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ON THE MULTIPLICITIES OF EIGENVALUES OF GRAPHS AND THEIR VERTEX DELETED SUBGRAPHS: OLD AND NEW RESULTS*

SLOBODAN K. SIMIĆ†, MILICA ANĐELIĆ‡, CARLOS M. DA FONSECA§, AND DEJAN ŽIVKOVIĆ¶

This paper is dedicated to Prof. Peter Rowlinson on the occasion of his 70th birthday.

Abstract. Given a simple graph G, let A_G be its adjacency matrix. A principal submatrix of A_G of order one less than the order of G is the adjacency matrix of its vertex deleted subgraph. It is well-known that the multiplicity of any eigenvalue of A_G and such a principal submatrix can differ by at most one. Therefore, a vertex v of G is a downer vertex (neutral vertex, or Parter vertex) with respect to a fixed eigenvalue μ if the multiplicity of μ in A_{G-v} goes down by one (resp., remains the same, or goes up by one). In this paper, we consider the problems of characterizing these three types of vertices under various constraints imposed on graphs being considered, on vertices being chosen and on eigenvalues being observed. By assigning weights to edges of graphs, we generalize our results to weighted graphs, or equivalently to symmetric matrices.

Key words. Graph, Adjacency matrix, Multiplicity, Downer vertex, Neutral vertex, Parter vertex, Cut vertex, Kronecker product.

AMS subject classifications. 15A18, 05C50.

1. Introduction. Let $G = (V(G), E(G))$ be a simple graph, with loops allowed in some considerations, but neither multiple edges nor multiple loops. As usual, $n = |V(G)| = |G|$ and $m = |E(G)| = ||G||$ are the order and size of G, respectively. If not otherwise specified, $V(G) = \{1, 2, \dots, n\}$. The adjacency matrix of G is the $(0, 1)$-matrix $A_G = (a_{ij})_{n\times n}$, with rows and columns indexed by the vertices of G, where $a_{ij} = 1$ if and only if $\{i, j\}$ (or ij for short) is an edge of G. The characteristic polynomial of A_G, i.e., $\det(xI - A_G)$, is also called the *characteristic polynomial* of G, and is denoted by $\phi_G(x)$. Its zeroes (with repetitions) comprise the spectrum of G, denoted by σ_G. Since A_G is symmetric, the spectrum of G is real, and consists of

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n eigenvalues

$$\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G).$$

Recall also that $\lambda_1(G) > \lambda_2(G)$ whenever G is connected. If $\lambda \in \sigma_G$, then $m_G(\lambda)$ denotes its multiplicity (algebraic, or geometric – note they coincide since A_G is symmetric). For convenience, if $\lambda \notin \sigma_G$, then we define that $m_G(\lambda) = 0$.

The equation $A_G\,\mathbf{x} = \lambda\,\mathbf{x}$ is called the *eigenvalue equation* of the (labeled) graph G. For a fixed $\lambda \in \sigma_G$, its non-trivial solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is called a λ-*eigenvector* of G. Note, in the scalar form, at each vertex it reads:

$$\lambda x_u = \sum_{uv \in E(G)} x_v \qquad (u \in V(G)). \tag{1.1}$$

The corresponding eigenspace is denoted by $\mathscr{E}_G(\lambda)$. Furthermore, each eigenvector may be interpreted as a mapping $\mathbf{x} : V(G) \to \mathbb{R}$. So if $v \in V(G)$, then $\mathbf{x}(v)$ and x_v can be identified. In particular, for connected graphs the largest eigenvalue is of multiplicity one, and then the corresponding eigenvector can be chosen to have positive entries (see, for example, [7, Theorem 1.3.6]), and then it is referred as the *principal eigenvector* of G.

It is well known that for each real symmetric matrix A, there exists an orthogonal matrix U such that $U^T A\ U = D$, where D is a diagonal matrix. If $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then the columns of U are the eigenvectors of A which form an eigenbasis of $\mathbb{R}^n$, and which are ordered to conform the eigenvalues. If

$$\mu_1(G) > \mu_2(G) > \cdots > \mu_r(G) \qquad (r \leqslant n)$$

are the distinct eigenvalues of A, then we may write $D = \mu_1 E_1 \dot{+} \mu_2 E_2 \dot{+} \cdots \dot{+} \mu_r E_r$, where E_i has the block diagonal form $\operatorname{diag}\underbrace{(0, \ldots, 0, I, 0, \ldots, 0)}_{r}$; here 0 is the all-zero matrix and I the identity matrix (the order of the matrix in the ith position along the diagonal is equal to $m_A(\mu_i)$). The *spectral decomposition* of A (or of G, if $A = A_G$) is given by:

$$A = \mu_1 P_1 + \mu_2 P_2 + \cdots + \mu_r P_r\,, \tag{1.2}$$

where $P_i = U E_i U^T$, for $i = 1, 2, \ldots, r$, is the *projection matrix* of the whole space $\mathbb{R}^n$ onto the eigenspace $\mathscr{E}_G(\mu_i)$. We also write P_μ, when $\mu = \mu_i$. If $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the standard (orthonormal) basis of $\mathbb{R}^n$, then $\alpha_{i,v} = \|P_i\,\mathbf{e}_v\|$ is called the *graph angle* between $\mathbf{e}_v$ and $\mathscr{E}_G(\mu_i)$ or, to be more precise, it is just a cosine of the corresponding angle (cf. [5, Chapter 4]). If $U \subseteq \{1, 2, \ldots, n\}$ (or if $U \subseteq V(G)$ in the context of graphs) let $\mathbf{e}_U = \sum_{u \in U} \mathbf{e}_u$; so $\mathbf{e}_U$ is the characteristic vector of the subset U. Then

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define $\rho_i(U) = ||P_i \mathbf{e}_U||$, and also $\alpha_i(U) = \frac{||P_i \mathbf{e}_U||}{||\mathbf{e}_U||}$ (clearly, the latter is the *graph angle* between $\mathbf{e}_U$ and $\mathscr{E}_G(\mu_i)$). If $U = \{j\}$, then we write ρ_{ij} for $\rho_i(U)$, and α_{ij} for $\alpha_i(U)$.

For simple graphs we denote by $G-u$ (resp., $G-U$) the subgraph of G obtained by deleting from G the vertex u (resp., the vertex subset U). Given a vertex $v \in V(G)$, let $\Gamma_G(v) = \{w \in V(G) \ : \ vw \in E(G)\}$, i.e., it is the (open) neighborhood of v in G. If $v \notin V(G)$ and $S \subseteq V(G)$, then $G + v$ is a supergraph of G obtained by adding to G the vertex v and edges vw, where $w \in S$. Clearly, $\Gamma_{G+v}(v) = S$. So, if $V(H) \subset V(G)$, then $H + v$ is well defined. Otherwise, S must be specified. If H is an induced subgraph of G, then $H \subseteq G$ (or $H \subset G$) means that H is an induced (resp., proper induced) subgraph of G.

