are no real eigenvalues. Also, with $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2$,

$$d(x) = (iGx, x)^2 - 4(x, x)(L_2x, x) = -g^2(x_1x_2 - x_1\bar{x}_2)^2 - 4\|x\|^4,$$

(and notice that $x_1x_2 - x_1\bar{x}_2$ is pure-imaginary). When $g > 2$ all eigenvalues are real and distinct. Indeed, there are four simple real eigenvalues which, when listed in increasing order have types $-, +, -, +$, respectively. When $g = 2$ there are defective eigenvalues $1$ and $-1$, each with algebraic multiplicity two and neutral type.

The eigencurves are shown in Figure 3.1 for the special case $g = 2$. In this case, we have (in Theorem 3.1) $k_- = -1$, $k_+ = +1$ and $\mu_1(k_-) = \mu_2(k_+) = 0$.

If $g > 2$ the two arcs of Figure 3.1 are displaced “bodily” downwards and there are four distinct real eigenvalues. It is easily seen that:

(a) If $y \in \mathbb{R}^2$ then $d(y) < 0$,

(b) Throughout this paper we use the Euclidean norm on $\mathbb{C}^n$, and the induced operator norm on $\mathbb{C}^{n \times n}$. 

**Fig. 3.1. Eigencurves for Example 3.3 at $g = 2$.**
(b) if $x = \begin{bmatrix} 1 \\ i \end{bmatrix}$ then $d(x) > 0$,

and, by Proposition 2.3, the system has waveguide-type.

If $g < 2$ the two arcs of Figure 3.1 are displaced “bodily” upwards and there are no real eigenvalues so (by Theorem 3.1) the system is not of waveguide-type.

When $g = 2$ it is easily verified that, since $L_2 = I$,

$$d(x) = (x^*(iG)x)^2 - 4\|x\|^2(L_2x,x),$$

(3.2)

$$= \{x^*(iG+2I)x\}\{x^*(iG-2I)x\}.$$  

But then we have $iG+2I \geq 0$ and $iG-2I \leq 0$ and it follows that $d(x) \leq 0$ for all $x \in \mathbb{C}^2$. Thus, when $g = 2$, the system is not of waveguide-type.

![Fig. 3.2. Eigencurves for Example 3.4](image-url)
example could be described as “contrived”. It shows that a system of the form (1.2) may have two real eigenvalues and \textit{not} have waveguide-type.

**Example 3.5.** (This is Example 4 of [12].) Let

$$G = \begin{bmatrix} 0 & \sqrt{4 + \sqrt{12}} \\ -\sqrt{4 + \sqrt{12}} & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{3.3}$$

Then it is found that the spectrum consists of two \textit{defective} real eigenvalues, each with algebraic multiplicity two (see Figure 3.3). It follows that, at an associated eigenvector \(y\) we have \(d(y) = 0\) and condition (b) of Proposition 2.3 is satisfied.

Furthermore, it can be shown that \(d(x) \leq 0\) for all \(x \neq 0\). Thus, condition (a) of Proposition 2.3 cannot be satisfied, and the system \textit{does not} have waveguide-type. Notice also that, in the context of Theorem 3.1 the sign-characteristic of each real eigenvalue (see Definition 1.3(d)) is the singleton \(\{0\}\). The real eigenvalues are at \(\lambda = \pm 3^{1/4}\).

![Fig. 3.3. Eigencurves for Example 3.5](image)

Suppose now that we make a \textit{small} perturbation of \(L_2\):

$$L_2' = \begin{bmatrix} 2 + \varepsilon & 1 \\ 1 & 2 + \varepsilon \end{bmatrix}$$

and consider the perturbed eigenfunctions \(\mu_1(\lambda, \varepsilon), \ \mu_2(\lambda, \varepsilon)\). It is found that (with \(g = \sqrt{4 + \sqrt{12}}\)) we may write

\[
\begin{align*}
\mu_1(\lambda, \varepsilon) & = \lambda^2 + 2 + \varepsilon + (g^2\lambda^2 + 1)^{1/2}, \\
\mu_2(\lambda, \varepsilon) & = \lambda^2 + 2 + \varepsilon - (g^2\lambda^2 + 1)^{1/2}.
\end{align*}
\]
If \( \varepsilon > 0 \) both curves of Figure 3.3 are displaced upwards, there are no real eigenvalues and, by Theorem 3.1, the system is not of waveguide-type.

