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A NOTE ON THE SPECTRAL RADIUS OF A PRODUCT OF COMPANION MATRICES*

E.S. KEY[†] AND H. VOLKMER[†]

Abstract. Conditions are given on the coefficients of the characteristic polynomials of a set of k companion matrices to ensure that the spectral radius of their product is bounded by t^k where $0 < t < 1$.

Key words. Companion matrices, Matrix products, Spectral radius.

AMS subject classifications. 15A42, 15B99.

1. Introduction. In a recent paper [2] on population dynamics, A. Blumenthal and B. Fernandez used a bound on the spectral radius of a finite product of companion matrices [2, Lemma 5.5]. In light of the authors' work [4] on products of companion matrices, B. Fernandez inquired of the authors if they could supply a proof of their Lemma 5.5, which we have done in Theorem 1 in Section 3 below. In Section 2, we point out that a special case of our result is connected to the well-known Eneström-Kakaya theorem [1, Theorem 1.2] on the location of zeros of a polynomial.

2. The Eneström-Kakaya theorem. Consider the n by n companion matrix

$$(2.1) \quad C = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Its characteristic polynomial is

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n.$$

By the Eneström-Kakaya theorem [1, Theorem 1.2], the assumption

$$(2.2) \quad 1 \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0.$$

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implies that all zeros λ of $p(\lambda)$ satisfy $|\lambda| \leq 1$. Therefore, under assumption (2.2), the spectral radius $\rho(C)$ of C is at most 1.

We can prove this result using matrix notation as follows. Let A be the $n + 1$ by $n + 1$ matrix

$$(2.3) \quad A = \begin{bmatrix} C & 0 \\ u & 1 \end{bmatrix},$$

where $u = (0, 0, \dots, 0, 1)$ is a row vector of dimension n . Let B be the $n + 1$ by $n + 1$ companion matrix

$$(2.4) \quad B = \begin{bmatrix} 1 - a_1 & a_1 - a_2 & a_2 - a_3 & \cdots & a_{n-1} - a_n & a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

and let L be the $n + 1$ by $n + 1$ matrix with 1's on the main diagonal, -1 's on the super diagonal and 0 entries everywhere else. By calculation, we verify that

$$AL = LB$$

so

$$(2.5) \quad L^{-1}AL = B.$$

Therefore, A , B are similar and so $\rho(A) = \rho(B)$. We obtain

$$\rho(C) \leq \rho(A) = \rho(B) = 1,$$

where B is a stochastic matrix [3, page 526], that is, a nonnegative matrix whose row sums are all 1.

3. An extension. We use the same idea to prove the following lemma.

LEMMA 3.1. *Let C_i , $i = 1, \dots, k$, be companion matrices of the form (2.1) with first rows $-(a_{i1}, a_{i2}, \dots, a_{in})$, respectively. Suppose that*

$$(3.1) \quad 1 \geq a_{i1} \geq a_{i2} \geq \cdots \geq a_{in} \geq 0 \text{ for } i = 1, 2, \dots, k.$$

Then

$$\rho(C_1 C_2 \cdots C_k) \leq 1.$$

Proof. We form the matrices A_i, B_i as before and note that

$$\rho(C_1 C_2 \cdots C_k) \leq \rho(A_1 A_2 \cdots A_k) = \rho(B_1 B_2 \cdots B_k) = 1$$

since $B_1 B_2 \cdots B_k$ is a row stochastic matrix. \square

We now obtain our main result.

THEOREM 3.2. *Let $C_i, i = 1, \dots, k$, be companion matrices of the form (2.1) with first rows $-(a_{i1}, a_{i2}, \dots, a_{in})$, respectively. Suppose that*

$$(3.2) \quad a_{i0} := 1 > a_{i1} > a_{i2} > \cdots > a_{in} \geq 0 \text{ for } i = 1, 2, \dots, k.$$

Define

$$t = \max_{i=1}^k \max_{j=1}^n \frac{a_{i,j}}{a_{i,j-1}} < 1.$$

Then

$$\rho(C_1 C_2 \cdots C_k) \leq t^k < 1.$$

Proof. We define $\tilde{a}_{i,j} = t^{-j} a_{i,j}$ and corresponding companion matrices \tilde{C}_i . Let $W = \text{diag}(1, t^{-1}, \dots, t^{-n+1})$. Then

$$C_i = tW\tilde{C}_iW^{-1}.$$

Therefore,

$$\rho(C_1 C_2 \cdots C_k) = t^k \rho(\tilde{C}_1 \tilde{C}_2 \cdots \tilde{C}_k) \leq t^k,$$

where we applied Lemma 3.1 to the matrices \tilde{C}_i . \square

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