A graph G is *bipartite* if its vertex set can be partitioned into two disjoint sets U and V such that each edge connects one vertex in U with one in V. The spectrum of bipartite graphs is symmetric with respect to the origin (see, for example, [7, p. 56]). The degree of a vertex v of G is the number of its neighbours. So $deg(v) = |\Gamma_G(v)|$. A vertex of degree one is called a pendant vertex. A pendant edge is an edge incident to a pendant vertex. A graph G is regular if all its vertices have the same degree. In particular, G is r-regular if each of its vertices has degree equal to r. In a connected graph a vertex v is a *cut vertex* if $G - v$ is disconnected.

Given a graph G, its *complement*, denoted by $\bar{G}$, is a graph with the same vertex set, and in which two vertices are adjacent if they were non-adjacent in G. The *line graph* of a graph H, denoted by $L(H)$, is a graph whose vertex set corresponds to the edge set of H, with two vertices being adjacent in $L(H)$ whenever the corresponding edges in H are adjacent, i.e., share a common vertex).

The (*Kronecker*) *product* of two graphs G and H is the graph $G \otimes H$, whose vertex set is equal to the Cartesian product $V(G) \times V(H)$, and in which vertices (u_1, v_1) and (u_2, v_2) are adjacent if $u_1 u_2 \in E(G)$ and $v_1 v_2 \in E(H)$. It is noteworthy that $A_{G \otimes H} = A_G \otimes A_H$ (here $\otimes$ stands for both, product of graphs and matrices).

We will also consider weighted graphs. For a given graph G (with loops allowed), $\hat{G}$ denotes its weighted counterpart. In other words, $\hat{G} = (G, \hat{w})$, where the mapping $\hat{w} : E(G) \to \hat{W}$ assigns to each edge its weight (a non-zero real number). G is also called the *underlying graph* of $\hat{G}$. The adjacency weighted matrix of $\hat{G}$ is the matrix $\hat{A}_{\hat{G}} = (\hat{a}_{ij})_{n \times n}$, where $\hat{a}_{ij} = \hat{w}(ij)$ if $ij \in E(G)$, or 0 otherwise. Note, since $0 \notin \hat{W}$, we can keep track of edges and non-edges of $\hat{G}$ (and also of its underlying graph G). Therefore, any symmetric matrix corresponds to some weighted graph, say $\hat{G}$, and it is equal to $\hat{A}_{\hat{G}}$. Hence, $\hat{A}_{\hat{G}}$ and $\hat{G}$ can be used interchangeably. If $\hat{W} = \{1\}$, then $\hat{G}$ and G coincide (and also, $\hat{A}_{\hat{G}}$ and A).

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FIG. 1.1. *Graph H and its line graph $L(H)$.*

For further details on graphs (or graph spectra), the readers are referred to [2] (resp., [7]). To gain the readability, we will usually omit further on the names of graphs and/or matrices from our notation whenever they being understood from the context.

The rest of the paper is organized as follows: In Section 2, to make the paper more self-contained, we first give some historical facts relevant to our main topic and then introduce a classification of vertices of (weighted) graphs by spectral means; afterwards we provide some results from spectral graph theory to be used further on. The main result of Section 3 is Theorem 3.1 (it will be deduced from the formulas for the characteristic polynomials of one-vertex deleted subgraph, or one-vertex extended supergraph (see also [12], Theorem 2.1 and Lemma 3.1 therein); some further results with focus on graphs are included as well. In Section 4, we focus on graphs with cut vertices (see Theorem 4.1 and Corollary 4.2; the related result from [12] was restricted on trees). In Section 5, we consider compound graphs (i.e., graphs obtained by some graph operations). In Section 6, we add several interesting observations related to graphs. In Section 7, we consider weighted graphs, or equivalently (symmetric) matrices, and generalize some of results from our previous sections to weighted graphs (in particular, Theorems 3.1 and 4.1). Finally, in Section 8, we add a few concluding remarks.

2. Preliminaries. It is well-known that the multiplicity of any eigenvalue of a graph (resp., symmetric matrix) changes by at most 1 if any vertex of G (resp., $\hat{G}$) is deleted. This fact is an immediate consequence of the Cauchy Interlacing Theorem (see, for example, [7, p. 18]). Henceforth, in general, for any vertex, say v, of a weighted graph $\hat{G}$ the following question arises:

> *Under which conditions does the multiplicity of a given eigenvalue go down by* 1*, remain the same, or go up by* 1 *in* $\hat{G} - v$*?*

The answer to this question, as it can be expected, depends on $\hat{G}$, v and $\lambda \in \sigma_{\hat{G}}$.

In what follows, the words index and vertex will define the same notion. However, the first will be used for matrices while the second for graphs.

In 1960, Parter [16] noticed that, for a given Hermitian matrix M whose underlying graph is a tree the following holds: If $m_M(\lambda) \geqslant 2$, then there exists an index v

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such that $m_{M(v)}(\lambda) \geqslant 3$, where $M(v)$ is the principal submatrix of M (with vth row and column deleted). Under the same condition, Wiener [20] proved that there is always an index v such that $m_{M(v)}(\lambda) = m_M(\lambda)+1$. Related to these two observations, we have the so-called Parter-Wiener Theorem (cf. [11]).

Theorem 2.1 (Parter-Wiener Theorem). *Let M be an acyclic Hermitian matrix. If there exists an index u such that λ is an eigenvalue of M and of $M(u)$, then there is an index v such that $m_{M(v)}(\lambda) = m_M(\lambda) + 1$.*

For this historical reason, several authors have recently started to call an index v from the above theorem the *Parter vertex of M for λ* [10, 11, 12, 13]. If $m_{M(v)}(\lambda) = m_M(\lambda)$ (resp., $m_{M(v)}(\lambda) = m_M(\lambda) - 1$), then the vertex v is called *neutral vertex* (resp., *downer vertex*) of M for λ. However, it is worth mentioning that Godsil (see, for example, [9]), motivated by [14], coined (in the context of matching polynomials of graphs) these vertices of a graph as λ-*essential*, λ-*neutral*, and λ-*positive* provided $m_{M(v)}(\lambda) = m_M(\lambda)-1$, $m_{M(v)}(\lambda) = m_M(\lambda)$, and $m_{M(v)}(\lambda) = m_M(\lambda)+1$, respectively (see also [8]). In addition, the vertices v, for which $m_{M(v)}(\lambda) \geqslant m_M(\lambda)$, are known as *Fiedler vertices* [13]. We point out that the main results in the literature regarding the classification of vertices of weighted graphs (or their matrices) have been confined to the cases when the underlying graph is a tree (or acyclic). For more general cases, only a few attempts were successful (see [10, 12]). For example, in [10], it was shown that every symmetric (or Hermitian) matrix contains an index corresponding to the downer vertex. The same fact was also observed by Cvetković et al. in [4], but in the context of graphs. Considering the concepts of star sets, star complements and star partitions of graphs (with straightforward generalizations to weighted graphs), they have proved many interesting facts on downer vertices (see, for example, [5, 7], and references therein).

In spite of the above differences in terminology, in the rest of the paper we refer to a vertex v of a (weighted) graph $\hat{G}$, with respect to $\lambda \in \sigma_{\hat{G}}$, as a

(a) *downer vertex* if $m_{\hat{G}-v}(\lambda) = m_{\hat{G}}(\lambda) - 1$;
(b) *neutral vertex* if $m_{\hat{G}-v}(\lambda) = m_{\hat{G}}(\lambda)$;
(c) *Parter vertex* if $m_{\hat{G}-v}(\lambda) = m_{\hat{G}}(\lambda) + 1$.