If \( \varepsilon < 0 \) the lower curve of Figure 3.3 is displaced downwards, four simple real eigenvalues are generated, and the system has waveguide-type.

4. Semisimple eigenvalues. Semisimple eigenvalues may, of course, arise whether a system has waveguide-type or not and, in this section, we first discuss a role played by the discriminant in the geometry of the eigenfunctions. As in (1.2), let

\[
\mathbb{L}(\lambda) = I\lambda^2 + iG\lambda + L_2, \quad G^T = -G \in \mathbb{R}^{n \times n}, \quad L_2^* = L_2 \in \mathbb{C}^{n \times n}
\]

and define \( d(x) \) as in (2.1). For an eigenvalue \( \lambda_0 \) of \( \mathbb{L}(\lambda) \) with eigenvector \( x_0 \), \( \|x_0\|^2 = x_0^*x_0 = 1 \), we have

\[
\mathbb{L}(\lambda_0)x_0 = (I\lambda_0^2 + iG\lambda_0 + L_2)x_0 = 0.
\]

Then

\[
\lambda_0^2 + (iGx_0, x_0)\lambda_0 + (L_2x_0, x_0) = 0
\]

and so, using (2.1),

\[
2\lambda_0 = -(iGx_0, x_0) \pm \sqrt{d(x_0)}.
\]

If eigenvector \( x_0 \in G' \) of (2.2) then \( d(x_0) \geq 0 \) and \( \lambda_0 \) takes a real value consistent with (1.2).

Now let \( \mu(\lambda) \) be an analytic eigenfunction for \( \mathbb{L}(\lambda) \) for which \( \mu(\lambda_0) = 0 \) and (as in Subsection 1.2)

\[
\mathbb{L}(\lambda)x(\lambda) = \mu(\lambda)x(\lambda), \quad \lambda \in \mathbb{R},
\]

with \( \|x(\lambda)\|^2 = x(\lambda)^*x(\lambda) = 1 \). Thus, (1.1) holds and, differentiating, we also have

\[
\mathbb{L}^{(1)}(\lambda)x(\lambda) + \mathbb{L}(\lambda)x^{(1)}(\lambda) = \mu^{(1)}(\lambda)x(\lambda) + \mu(\lambda)x^{(1)}(\lambda)
\]

on a real neighbourhood of \( \lambda_0 \).

It is known (Theorem 12.2.1 of [8]) that, because all elementary divisors associated with \( \lambda_0 \) are linear, \( x_0 = x(\lambda_0) \) can be chosen in (1.1) so that

\[
x_0^*L^{(1)}(\lambda_0)x_0 = x_0^*(2I\lambda_0 + iG)x_0 \neq 0.
\]

Indeed, if \( L^{(1)}(\lambda_0) \) has \( p_+ \) positive eigenvalues and \( p_- \) negative eigenvalues, then there are \( p_+ \) (resp., \( p_- \)) linearly independent eigenvectors associated with \( \lambda_0 \) for which
\[ x^*_0 L^{(1)}(\lambda_0)x_0 > 0 \text{ (resp., } x^*_0 L^{(1)}(\lambda_0)x_0 < 0). \] Consequently, there are \( p_+ \) (resp., \( p_- \)) choices of \( x_0 \) for which \( \mu^{(1)}(\lambda_0) > 0 \) (resp., \( \mu^{(1)}(\lambda_0) < 0 \)).

Premultiply (4.4) by \( x(\lambda)^* \) and set \( \lambda = \lambda_0, x_0 = x(\lambda_0) \) (as in (4.1)). Then, using (4.2) and \( \mu(\lambda_0) = 0 \), we obtain
\[
\mu^{(1)}(\lambda_0) = x^*_0 (2I\lambda_0 + iG)x_0,
= 2\lambda_0 + x^*_0 (iG)x_0,
= \pm \sqrt{d(x_0)},
\]
and note that, if Ker \( L(\lambda_0) \) has dimension \( \delta \geq 1 \) then \( \mu^{(1)}(\lambda_0) \) may take as many as \( \delta \) distinct values determined by the discriminant.

