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A NOTE ON THE SPECTRAL RADIUS OF A PRODUCT OF COMPANION MATRICES

E.S. KEY† AND H. VOLKMER†

Abstract. Conditions are given on the coefficients of the characteristic polynomials of a set of $k$ companion matrices to ensure that the spectral radius of their product is bounded by $t^k$ where $0 < t < 1$.

Key words. Companion matrices, Matrix products, Spectral radius.

AMS subject classifications. 15A42, 15B99.

1. Introduction. In a recent paper [2] on population dynamics, A. Blumenthal and B. Fernandez used a bound on the spectral radius of a finite product of companion matrices [2, Lemma 5.5]. In light of the authors’ work [4] on products of companion matrices, B. Fernandez inquired of the authors if they could supply a proof of their Lemma 5.5, which we have done in Theorem 1 in Section 3 below. In Section 2, we point out that a special case of our result is connected to the well-known Eneström-Kakaya theorem [1, Theorem 1.2] on the location of zeros of a polynomial.

2. The Eneström-Kakaya theorem. Consider the $n$ by $n$ companion matrix

$$C = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \quad (2.1)$$

Its characteristic polynomial is

$$p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n.$$

By the Eneström-Kakaya theorem [1, Theorem 1.2], the assumption

$$1 \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0. \quad (2.2)$$

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implies that all zeros \( \lambda \) of \( p(\lambda) \) satisfy \( |\lambda| \leq 1 \). Therefore, under assumption (2.2), the spectral radius \( \rho(C) \) of \( C \) is at most 1.

We can prove this result using matrix notation as follows. Let \( A \) be the \( n + 1 \) by \( n + 1 \) matrix

\[
A = \begin{bmatrix}
C & 0 \\
u & 1
\end{bmatrix},
\]

(2.3)

where \( u = (0, 0, \ldots, 0, 1) \) is a row vector of dimension \( n \). Let \( B \) be the \( n + 1 \) by \( n + 1 \) companion matrix

\[
B = \begin{bmatrix}
1 - a_1 & a_1 - a_2 & a_2 - a_3 & \cdots & a_{n-1} - a_n & a_n \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix},
\]

(2.4)

and let \( L \) be the \( n + 1 \) by \( n + 1 \) matrix with 1's on the main diagonal, \(-1\)'s on the super diagonal and 0 entries everywhere else. By calculation, we verify that

\[
AL = LB
\]

so

\[
L^{-1}AL = B.
\]

Therefore, \( A, B \) are similar and so \( \rho(A) = \rho(B) \). We obtain

\[
\rho(C) \leq \rho(A) = \rho(B) = 1,
\]

where \( B \) is a stochastic matrix [3 page 526], that is, a nonnegative matrix whose row sums are all 1.

**3. An extension.** We use the same idea to prove the following lemma.

**Lemma 3.1.** Let \( C_i, i = 1, \ldots, k, \) be companion matrices of the form (2.1) with first rows \( -(a_{i1}, a_{i2}, \ldots, a_{in}) \), respectively. Suppose that

\[
1 \geq a_{i1} \geq a_{i2} \geq \cdots \geq a_{in} \geq 0 \quad \text{for} \quad i = 1, 2, \ldots, k.
\]

(3.1)

Then

\[
\rho(C_1C_2 \cdots C_k) \leq 1.
\]
Proof. We form the matrices $A_i, B_i$ as before and note that

$$\rho(C_1C_2\cdots C_k) \leq \rho(A_1A_2\cdots A_k) = \rho(B_1B_2\cdots B_k) = 1$$

since $B_1B_2\cdots B_k$ is a row stochastic matrix. □

We now obtain our main result.

**Theorem 3.2.** Let $C_i, i = 1, \ldots, k$, be companion matrices of the form (2.1) with first rows $-(a_{i1}, a_{i2}, \ldots, a_{in})$, respectively. Suppose that

$$a_{i0} := 1 > a_{i1} > a_{i2} > \cdots > a_{in} \geq 0 \text{ for } i = 1, 2, \ldots, k. \quad (3.2)$$

Define

$$t = \max_{i=1}^{k} \max_{j=1}^{n} \frac{a_{i,j}}{a_{i,j-1}} < 1.$$

Then

$$\rho(C_1C_2\cdots C_k) \leq t^k < 1.$$

Proof. We define $\tilde{a}_{i,j} = t^{-j}a_{i,j}$ and corresponding companion matrices $\tilde{C}_i$. Let $W = \text{diag}(1, t^{-1}, \ldots, t^{-n+1})$. Then

$$C_i = tW\tilde{C}_iW^{-1}.$$

Therefore,

$$\rho(C_1C_2\cdots C_k) = t^k\rho(\tilde{C}_1\tilde{C}_2\cdots \tilde{C}_k) \leq t^k,$$

where we applied Lemma 3.1 to the matrices $\tilde{C}_i$. □

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**REFERENCES**