In addition, a *Fiedler vertex* is a vertex which is either neutral vertex or Parter vertex.

In what follows, we provide several results from the literature on spectral graph theory to be used further on.

The following (recursive) formula for computing the characteristic polynomial of a graph is due to Schwenk (see, for example, [17, p. 37]). Given a simple graph G,

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let v be its fixed vertex. Then

$$\phi_G(x) = x\phi_{G-v}(x) - \sum_{w\in\Gamma_G(v)} \phi_{G-v-w}(x) - 2\sum_{C\in\mathscr{C}(v)} \phi_{G-V(C)}(x), \tag{2.1}$$

where $\mathscr{C}(v)$ denotes the set of all cycles (of length at least three) passing through v. Here we assume that $\phi_H(x) = 1$, if $|H| = 0$.

The next formula expresses the characteristic polynomial of a vertex deleted subgraph of a graph G in terms of eigenvalues and angles of G (see, for example, [7, p. 33]).

$$\phi_{G-v}(x) = \phi_G(x)\sum_{i=1}^{r} \frac{||P_i\mathbf{e}_v||^2}{x-\mu_i}\,. \tag{2.2}$$

Similarly, the next formula, due to Rowlinson (see, for example, [7, p. 33]) expresses the characteristic polynomial of an one-vertex extension of a graph G in terms of eigenvalues and some "angle-like" quantities of G. Let $G' = G + v$ and $U = \Gamma_{G'}(v)$. Then, taking that $\mathbf{e}_U = \sum_{u\in U}\mathbf{e}_u$ (see Section 1), we have:

$$\phi_{G'}(x) = \phi_G(x)\left(x - \sum_{i=1}^{r} \frac{||P_i\mathbf{e}_U||^2}{x-\mu_i}\right). \tag{2.3}$$

Some generalizations of these formulas will be given in Section 7.

3. The vertex classification theorem. Without loss of generality, we first assume that G is connected. For weighted graphs the generalizations are almost literally the same (see Section 7). For disconnected graphs we can consider each component separately.

The following simple observations based on the Interlacing Theorem deserve to be mentioned:

(i) If μ_1 is the largest eigenvalue of a (connected) graph G, then the multiplicity of μ_1 is 1, and it drops to 0 in $G - v$, for any v. It is worth mentioning that the same is true for weighted graphs if all edges have positive weights, and if the underlying graph is connected. Then, besides the interlacing, one needs more arguments (see, for example, [3, Theorems 0.3 and 0.6, pp. 18–19]).

(ii) Analogously, for the smallest eigenvalue of a (connected) graph G, the multiplicity of $G - v$ either remains the same or goes down by 1 (by the interlacing). To see that both possibilities can occur, just see Theorem 3.9 below.

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(iii) For the other eigenvalues, in general, all three possibilities can arise. On the other hand, an interesting situation occurs in bipartite graphs with respect to the eigenvalue equal to 0. Then the multiplicity of 0 cannot remain the same, as can be easily seen by the interlacing, and the fact that the spectrum of any bipartite graph is symmetric with respect to the origin (as already mentioned in Section 1).

We are now ready to state the following important result:

THEOREM 3.1. *Let G be a graph of order $n > 1$, v a vertex of G, and let $G' = G - v$. Let*

$$A_G = \sum_{\mu\in\sigma_G} \mu P_\mu \qquad \text{and} \qquad A_{G'} = \sum_{\mu'\in\sigma_{G'}} \mu' P'_{\mu'}$$

be the spectral decompositions of G and G', respectively. If $\mu \in \sigma_G$, then v is a

(i) downer vertex for μ if and only if $||P_\mu \mathbf{e}_v|| \neq 0$;
(ii) neutral vertex for μ if and only if $||P_\mu \mathbf{e}_v|| = 0$ and $||P'_\mu \mathbf{e'}_U|| = 0$;
(iii) Parter vertex for μ if and only if $\|P'_\mu \mathbf{e'}_U\| \neq 0$,

where $\mathbf{e'}_U = \sum_{u\in U} \mathbf{e'}_u$, and $U = \Gamma_G(v)$.[1]

Proof. Let $L_1 = \lim_{x\to\mu} \frac{\phi_{G-v}(x)}{\phi_G(x)}$. Then $m_{G-v}(\mu) < m_G(\mu)$ if and only if $|L_1| = +\infty$, and the latter happens if and only if $\|P_\mu \mathbf{e}_v\| \neq 0$ (see (2.2)). So v is a downer vertex for μ in G, and (i) follows.

Similarly, let $L_2 = \lim_{x\to\mu} \frac{\phi_G(x)}{\phi_{G-v}(x)}$. Then $m_{G-v}(\mu) > m_G(\mu)$ if and only if $|L_2| = +\infty$, and the latter happens if and only if $||P'_\mu \mathbf{e}_U|| \neq 0$ (see (2.3)). So v is a Parter vertex for μ in G, and (iii) follows.

In what remains, we have that $\|P_\mu \mathbf{e}_v\| = 0$ and $\|P'_\mu \mathbf{e'}_U\| = 0$, and then neither $m_{G-v}(\mu) < m_G(\mu)$ nor $m_{G-v}(\mu) > m_G(\mu)$ holds. Consequently, $m_{G-v}(\mu) = m_G(\mu)$ holds and, therefore, v is a neutral vertex. Thus, (ii) also holds. ☐

COROLLARY 3.2. *Under the assumptions of Theorem 3.1, if v is a pendant vertex of G, while u is its neighbor, then, with respect to μ, v is a*

(i) downer vertex if $\alpha_{\mu,v} \neq 0$;
(ii) neutral vertex if $\alpha_{\mu,v} = \alpha'_{\mu,u} = 0$;
(iii) Parter vertex if $\alpha'_{\mu,u} \neq 0$.

The following two remarks are appropriate at this point:

REMARK 3.3. For a fixed $v \in V(G)$, if $\mathbf{x} \in \mathbb{R}^n$, then $\dot{\mathbf{x}}$ means that $\mathbf{x}(v) = 0$, while $\mathbf{x}'$ is the restriction of $\mathbf{x}$ to $\mathbf{x}' \in \mathbb{R}^{\mathbf{n-1}}$ (so vth component of $\mathbf{x}$ is discarded).

[1] Observe that $\mathbf{e}'_s \in \mathbb{R}^{n-1}$, while $\mathbf{e}_t \in \mathbb{R}^n$.