Furthermore, (in contrast with Example 3.5) when \( \lambda_0 \) is semisimple these values are nonzero. Indeed, the sign of \( \mu^{(1)}(\lambda_0) \) determines a member of the sign characteristic of \( \mathbb{L}(\lambda) \), either +1 or -1. There are precisely \( \delta \) such signs and they are stable under small perturbations of \( \mathbb{L}(\lambda) \) (see Section 5.9 and Proposition 12.2.1 of [8]). Thus, the discriminant plays a role in the geometry of the eigenfunctions. In particular, (4.5) provides a geometric interpretation for the magnitude of the discriminant functional evaluated at an eigenvector: The slope of an eigencurve at an eigenvalue (see Subsection 1.2) is determined by the square root of the discriminant at a corresponding eigenvector \( x_0 \) for which \( ||x_0|| = 1 \).

**Proposition 4.1.** Let \( \lambda_0 \) be a semisimple real eigenvalue of \( \mathbb{L}(\lambda) \) of (1.2) with algebraic multiplicity \( \delta \geq 1 \). Then exactly \( \delta \) eigencurves have zeros at \( \lambda_0 \). These zeros are all simple and the slopes of these eigencurves at \( \lambda_0 \) are determined by the discriminant as in (4.5).

For comparison with waveguide-type, the significance of strong stability is clarified in the next proposition (and recall that systems with strong stability do not have waveguide-type). Illustrations for systems of category (2) appear in Examples 3.3 and 5.1.

**Proposition 4.2.** Consider the following statements:

(1) The gyroscopic system \( \mathbb{L}(\lambda) \) of (4.1) is strongly stable.
(2) All eigenvalues of \( \mathbb{L}(\lambda) \) of (4.1) are real and semisimple.
(3) All solutions of the differential system
\[
Ix^{(2)}(t) - Gx^{(1)}(t) + L_2 x(t) = 0
\]
are bounded on the real line.

Then (1) \( \Rightarrow \) (2) and (2) \( \iff \) (3).

**Proof.** Let \( \lambda_0 \) be a real eigenvalue of \( \mathbb{L}(\lambda) \) of (4.2) and let \( \mathbb{L}(\lambda) \) be strongly stable.
Then \( d(x) > 0 \) for all \( x \neq 0 \). It follows immediately from (4.5) (and the discussion of eigencurves) that for each eigencurve \( \mu_j(\lambda) \) with \( \mu_j(\lambda_0) = 0 \), we have \( \mu_j^{(1)}(\lambda_0) \neq 0 \). This implies that \( \lambda_0 \) is a semisimple real eigenvalue of \( L(\lambda) \) and, hence, (1) \( \Rightarrow \) (2).

The equivalence of (2) and (3) follows from Theorem 2.3 (p. 155) of [7].

Concerning the converse statement, (2) \( \Rightarrow \) (1) in Proposition 4.2, the semisimple property ensures that \( \mu^{(1)}(\lambda_0) \neq 0 \) for each eigencurve \( \mu_j(\lambda_0) = 0 \) and then, from (4.5), \( d(x_0) > 0 \) for corresponding eigenvectors \( x_0 \). But this does not guarantee \( d(x) > 0 \) for all \( x \neq 0 \), so the system is not necessarily strongly stable. This is confirmed by an example of Müller [15], which appears again as Example 7 of [12]. More detail will be given in the next section.

In the two following examples, \( L_2 \) is real-symmetric, so that the spectra have Hamiltonian symmetry. Consequently, the real-analytic eigenfunctions \( \mu(\lambda) \) (see (4.3)) are even functions of \( \lambda \). Example 4.3 has waveguide-type and Example 4.4 is not of waveguide-type.

**Example 4.3.** Let

\[
G = \frac{2\sqrt{3}}{3} \begin{bmatrix} 0 & 3 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.
\]

Then \( L \) has (truncated) real eigenvalues

\(-1.732, \ -1.732, \ -1.196, \ 1.196, \ 1.732, \ 1.732,\)

---

**Fig. 4.1. Eigencurves for Example 4.3**
and non-real eigenvalues $\pm 1.4481i$ (see Figure 4.1). Furthermore, the double real eigenvalues are not semisimple. It follows immediately from Proposition 3.2 that the system has waveguide-type.

The presence of non-real eigenvalues and of real eigenvalues which are not semisimple show that the system is not strongly stable.

Example 4.4. Let

$$G = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{bmatrix}. $$

Then $L(\lambda)$ of (2.7) has the six (truncated) eigenvalues

$$\pm 0.7071 \pm i(1.8708), \quad 0, \quad 0,$$

and the zero eigenvalue is defective, with algebraic and geometric multiplicities two and one, respectively. Notice that the hypothesis (b) of Proposition 3.2 is not satisfied.

Then observe that $L_2 = -2G^2$ so that we may write

\begin{equation}
L(\lambda) = I\lambda^2 + iG\lambda + 2(iG)^2
\end{equation}

From the definition (2.1),

\begin{equation}
d(x) = (iGx, x)^2 - 4||x||^2(-2G^2x, x)
\end{equation}

\begin{equation}
= 8||x||^2(G^2x, x) - (Gx, x)^2.
\end{equation}
Then we have $\langle G^2x, x \rangle = \langle -G^T Gx, x \rangle = -(Gx, Gx) = -\|Gx\|^2$, so that
\[
d(x) = -8\|x\|^2\|Gx\|^2 - (Gx, x)^2,
\]
and it follows that $d(x) \leq 0$ for all $x \neq 0$. Thus, by Proposition 2.3 the system does not have waveguide-type.

Examples 3.3 and 4.3 illustrate the fact that, for Hamiltonian systems of the form (1.2), waveguide-type admits either semisimple or defective real eigenvalues. In contrast, the multiple real eigenvalues of Examples 3.5 and 4.4 are defective and the systems are not of waveguide-type.

**Proposition 4.5.** (a) Systems of the form (1.2) with waveguide-type have real spectrum and eigenvalues may be either semisimple or defective.

(b) There are systems of the form (1.2) with $\sigma(L)$ contained in $\mathbb{R}$, all eigenvalues are defective, and the system is not of waveguide-type.

5. Waveguide-type and real spectrum. As we have seen, an important property of systems of waveguide-type is the guarantee of two real eigenvalues (Theorem 3.1 and Proposition 4.5) and it is natural to ask: Does the existence of real spectrum imply waveguide-type? Or strong stability? We have seen in Examples 3.5 and 4.4 that the presence of real spectrum need not imply waveguide-type, but they both include defective eigenvalues. In contrast, the following example of P.C. Müller [15] (see also [12]) has entirely real semisimple spectrum and is of waveguide-type.

By definition, systems of waveguide-type cannot be strongly stable - because there is a $y$ such that $d(y) \leq 0$. However, it is possible that $d(x) > 0$ for all eigenvectors and an $x$ for which $d(x) \leq 0$ must be sought elsewhere. Müller’s example is of this kind. (Notice also that, in this case, all eigenvalues are real and simple (unrepeated), so Proposition 3.2 does not apply.) We see “at a glance” (Figure 5.1) that all eigenvalues are real and semisimple.

The numerical range, $NR(L)$, contains non-real points and, even though $d(x) > 0$ at all eigenvectors $x$, there are other vectors, $y$, for which $d(y) \leq 0$, to show that the system is not strongly stable (see Proposition 4.2). Indeed, Figure 7 of [12] shows that there are non-real points in $NR(L)$ (i.e. there are vectors $y$ for which $d(y) < 0$). Then Proposition 2.3 can be applied to establish waveguide-type.

**Example 5.1.** We have:
\[
M = \begin{bmatrix} M_1 & 0 \\ 0 & M_1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & G_1 \\ -G_1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & 0 \\ 0 & K_1 \end{bmatrix},
\]
where \( M_1 = \text{diag}[0.2, 0.8, 0.2, 1/9] \), \( G_1 = 150 \text{diag}[0.4, 1.6, 0.4, 7/36] \), and
\[
K_1 = \begin{bmatrix}
-2800 & -1200 & 0 & -1200 \\
-1200 & -15600 & -1200 & 0 \\
0 & -1200 & -2800 & 1200 \\
-1200 & 0 & 1200 & 561.48
\end{bmatrix}.
\]

Note that \( K_1 \) and \( K \) are indefinite.

6. **Canonical forms.** Canonical forms (in the sense of Jordan canonical forms) play an important part in computation with matrix polynomials, and waveguide systems of the form (1.2) are included in the recent analysis of [13]. We indicate the nature of these results here, and observe that they are consistent with the definitions of Sections 1 and 3.

Canonical forms are arrived at using a “linearization” of \( L(\lambda) \) of (1.2), namely linear pencils \( \lambda A - C_R \), where
\[
A = \begin{bmatrix}
iG & I \\
I & 0
\end{bmatrix}, \quad C_R = \begin{bmatrix}
0 & I \\
-L_2 & -iG
\end{bmatrix},
\]
Observe that \( A^* = A, (AC_R)^* = AC_R \), and \( \lambda A - C_R \) is a “linearization” of \( L(\lambda) \) with Hermitian coefficients.