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Now, for a graph G and $\mu \in \sigma_G$, we have:

(i) if v is a downer vertex, then there exists a basis of $\mathscr{E}_G(\mu)$ such as $\{\dot{\mathbf{x}}_1, \ldots, \dot{\mathbf{x}}_{k-1}, \mathbf{x}_k\}$, where $\mathbf{x}_k(v) \neq 0$; in addition, $\{\dot{\mathbf{x}}'_1, \ldots, \dot{\mathbf{x}}'_{k-1}\}$ is a basis for $\mathscr{E}_{G-v}(\mu)$;

(ii) if v is a neutral vertex, then there exists a basis of $\mathscr{E}_G(\mu)$ such as $\{\dot{\mathbf{x}}_1, \ldots, \dot{\mathbf{x}}_k\}$; in addition, $\{\dot{\mathbf{x}}'_1, \ldots, \dot{\mathbf{x}}'_k\}$ is a basis for $\mathscr{E}_{G-v}(\mu)$;

(iii) if v is a Parter vertex, then there exists a basis of $\mathscr{E}_G(\mu)$ such as $\{\dot{\mathbf{x}}_1, \ldots, \dot{\mathbf{x}}_k\}$; in addition, $\{\dot{\mathbf{x}}'_1, \ldots, \dot{\mathbf{x}}'_k, \mathbf{y}_{k+1}\}$ is a basis for $\mathscr{E}_{G-v}(\mu)$ (now, if $\mathbf{y}_{k+1}$ is lifted into $\mathbb{R}^n$, it is not a μ-eigenvector for G, even if its vth coordinate is 0 – then the corresponding vector satisfies all eigenvalue equations from (1.1) but one at v).

Remark 3.4. Since $\lim_{x \to x_0} r(x) = 0$ is equivalent to $\lim_{x \to x_0} |\frac{1}{r(x)}| = +\infty$, for any rational function $r(x)$, we can also express items (i–iii) from the above theorem as follows:

(i)′ v is a downer vertex if and only if $\|P'_\mu \mathbf{e}'_U\| = 0$ and $\displaystyle\sum_{\mu'_i \neq \mu} \frac{\|P'_{\mu'_i} \mathbf{e}'_U\|^2}{\mu - \mu'_i} = \mu$;

(ii)′ v is a neutral vertex if and only if $\|P_\mu \mathbf{e}_v\| = 0$ and $\displaystyle\sum_{\mu_i \neq \mu} \frac{\|P_{\mu_i} \mathbf{e}_v\|^2}{\mu - \mu_i} \neq 0$, or if and only if $\|P'_\mu \mathbf{e}'_U\| = 0$ and $\displaystyle\sum_{\mu'_i \neq \mu} \frac{\|P'_{\mu'_i} \mathbf{e}'_U\|^2}{\mu - \mu'_i} \neq \mu$;

(iii)′ v is a Parter vertex if and only if $\|P'_\mu \mathbf{e}'_U\| \neq 0$ and $\displaystyle\sum_{\mu_i \neq \mu} \frac{\|P'_{\mu_i} \mathbf{e}'_U\|^2}{\mu - \mu_i} = \mu$.

Clearly, the above conditions are somehow awkward to be used compared with those from Theorem 3.1. On the other hand, at this place, it is noteworthy to say that $\|P_\mu \mathbf{e}_v\| \|P'_\mu \mathbf{e}'_U\| \neq 0$ cannot hold for any graph G.

As already noted any vertex of a connected graph is downer vertex for the largest eigenvalue. For the second largest eigenvalue we have the following situation:

Theorem 3.5. *Let G be a connected graph whose second largest eigenvalue is μ. Then v is a Parter vertex (in G) for μ if and only if v is*

(i) a cut vertex,
(ii) there are no components of $G - v$ with largest eigenvalue greater than μ, and
(iii) at least two of them have the largest eigenvalue equal to μ.

Proof. Assume first that v is a Parter vertex. Let the multiplicity of μ in G be $s \geqslant 1$ (so $m_G(\mu) = s$). Then $m_{G-v}(v) = s + 1$. Therefore, by the interlacing, and in addition since μ is the second largest eigenvalue of G, the largest eigenvalue of $G - v$ is μ but with multiplicity $s + 1$. So the largest eigenvalue of $G - v$ is of multiplicity

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greater than or equal to 2, and $G - v$ is disconnected. Recall, the largest eigenvalue of any connected graph is of multiplicity 1. Consequently, v is a cut vertex in G, and (i) follows. Next, since v is a cut vertex in G, we immediately deduce (ii) and (iii). Otherwise, by the interlacing, $m_{G-v}(v) \leqslant s$.

Conversely, assume that conditions (i)–(iii) from the theorem hold. Let $t \geqslant 2$ be the number of components in $G' = G - v$ whose largest eigenvalue is equal to μ. It is now easy to see that the condition from Theorem 3.1(iii) holds, i.e. that under the notation of that theorem we have that $\|P'_\mu \mathbf{e}'_U\| \neq 0$. The main argument for the latter claim stems from the fact that all entries in P' are non-negative (recall, μ is the largest eigenvalue of G'). So $m_G(\mu) = t - 1$, as required.

This completes the proof. □

Remark 3.6. In view of the second part of the above proof, it is worth mentioning that one collection of $t - 1$ linearly independent μ-eigenvectors for G can be easily constructed by making use of the principal eigenvectors of the observed t components of $G - v$ (then the principal eigenvector of the first component is combined with principal eigenvectors of the remaining $t - 1$ components to get $t - 1$ in total linearly independent eigenvectors).

Note also that the condition (iii) in the theorem above is necessary. Namely, if only one component with the largest eigenvalue μ exists, then it can be easily shown by (2.1) that $\mu \notin \sigma_G$ (i.e., $m_G(\mu) = 0$). On the other hand $m_{G-v}(\mu) = 1$. So the theorem can be extended to hold even in this case if, as already done, we adopt that each number which is not in the spectrum of a graph has it as an eigenvalue of multiplicity equal to 0.

For some additional facts relevant to these considerations the readers are referred to [18].

For bipartite graphs, as already observed, there are no neutral vertices for $\mu = 0$. Moreover, from Theorem 3.1 we immediately obtain the following result:

Corollary 3.7. *If G is a bipartite graph, and if $\mu = 0$, then we have:*

(i) v is a downer vertex for μ if and only if $\|P_\mu \mathbf{e}_v\| \neq 0$ or, equivalently, $\|P'_\mu \mathbf{e}'_U\| = 0$;
(ii) v is a Parter vertex for μ if and only if $\|P_\mu \mathbf{e}_v\| = 0$ or, equivalently, $\|P'_\mu \mathbf{e}'_U\| \neq 0$.

Remark 3.8. For non-zero eigenvalues, due to the symmetry of the spectrum of bipartite graphs with respect to the origin and the fact that the subgraphs of bipartite graphs are also bipartite, we have that all "angles" which appear in Theorem 3.1 are, for $\pm\mu$, the same (by (2.2) and (2.3)). Therefore, each vertex of the graph in question

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is of the same type for such eigenvalues. Note also that the above claim on angles can be obtained geometrically, since the eigenvectors in question (for $\pm\mu$) differ by a reflection in respect to an appropriate hyperplane.