Then (see Theorem 4.3 of [13]), there is a (selfadjoint Jordan) triple \( (X, J, PX^*) \), where \( P \) and \( J \) (a Jordan form for \( A^{-1}C_R \)) are block diagonal. We sketch the block structure of \( J \) and \( P \):
(a) If $\alpha$ is a real eigenvalue with a partial multiplicity three (for convenience) and associated sign characteristic $\epsilon$, ($\epsilon = +1$ or $-1$), then $P$ and $PJ$ have associated blocks on their main diagonal:

\[
P_j = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad P_jJ_j = \begin{pmatrix}
0 & 1 & \alpha \\
1 & \alpha & 0 \\
\alpha & 0 & 0
\end{pmatrix}.
\]

(For a linear elementary divisor the corresponding entries are $\epsilon$, $\epsilon\alpha$.)

(b) If $\beta \neq \overline{\beta}$ is a non-real conjugate eigenvalue pair with partial multiplicity two, then $P$ and $PJ$ have a diagonal block structure with associated (Hermitian) diagonal blocks:

\[
P_k = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad P_kJ_k = \begin{pmatrix}
0 & 0 & 1 & \beta \\
0 & 0 & \beta & 0 \\
1 & \overline{\beta} & 0 & 0 \\
\overline{\beta} & 0 & 0 & 0
\end{pmatrix}.
\]

(For a linear elementary divisor the corresponding entries are \([0 1 \alpha], [0 \beta \overline{\beta}]\).) The extension to elementary divisors of general degree is natural, and the reader is referred to [13] for more details.

7. Conclusions. There is a considerable literature on systems of waveguide-type, and the natural mathematical models for such systems are set in spaces of infinite dimension. It is also the case that computational methods are frequently used in the detailed examination of such systems. This generally requires the formulation of systems acting on a finite dimensional space, and such a system should retain essential spectral properties of waveguides. Beginning with a review of the basic spectral properties of matrix-valued functions, we have examined the spectra of finite-dimensional waveguide systems and provided interesting examples.

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REFERENCES

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Appendix A. Abramov’s existence theorem.

Theorem A.3 below is a finite-dimensional version of Abramov’s theorem. It includes the existence of real eigenvalues for systems of waveguide-type as forecast in Theorem 3.1. The proof is essentially that of Abramov, but it is adapted to the present context and includes some small refinements. The concepts introduced in Section 1.3 play an important role in this discussion. It will be seen that the five items of Theorem A.3 (below) are consistent with our discussion of Examples 3.3–5.1.

Recalling equation (2.3) and Definition 2.2 we make a further subdivision of the real line. Define

\[ \delta_- := \inf_{\mathbb{G}} p_+ \quad \delta_+ := \sup_{\mathbb{G}} p_- . \]

Then we have:

**Lemma A.1.** If \( L(\lambda) \) of (1.2) has waveguide-type, then \( \delta_- \leq \delta_+ \).

**Proof.** We assume that there is a vector \( x_2 \) such that \( d(x_2) < 0 \) (i.e., \( \mathbb{G}' \neq \mathbb{C}^n \setminus \{0\} \)). Since \( \mathbb{G} \neq \emptyset \), there exist \( x_1 \) such that \( d(x_1) > 0 \). We set \( z_t = tx_1 + (1-t)x_2, \quad t \in [0,1], \) and consider the polynomial \( f(t) = d(z_t) \). Since \( f(1) > 0 \) and \( f(0) < 0 \), it has a zero in \((0,1)\). We denote by \( t_* \) the zero nearest to 1. Since \( f(t) > 0, \forall t \in (t_*,1] \), we have \( z_t \in \mathbb{G} \) for \( t \) from \((t_*,1] \). But \( d(z_{t_*}) = 0 \) and, therefore,

\[ \delta_- = \inf_{\mathbb{G}} p_+ \leq \lim_{t \to t_*} p_+(z_t) = p_+(z_{t_*}) = p_-(z_{t_*}) = \lim_{t \to t_*} p_-(z_t) \leq \sup_{\mathbb{G}} p_- = \delta_+. \]

The required inequality follows.

Assume now that \( \mathbb{G}' = \mathbb{C}^n \setminus \{0\} \). Since \( \mathbb{G} \neq \mathbb{C}^n \setminus \{0\} \), there exist a vector \( x_3 \neq 0 \) such that \( d(x_3) = 0 \) (i.e., \( x_3 \in \mathbb{G}' \setminus \mathbb{G} \)). Assume that \( x_1 \in \mathbb{G} \). We set \( z_\alpha = x_3 + \alpha x_1, \quad \alpha \in \mathbb{R} \) and we consider the nonnegative polynomial \( d(z_\alpha) \) with leading coefficient \( d(x_1) > 0 \). Thus, for \( \alpha > 0 \) sufficiently small, we have \( d(z_\alpha) > 0 \) and \( z_\alpha \in \mathbb{G} \). Since \( z_\alpha \) tends to \( x_3 \) when \( \alpha \) goes to zero and \( d(x_3) = 0 \), the proof concludes as in the previous case.

Now if \( L(\lambda) \) has waveguide-type then, using (2.4),

\[ -\infty < k'_- \leq k_- \leq \delta_- \leq \delta_+ \leq k_+ \leq k'_+ < +\infty. \]

See Figure A.1 below.

Abramov’s classification of real eigenvalues is based on (A.2) in a natural way:

\[ \sigma'_-, \quad \sigma_-, \quad \sigma_0, \quad \sigma_+, \quad \sigma'_+ \]
are the (possibly empty) sets of real eigenvalues in the five finite intervals of the real line defined by \((A.2)\). Thus, if \(\sigma_R\) denotes the set of all real eigenvalues of \(L\), then

\[
\sigma'_- := \sigma_R \cap [k'_-, k_-], \quad \sigma_- := \sigma_R \cap [k_-, \delta_-], \quad \sigma_0 := \sigma_R \cap [\delta_-, \delta_+],
\]

\[
\sigma'_+ := \sigma_R \cap (\delta_+, k'_+], \quad \sigma'_+ := \sigma_R \cap (k_+, k'_+].
\]

If, in the definition of \(\sigma'_-\), we have \(k'_- = k_-\), then we consider the set \(\sigma'_-\) to be empty, and similarly for \(\sigma'_+\) and \(k'_+, k_+\).

It is our next objective to associate real eigenvalues of the three types in Definition 1.3 with these subintervals of the real line but bear in mind that, as in our Examples 3.3–5.1, some or all of these subsets of real eigenvalues may be empty. In particular, thinking in terms of eigenfunctions for the system \((1.2)\), it is obvious that the right-most real eigenvalue has either positive or neutral type, with a similar property for the left-most real eigenvalue. (See Figures 3.3 and 4.1 above, and also Theorem 3.1 of [14].)

We refer to \((2.2)\) and \((2.3)\) for the definitions of \(G\) and \(p_\pm(x)\), and give the following lemma without proof.

**Lemma A.2.** For \(L(\lambda)\) of \((1.2)\) and \(\lambda \in \mathbb{R}\) we have:

(a) If \((L(\lambda)x, x) < 0\) then \(x \in G\) and \(\lambda \in (p_- (x), p_+(x))\).

(b) If \(x \in G'\) then \((L^{(1)}(p_\pm(x))x, x) = \pm \sqrt{d(x)}\).

(c) If \(\lambda \in (-\infty, k_-) \cup [k_+, +\infty)\), then \(L(\lambda) \geq 0\).

(d) If \(\lambda \in (-\infty, k'_-) \cup (k'_+, +\infty)\), then \(L(\lambda) > 0\).

(e) The functionals \(p_+, p_-\) and \(d\) are continuous on \(G'\).

Recalling Definitions 1.3(a) and 1.3(b), we have:

**Theorem A.3.** If \(L(\lambda)\) of \((1.2)\) has waveguide-type, then:

(a) \(k_+\) and \(k_-\) are real eigenvalues.

(b) If \(\sigma_+ \neq \emptyset\), then \(k_+ \in \sigma_+\) (and similarly for \(k_-\) and \(\sigma_-\)).

(c) \(\sigma_+ (\sigma_-)\) consists of eigenvalues of positive (resp., negative) type.

(d) \(\sigma'_+ \cup \sigma'_-\) consists of eigenvalues of neutral type.
A multiple real eigenvalue of mixed type has a neutral eigenvector.