Recall, for the least eigenvalue of a graph there are no Parter vertices. In particular, for graph with least eigenvalue equal to -2 we can say more. Then G is either a generalized line graph (in particular a line graph) or an exceptional graph (for more details, see [6]). If G is a line graph of H (i.e., if $G = L(H)$), then both possibilities can arise:

THEOREM 3.9. *Let $G = L(H)$ be a line graph of a connected graph H. If v is a vertex in G, or equivalently, an edge e in H which corresponds to v, then the following holds for the eigenvalue -2 of G: If H is bipartite, then v is a*

- *downer vertex if $H - e$ is connected;*
- *neutral vertex if e is a bridge in H;*

otherwise, if H is non-bipartite, then v is a

- *downer vertex if $H - e$ is either connected and non-bipartite, or e is a bridge and both components of $H - e$ are non-bipartite;*
- *neutral vertex if either $H - e$ is connected and bipartite, or e is a bridge and one component of $H - e$ is bipartite while the other non-bipartite.*

Proof. Observe first that a bridge in H is a cut vertex in G (see for example Figure 1.1). The rest of the proof immediately follows from the following facts: If H is connected, then $m_{L(H)}(-2) = ||H|| - |H| + 1$ if H is bipartite, or $m_{L(H)}(-2) = ||H|| - |H|$ if H is non-bipartite (see, [6, Theorem 2.2.4]). $\square$

The similar result can be obtained for generalized line graphs (this is left to readers; see [6, Theorem 2.2.8]).

4. Graphs with a cut vertex. Let v be a cut vertex of a connected (simple) graph G. Then

$$G - v = H_1 \cup H_2 \cup \cdots \cup H_k \quad (k > 1),$$

where each H_i is connected ($i = 1, 2, \ldots, k$). For a fixed i, let $G_i = H_i + v$ (note, G_i is the subgraph of G induced by the vertex set $V(H_i) \cup \{v\}$ and $\Gamma_{H_i+v}(v) = \Gamma_G(v) \cap H_i$) and let μ_{ij}'s and P_{ij}'s ($j = 1, 2, \ldots, r_i$) be the (distinct) eigenvalues and the corresponding projection matrices of H_i's. Also let $\rho_{ij} = ||P_{ij}\mathbf{e}_{U_i}||$, where $U_i = \Gamma_{G_i}(v)$. Then, using (2.3), for a fixed $i \in \{1, \ldots, k\}$, we obtain

$$\phi_{G_i}(x) = \phi_{H_i}(x) \left(x - \sum_{j=1}^{r_i} \frac{\rho_{ij}^2}{x - \mu_{ij}} \right). \tag{4.1}$$

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More generally, we have the following formula:

THEOREM 4.1. *Let G be a graph and v its cut vertex. Under the above notation we have:*

$$\phi_G(x) = \phi_{G-v}(x)\left(x - \sum_{i=1}^{k}\sum_{j=1}^{r_i}\frac{\rho_{ij}^2}{x-\mu_{ij}}\right). \tag{4.2}$$

Proof. Applying (2.1) (the Schwenk formula) we obtain

$$\phi_G(x) = x\phi_{G-v}(x) - \sum_{w\in\Gamma_G(v)}\phi_{G-v-w}(x) - 2\sum_{C\in\mathscr{C}(v)}\phi_{G-V(C)}(x),$$

where $\mathscr{C}(v)$ is the set of all undirected cycles in G of length at least three passing through v. Since v is a cut vertex we easily obtain

$$\phi_G(x) = \prod_{i=1}^{k}\phi_{H_i}(x)\left(x - \sum_{i=1}^{k}\left(\sum_{w\in\Gamma_{H_i}(v)}\frac{\phi_{H_i-w}(x)}{\phi_{H_i}(x)} + 2\sum_{C\in\mathscr{C}_i(v)}\frac{\phi_{G_i-V(C)}(x)}{\phi_{H_i}(x)}\right)\right),$$

where $\mathscr{C}_i(v)$ is the set of all undirected cycles in G_i with length at least three passing through v. Applying the Schwenk formula to G_i, we next obtain

$$\phi_G(x) = \prod_{i=1}^{k}\phi_{H_i}(x)\left(x - \sum_{i=1}^{k}\frac{x\phi_{H_i}(x)-\phi_{G_i}(x)}{\phi_{H_i}(x)}\right),$$

or equivalently,

$$\phi_G(x) = \prod_{i=1}^{k}\phi_{H_i}(x)\left(x(1-k) + \sum_{i=1}^{k}\frac{\phi_{G_i}(x)}{\phi_{H_i}(x)}\right).$$

Using (4.1), we obtain

$$\phi_G(x) = \prod_{i=1}^{k}\phi_{H_i}(x)\left(x - \sum_{i=1}^{k}\sum_{j=1}^{r_i}\frac{\rho_{ij}^2}{x-\mu_{ij}}\right),$$

as required. ∎

Assume that $m_{H_i}(\mu) > 0$, for at least one $i \in \{1,\dots,k\}$. Then, from (4.2), it follows:

$$\phi_G(x) = \frac{\prod_{i=1}^{k}\phi_{H_i}(x)}{x-\mu}\left(-\sum_{i=1}^{k}\sum_{j:\,\mu_{ij}=\mu}\rho_{ij}^2 + (x-\mu)\left(x - \sum_{i=1}^{k}\sum_{j:\,\mu_{ij}\neq\mu}\frac{\rho_{ij}^2}{x-\mu_{ij}}\right)\right).$$

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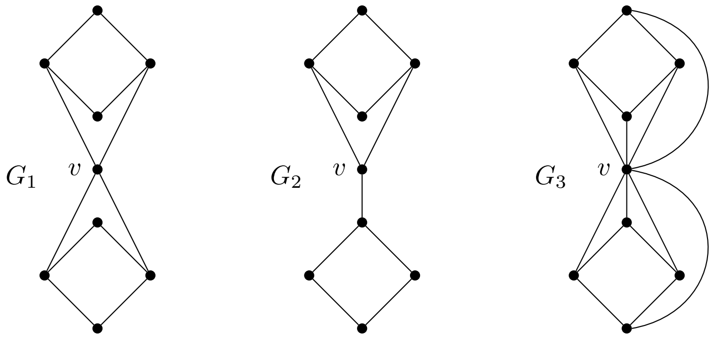

FIG. 4.1. *Graphs of Example 4.3.*

Setting

$$\rho_\mu^2 = \sum_{i=1}^{k} \sum_{j:\, \mu_{ij}=\mu} \rho_{ij}^2,$$

and using the above formula for ρ_μ, we immediately arrive at the following result about Parter vertices:

COROLLARY 4.2. *Under the above assumptions on G and H_i's, the cut vertex v of G is a Parter vertex with respect to μ if and only if for at least one $i \in \{1, \ldots, k\}$, v is a Parter vertex in $G_i = H_i + v$.*

In this context, for the other two types of vertices we have $\rho_\mu = 0$.

EXAMPLE 4.3. Let us consider the graphs in Figure 4.1. Let v be the cut vertex in each G_i above for which $G_i - v = 2C_4$ $(i = 1, 2, 3)$. Then $\sigma_{G_1} = [-2.83, -2, 0^5, 2, 2.83]$, $\sigma_{G_2} = [-2.59, -2, -0.55, 0^3, 0.55, 2, 2.59]$ and $\sigma_{G_3} = [-2^3, 0^4, 2^3, 4]$, while $\sigma_{2C_4} = [-2^2, 0^4, 2^2]$. Therefore, v is a downer vertex in G_1 for $\mu = 0$, while Parter vertex in G_2 also for $\mu = 0$ – this follows also by using Corollary 4.2. On the other hand, since G_3 is not bipartite, then v can be a neutral vertex for $\mu = 0$, as is true for G_3.