Proof. (a) We first show that $k_+$ is an eigenvalue. Since $k_+ = \sup_G p_+$ and the unit sphere in $\mathbb{C}^n$ is compact, it follows from (2.3) that there is a sequence $x_n \subset G$ such that

$$\|x_n\| = 1, \quad p_+(x_n) \to k_+, \quad \text{and} \quad x_n \to x.$$  

Since $(L(p_+(x_n))x_n, x_n) = 0$, it follows that

$$|\langle L(k_+)x_n, x_n \rangle| = |\langle L(k_+)x_n, x_n \rangle - \langle L(p_+(x_n))x_n, x_n \rangle|$$
$$= |\langle [L(k_+) - L(p_+(x_n))]x_n, x_n \rangle|$$
$$\leq \|L(k_+) - L(p_+(x_n))\| \|x_n\|\|x_n\|$$
$$= \|L(k_+) - L(p_+(x_n))\|$$

and we have

$$\lim_{n \to \infty} (L(k_+)x_n, x_n) = 0.$$  

By part (c) of Lemma A.2, $L(k_+) \geq 0$, hence

$$|\|L(k_+)x_n\| \leq \|L(k_+)\|^{1/2} (L(k_+)x_n, x_n)$$

and it follows that

$$\|L(k_+)x_n\| \to 0, \quad L(k_+)x = 0.$$  

From (A.3), it follows that $x \neq 0$ and consequently $k_+$ and $x$ form an eigenpair for $L$. In a similar way one shows that $k_-$ is an eigenvalue.

(b) Follows from (a) and the definitions of $\sigma_+$ and $\sigma_-$. 

(c) We show that $\sigma_+$ consists of eigenvalues of positive type. First, we establish that $k_+$ has an eigenvector of positive type. We select a sequence $\{x_n\}$ with the properties (A.3) and (A.4). From part (e) of Lemma A.2 it follows that

$$d(x) = \lim_{n \to \infty} d(x_n) \geq 0, \quad k_+ = p_+(x).$$

Assume that $d(x) = 0$, then $p_+(x) = p_-(x)$ and

$$k_+ = p_+(x) = p_-(x) = \lim_{n \to \infty} p_-(x_n) \leq \delta_+$$

which contradicts the fact that $\delta_+ < k_+$ (in (c) we accept that $\sigma_+ \neq \emptyset$). Consequently $d(x) > 0$ and $x \in G$. By part (b) of Lemma A.2, $k_+$ and $x$ form an eigenpair of positive type.
Second, we show that \( k_+ \) is an eigenvalue of positive type. Let \( z \) be any eigenvector corresponding to \( k_+ \). If \( k_+ \) and \( z \) form an eigenpair of negative type then we have \( (L(k_+)z, z) = 0 \) and \( (L^{(1)}(k_+)z, z) < 0 \). It follows that \( k_+ < p_+(z) \) which contradicts the fact that \( k_+ \) is the least upper bound of \( p_+ \) on \( G \).

Now we want to show that the eigenpair \( k_+ \), \( z \) cannot be neutral (see Definition \( 1.2 \)). If so, we let \( z_\alpha = z + \alpha x \) with \( \alpha \in \mathbb{R} \), where \( x \) is the eigenvector of positive type corresponding to \( k_+ \). Then \( L(k_+)z_\alpha = 0 \) and we have \( z_\alpha \in G' \) for \( \alpha \in \mathbb{R} \). Note that (in \( 2.1 \)) \( d(z_\alpha) \) is a nonnegative polynomial in \( \alpha \) of fourth degree, with leading coefficient \( d(x) > 0 \). Therefore, for sufficiently small \( \alpha \neq 0 \), the vector \( z_\alpha \in G \) and \( k_+ = p_+(z) = \lim_{\alpha \to 0^+} p_-(z_\alpha) \leq \delta_+ \), which leads to a contradiction. Consequently, \( k_+ \) is an eigenvalue of positive type.

Let \( \lambda \in \sigma_+ \), \( \lambda \neq k_+ \), that is \( \delta_+ < \lambda < k_+ \), and let \( z \) be a corresponding eigenvector. Since \((L(\lambda)z, z) = 0\), we have \( z \in G' \). The eigenpair \( \lambda, z \) cannot be of neutral type, since if \((L^{(1)}(\lambda)z, z) < 0\) then \( z \in G \) and \( \lambda = p_-(z) \), contradicting the fact that \( \delta_+ < \lambda \).