REMARK 4.4. From the above corollary, it also follows that we can easily construct a family of graphs such that for each member of the family, one prescribed vertex (which is as well a cut vertex) is always neutral vertex (or Parter).

REMARK 4.5. Recall, in the case of the second largest eigenvalue, a cut vertex is Parter vertex if the assumptions of Theorem 3.5 hold. In this situation, Remark 3.6 also becomes interesting.

5. Vertex types in compound graphs. In this section, we focus our attention only on simple graphs without loops. For an r-regular graph G of order n, let $\bar{G}$ denote

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its complement. Hence, $\bar{G}$ is an $(n-1-r)$-regular graph. If $\sigma_G = \{r, \lambda_2, \ldots, \lambda_n\}$, then $\sigma_{\bar{G}} = \{n-1-r, -\lambda_2-1, \ldots, -\lambda_n-1\}$, see [7, p. 25]. In addition, $\mathscr{E}_G(\lambda_i) = \mathscr{E}_{\bar{G}}(-\lambda_i-1)$, and $\mathscr{E}_G(r) = \mathscr{E}_{\bar{G}}(n-1-r)$. Therefore, it immediately follows:

THEOREM 5.1. *A vertex v is a downer vertex of a regular graph G of degree r with respect to $\mu \in \sigma_G \setminus \{r\}$ if and only if v is a downer vertex in $\bar{G}$ with respect to $-\mu - 1 \in \sigma_{\bar{G}} \setminus \{n-1-r\}$. In addition, v is a downer vertex in G with respect to r if and only if v is a downer vertex in $\bar{G}$ with respect to $n-1-r$.*

REMARK 5.2. It is worth mentioning at this place that in line graphs the classification of vertices corresponds to classification of edges in the root graph, but with respect to the signless Laplacian spectrum (see [7], for definitions and other related details). For these considerations, note that for the Q-spectrum (i.e., signless Laplacian spectrum) the Interlacing Theorem holds for edges (instead of vertices), and also that there exists a simple relationship between the eigenvalues of a graph H with respect to signless Laplacian matrix and the eigenvalues of the graph G $(= L(H))$ with respect to adjacency matrix (see [7, Chapter 7]).

For the Kronecker product of two graphs G and H we have that following holds (see, for example, [7, Chapter 2]:

(i) If A_G and A_H are the adjacency matrices of the graphs G, and H, respectively, then the adjacency matrix of their Kronecker product (i.e., of $K = G \otimes H$) reads:

$$A_K = A_G \otimes A_H.$$

(ii) If $\sigma_G = \{\lambda_1(G), \ldots, \lambda_g(G)\}$ and $\sigma_H = \{\lambda_1(H), \ldots, \lambda_h(H)\}$, then the spectrum of $K = G \otimes H$ consists of all products

$$\lambda_{ij}(K) = \lambda_i(G)\lambda_j(H),$$

where $i = 1, \ldots, g$ and $j = 1, \ldots, h$.

(iii) If $\mathbf{x}_1, \ldots, \mathbf{x}_g$ and $\mathbf{y}_1, \ldots, \mathbf{y}_h$ are eigenbases for G and H, respectively, then $A_K \mathbf{z}_{ij} = \lambda_{ij}(K)\mathbf{z}_{ij}$, where $\mathbf{z}_{ij} = \mathbf{x}_i \otimes \mathbf{y}_j$. Moreover, the eigenbasis of K consists of the vectors

$$\mathbf{z}_{ij} = \mathbf{x}_i \otimes \mathbf{y}_j, \quad (i = 1, \ldots, g \text{ and } j = 1, \ldots, h).$$

Assume now that $\mu_1(G), \ldots, \mu_{r_g}(G)$ and $\mu_1(H), \ldots, \mu_{r_h}(H)$ are the distinct eigenvalues of the corresponding graphs. For a fixed i and j, let $\mathscr{E}_G(\mu_i(G))$ and $\mathscr{E}_H(\mu_j(H))$ be the eigenspaces of the corresponding graphs, and let $\mathscr{B}_G(\mu_i(G))$ and $\mathscr{B}_H(\mu_j(H))$ be the bases of these two eigenspaces. Then the basis of $\mathscr{B}_K(\mu_i(G)\mu_j(H))$ consists of all vectors $\mathbf{z} = \mathbf{x} \otimes \mathbf{y}$, where $\mathbf{x} \in \mathscr{B}_G(\mu_i(G))$ and $\mathbf{y} \in \mathscr{B}_H(\mu_j(H))$ provided the

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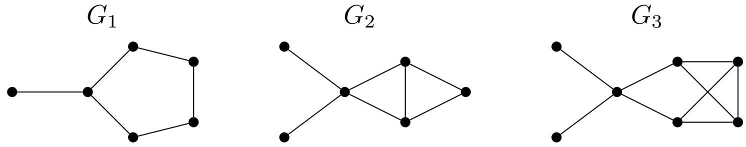

Fig. 5.1. *Graphs of Example 5.4.*

Kronecker product of these two graphs is coincidence-free, that is, $\mu_{ij}(K) \neq \mu_{i'j'}(K)$ whenever $(i,j) \neq (i',j')$. Then we have that

$$\alpha^2_{\mu_{ij}(K),w} = \alpha^2_{\mu_i(G),u}\alpha^2_{\mu_j(H),v},$$

where $w = (u,v)$, $\mu_{ij}(K) = \mu_i(G)\mu_j(H)$ ([5, Proposition 4.3.12]).

More generally, by the same proposition, we have that

$$\alpha^2_{\mu_k(K),w} = \sum_{i\in I}\alpha^2_{\mu_i(G),u}\alpha^2_{\mu_{i^*}(H),v},$$

where $w = (u,v)$ and $I = \{i : \mu_i(G)\mu_{i^*}(H) = \mu_k(K)\}$.

Thus, in general we have:

Theorem 5.3. *Let u be a vertex of G which is a downer in G with respect to $\mu(G)$ and let v be a vertex of H which is a downer in H with respect to $\mu(H)$. Then $w = (u,v)$ is a downer vertex in $G \otimes H$ with respect to $\mu(G)\mu(H)$.*

The converse of the above theorem does not hold. Moreover, from the next example we see that for all other cases concerning vertex types of u and v, nothing can be definitely concluded about the type of $w = (u,v)$.

Example 5.4. In this example, we consider three graphs and Kronecker products of some combinations of them which give rise to all types of vertices (downer, neutral, and Parter); see Figure 5.1.

In the following tables, row and column headers contain possible types of vertices of the first and second graph of the Kronecker product, while each entry shows possible types of the corresponding vertices in the Kronecker product graph. For instance, the third table for $G_2 \otimes G_3$ shows that a downer vertex u for some eigenvalue of G_2 (the row d) and a neutral vertex v for some eigenvalue of G_3 (the column n) can produce downer, neutral, or Parter vertex $u \otimes v$ of the Kronecker product graph $G_2 \otimes G_3$ (the entries d,n,p).

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$G_1 \otimes G_1$

	d	n	p
d	d	d,n	d,n
n	d,n	d,n	d,n
p	d,n	d,n	n

$G_2 \otimes G_2$

	d	n	p
d	d	d,n,p	d,n,p
n	d,n,p	d,n	d,p
p	d,n,p	d,p	d,p

$G_2 \otimes G_3$

	d	n	p
d	d	d,n,p	d,p
n	d,n,p	d,n,p	d,p
p	d,n,p	d,n,p	p

In general, based on the observations derived for simple graphs, we can clearly say that the same conclusions as in Theorem 5.3 and Example 5.4 are valid for the Kronecker product of two (symmetric) matrices.

6. Some further observations. As in the previous sections we now consider only simple graphs and provide some further interesting observations related to the topic in question. The main focus will be put not on a single vertex, but rather on some collections of vertices of a graph.

First, it is obvious that two vertices of some graph have the same type for any eigenvalue if they belong to the same orbit under the action of its automorphism group. So the vertices of the graphs which are, say vertex transitive (i.e., with only one orbit) are all downer vertices (for each eigenvalue).

Secondly, many ideas arising from star sets and star complements, or star partitions (see, for example, [7, Chapter 5]) can be exploited here. We will introduce these concepts only for simple graphs (having in view that they can be easily extended to weighted graphs, or symmetric matrices). Given a graph G, let P be the (orthogonal) projection of the whole space (say, $\mathbb{R}^n$) to the eigenspace of some eigenvalue (say μ) of G. Then, the vectors of the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ are projected in the collection of n vectors, also called (due to Seidel, see [19]) a *eutactic star*. Clearly, all these vectors $P\mathbf{e}_1, P\mathbf{e}_2, \ldots, P\mathbf{e}_n$ span the eigenspace $\mathscr{E}_G(\mu)$. Let $X \subseteq V(G)$ be the set of indices of those vectors $P\mathbf{e}_i$ $(i \in X)$ which form a basis of $\mathscr{E}_G(\mu)$. Then the vertices corresponding to X represent a *star set* of G. The subgraph induced by the set $\bar{X} = V(G) \setminus X$ is called the *star complement* of G. The partition $X_1 \cup X_2 \cup \cdots \cup X_r$ of $V(G)$ such that X_i is a star set for μ_i $(i = 1, 2, \ldots, r)$ is called a *star partition* of G.

It is noteworthy that X is a star set for μ (of multiplicity $k > 0$) in G if and only if $|X| = k$ and μ is not an eigenvalue of $G - X$ (see again [7, Chapter 5]). So it immediately follows that any vertex in X is a downer vertex, not only in G, but also in $G - Y$ for any $Y \subset X$. In other words, for a fixed μ, G has at least k downer vertices.

Needless to add, from the above (geometric) approach to star partitions and star complements, it follows that both of them do exist in any graph for any eigenvalue.

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On the other hand, for star partitions this is also true, but not so evident (see, for example, [5, Chapter 7] for more details). So vertices of any graph, say G, can be partitioned into downer vertices such that the star set X_i (i.e. the corresponding star set for for μ_i) contains exactly $m_G(\mu_i)$ downer vertices for μ_i $(i = 1, 2, \ldots, r)$.

Recall, two (distinct) eigenvalues of a graph are *conjugate* if they are conjugate as algebraic numbers or, equivalently, if they have the same minimal polynomial over the field of rational numbers. Therefore, we have:

THEOREM 6.1. *Let μ and $\bar{\mu}$ be two conjugate eigenvalues of a graph G. Then a vertex v in G is a downer (neutral, or Parter) vertex with respect to μ if and only if it is a downer (neutral, or Parter) vertex with respect to $\bar{\mu}$.*

Proof. Consider the characteristic polynomials of G and $G - v$. Then $\phi_G(x) = 0$ and $\phi_{G-v}(x) = 0$, for $x = \mu$, if and only if the same holds for $x = \bar{\mu}$. Moreover, their algebraic multiplicities are the same, and so the conclusion follows. □

As an immediate consequence we now obtain:

COROLLARY 6.2. *If $\mu \in \sigma_G$ is an algebraic number whose minimal polynomial is of degree $d(\mu)$, then the number of downer vertices for μ is at least equal to $d(\mu)m_G(\mu)$.*

It is also noteworthy that for a given size of the star complement, say t, the size of the star cell (provided $\mu \neq -1, 0$) is bounded. Namely, if μ is not an integer (i.e., if it is an algebraic irrational number), then $|X| < t$ (since a star cell for its algebraic conjugate is contained in the corresponding star complement), while otherwise if μ is an integer, and if $|G| > 4$, then $|X| \leqslant \binom{t}{2}$ (cf. [7, Chapter 5]).

Apart from these results, there are many others and in sequel a few more will be mentioned. Namely, for a given graph G, its eigenvalue μ (of multiplicity $k > 0$) we also have:

(i) if $\mu \neq 0$ and G is connected, then any vertex from a star set (so a downer vertex) has a neighbour in the corresponding star complement (see [7, Theorem 5.1.4.]);
(ii) if $\mu \neq 0, -1$ any two vertices in the star set (so downer vertices) cannot have the same neighbours in the star complement (see, for example, [7, Theorem 5.1.4.]);
(iii) each Fiedler vertex for μ belongs to each star complement of G for μ;
(iv) if G is connected, then it contains a connected star complement for each μ (so k downer vertices can be deleted from a connected graph G to obtain also a connected graph) (see [7, Theorem 5.1.6.]).

The most striking one (the Reconstruction Lemma) reads: Given a graph H as a

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star complement of G, X as a corresponding star set and the neighbours in H of all vertices from X (so for downer vertices for μ), then for any two vertices in X we can decide (uniquely) if they are adjacent or not.

Finally, we can add that the vertices as classified in this paper have an important role in various structural properties of graphs such as domination properties (see, [5, 7] and references therein), or connectivity properties (see [15]), etc., mostly studied by Rowlinson. Needless to add, most of the results above can be easily generalized to weighted graphs. In the next section we will not pursue these kind of results, but rather postpone them for some of forthcoming paper(s).

7. Weighted graphs. In this section, we generalize some of the results obtained for simple graphs to weighted graphs. Most of them are technical, and so stated without proofs. Besides replacing G with $\hat{G}$, usually only some simple adjustments are needed.

We first generalize the results introduced in Section 2. The generalized form of the Schwenk formula reads (see, for example, [1]):

$$\begin{aligned}\phi_{\hat{G}}(x) &= (x - a_{vv})\phi_{\hat{G}-v}(x) \\ &\quad - \sum_{w\in\Gamma_G(v)} a_{vw}^2 \phi_{\hat{G}-v-w}(x) - 2\sum_{C\in\mathscr{C}(v)} \left(\prod_{ij\in E(C)} a_{ij}\right)\phi_{\hat{G}-V(C)}(x), \end{aligned} \tag{7.1}$$

where $\mathscr{C}(v)$ is the set of all undirected cycles in G with length greater than 2 passing through v. Here we also assume that $\phi_{\hat{H}}(x) = 1$ if $|H| = 0$.