Now we show that the eigenpair \( \lambda, z \) cannot be neutral. If so, then \((L(\lambda)z, z) = 0\) and \((L^{(1)}(\lambda)z, z) = 0\), so that \( \lambda = p_+(z) \). Consider an eigenpair \( k_+, x \) of positive type, and set \( z_\alpha = z + \alpha x \). We can assume that \( \text{Re}(L(\lambda)z, x) \leq 0 \). (If not we can replace \( x \) by \(-x\).) Note that, since \( \delta_+ < \lambda < k_+ \), we have \((L(\lambda)x, x) < 0\), and therefore,

\[
(L(\lambda)z_\alpha, z_\alpha) = (L(\lambda)z, z) + 2\alpha \text{Re}(L(\lambda)z, x) + \alpha^2(L(\lambda)x, x) < 0
\]

for \( \alpha > 0 \). We have \( z_\alpha \in G \), for \( \alpha > 0 \) and we obtain \( \lambda = p_-(z) = \lim_{\alpha \to 0^+} p_-(z_\alpha) \leq \delta_+ \), which contradicts the fact that \( \lambda \in \sigma_+ \). Thus, \( \lambda \) is an eigenvalue of positive type.

The case in which \( \lambda \in \sigma_- \) can be proved in a similar way.

(d) Let \( \lambda, x \) be an eigenpair with \( \lambda \in \sigma'_+ \). Then \( x \in G' \). But \( x \) cannot be in \( G \) since, by definition, \( \lambda > k_+ \). So we have \( d(x) = 0 \) and by part (b) of Lemma \( A.2 \) the eigenpair \( \lambda, x \) is neutral. It can be shown in a similar way that eigenvalues in \( \sigma'_- \) are also neutral.

(e) Assume that \( \lambda \in \sigma_0 \) is an eigenvalue of mixed type. Then \( \lambda \) has eigenvectors of both positive and negative type. Let \( \lambda, x \) and \( \lambda, y \) be eigenvectors of positive and negative type, respectively. Then \( z_t = tx + (1 - t)y, t \in [0, 1] \) is also an eigenvector corresponding to \( \lambda \). The polynomial \( q(t) = (L^{(1)}(\lambda)z_t, z_t) \) has opposite signs at the endpoints of interval \([0, 1]\). Therefore, there exists a \( t_* \in (0, 1) \) such that \( q(t_*) = (L^{(1)}(\lambda)z_{t_*}, z_{t_*}) = 0 \). Consequently, \( \lambda, z_{t_*} \) is a neutral eigenpair. 


Example A.4. We return to the system discussed in Example 4.3 and add some
details to Figure 4.1 (see Figure A.2). Note that, in (2.4) and (A.2),
\[ k'_- = -1.732, \quad k_- = -1.196, \quad k_+ = 1.196, \quad k'_+ = 1.732, \]
with \( k'_- < k_- < k_+ < k'_+ \). Note also that the eigenvalues \( k'_- = -1.732 \) and \( k'_+ = 1.732 \)
are not semisimple, and items (a)–(e) of Theorem A.3 are illustrated.

\[ \begin{align*}
\mu & \quad 5 \\
4 & \quad 3 \\
2 & \quad 1 \\
1 & \quad 0 \\
\lambda & \quad 1 \\
\sigma_0 & \quad k'_- \\
\sigma_0 & \quad k_- \\
\sigma_0 & \quad k_+ \\
\sigma_0 & \quad k'_+ \\
\end{align*} \]

\[ \begin{align*}
\text{Fig. A.2. The boundary and the real part of } NR(L). \end{align*} \]

Appendix B. A Venn diagram.

The Venn diagram of Figure B.1 illustrates the variety of gyroscopic systems
which have arisen in our study of equation (1.2). In particular, we have:
\[ \{ \text{strongly stable systems} \} \subset \{ \text{systems with real and semisimple spectrum} \} \]
\[ \subset \{ \text{systems with all real spectrum} \} \subset \{ \text{gyroscopic systems} \} \]
There are systems of waveguide type in each of the last three subsets.

\[ \begin{align*}
\text{gyroscopic systems} \\
\text{all real spectrum} \\
\text{all real and semisimple spectrum} \\
\text{strongly stable} \\
\text{systems of waveguide type} \\
\end{align*} \]

\[ \begin{align*}
\text{Fig. B.1. Gyroscopic and waveguide systems.} \end{align*} \]