In the case of weighted graphs, formula (2.2) holds as well and reads:

$$\phi_{\hat{G}-v}(x) = \phi_{\hat{G}}(x)\sum_{i=1}^{r} \frac{\|P_i\mathbf{e}_v\|^2}{x-\mu_i}\,. \tag{7.2}$$

In contrast, (2.3) has to be modified.

THEOREM 7.1. *Let $\hat{G}$ be a weighted graph, and $\sum_{i=1}^{m}\mu_i P_i$ its spectral decomposition. Let $\hat{G}' = \hat{G} + v$ be the one-vertex extension of $\hat{G}$, a_{vv} the weight of a loop at v (if any), and $\mathbf{w}$ a vector containing the weights of the edges incident to v. Then*

$$\phi_{\hat{G}'}(x) = \phi_{\hat{G}}(x)\left(x - a_{vv} - \sum_{i=1}^{r}\frac{\|P_i\mathbf{w}\|^2}{x-\mu_i}\right). \tag{7.3}$$

Proof. Since $\mathrm{adj}(xI - \hat{A}) = \det(xI-\hat{A})(xI-\hat{A})^{-1} = \det(xI-\hat{A})\sum_{i=1}^{r}\frac{P_i}{x-\mu_i}$, using the well-known determinant expansion formula (see [7, p. 38]) for 2×2 block

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matrices, we obtain

$$\begin{aligned}\phi_{\hat{G}'}(x) &= \begin{vmatrix} x - a_{vv} & -\mathbf{w}^T \\ -\mathbf{w} & xI - \hat{A} \end{vmatrix} \\ &= (x - a_{vv}) \det(xI - \hat{A}) - \mathbf{w}^T \operatorname{adj}(xI - \hat{A})\mathbf{w} \\ &= \phi_{\hat{G}}(x) \left(x - a_{vv} - \sum_{i=1}^{r} \frac{\|P_i \mathbf{w}\|^2}{x - \mu_i} \right). \quad \square\end{aligned}$$

The weighted counterpart of Theorem 3.1 now reads:

THEOREM 7.2. *Let $\hat{G}$ be a weighted graph of order $n > 1$, v a vertex of $\hat{G}$, and let $\hat{G}' = \hat{G} - v$. Let*

$$\hat{A}_{\hat{G}} = \sum_{\mu \in \sigma_G} \mu P_\mu \qquad \text{and} \qquad \hat{A}_{\hat{G}'} = \sum_{\mu' \in \sigma_{G'}} \mu' P'_{\mu'}$$

be the spectral decompositions of $\hat{G}$ and $\hat{G}'$, respectively. If $\mu \in \sigma_{\hat{G}}$, then

(i) v is a downer vertex if and only if $\|P_\mu \mathbf{e}_v\| \neq 0$;
(ii) v is a neutral vertex if and only if $\|P_\mu \mathbf{e}_v\| = 0$ and $\|P'_\mu \mathbf{w}'\| = 0$;
(iii) v is a Parter vertex if and only if $\|P'_\mu \mathbf{w}'\| \neq 0$,

where $\mathbf{w}' = \sum_{i \in \{1,\dots,n\} \setminus \{v\}} \hat{a}_{iv} \mathbf{e}'_i$. [2]

We next consider the weighted counterparts of results from Section 4. Let v be a cut vertex of a connected weighted graph $\hat{G}$. We understand here that v is a cut vertex of $\hat{G}$ if it is a cut vertex of G and $\hat{G}$ is connected if G is connected. Suppose that

$$\hat{G} - v = \hat{H}_1 \cup \hat{H}_2 \cup \cdots \cup \hat{H}_k \,,$$

where each $\hat{H}_i$ is connected, for $i = 1, \dots, k$. Let $\hat{G}_i = \hat{H}_i + v$. So, G_i is the subgraph of G induced by the vertex set $V(H_i) \cup \{v\}$.

The weight functions of $\hat{H}_i$ and $\hat{G}_i$ are the restrictions of $\hat{w}$, the weight function of $\hat{G}$, on $E(H_i)$ and $E(G_i)$, respectively. We can also assume that a vertex corresponding to v has a loop of weight $\hat{a}_{vv}^{(i)}$, where $\sum_{i=1}^{k} \hat{a}_{vv}^{(i)} = \hat{a}_{vv}$. Using (7.3), for a fixed i in $\{1, \dots, k\}$, we obtain

$$\phi_{\hat{G}_i}(x) = \phi_{\hat{H}_i}(x) \left(x - \hat{a}_{vv}^{(i)} - \sum_{j=1}^{r_i} \frac{\rho_{ij}^2}{x - \mu_{ij}} \right), \tag{7.4}$$

where $\rho_{ij} = \|P_{ij} \mathbf{w}_i\|$, while μ_{ij}, P_{ij}, and $\mathbf{w}_i$ refer to $\hat{H}_i$ (see Theorem 7.1).

[2] Observe that $\mathbf{e}'_s \in \mathbb{R}^{n-1}$, while $\mathbf{e}_t \in \mathbb{R}^n$.

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Next we generalize (2.3).

THEOREM 7.3. *Under the above notation we have:*

$$\phi_{\hat{G}}(x) = \phi_{\hat{G}-v}(x) \left(x - \hat{a}_{vv} - \sum_{i=1}^{k} \sum_{j=1}^{r_i} \frac{\rho_{ij}^2}{x - \mu_{ij}} \right). \tag{7.5}$$

Proof. Applying the general form of the Schwenk formula (cf. [1, Theorem 3.2]), we have

$$\phi_{\hat{G}}(x) = (x - \hat{a}_{vv})\phi_{\hat{G}-v}(x) - \sum_{u:\, u \neq v} (\hat{a}_{uv})^2 \phi_{\hat{G}-v-u}(x) - 2 \sum_{C \in \mathscr{C}(v)} \left(\prod_{ij \in E(C)} \hat{a}_{ij} \right) \phi_{\hat{G}-V(C)}(x),$$

where $\mathscr{C}(v)$ is the set of all undirected cycles in G of length at least three passing through v. Since v is a cut vertex we easily obtain

$$\begin{aligned} &\phi_{\hat{G}}(x) \\ &= \prod_{i=1}^{k} \phi_{\hat{H}_i}(x) \left(x - \hat{a}_{vv} - \sum_{i=1}^{k} \left(\sum_{\substack{u \sim v \\ u \in V(\hat{H}_i)}} \hat{a}_{uv}^2 \frac{\phi_{\hat{H}_i - u}(x)}{\phi_{\hat{H}_i}(x)} + 2 \sum_{C \in \mathscr{C}_i(v)} \prod_{ij \in E(C)} \hat{a}_{ij} \frac{\phi_{G_i - V(C)}(x)}{\phi_{H_i}(x)} \right) \right), \end{aligned}$$

where $\mathscr{C}_i(v)$ is the set of all undirected cycles in G_i with length at least three passing through v. Applying now the Schwenk formula to G_i, we next obtain

$$\phi_{\hat{G}}(x) = \prod_{i=1}^{k} \phi_{\hat{H}_i}(x) \left(x - \hat{a}_{vv} - \sum_{i=1}^{k} \frac{(x - a_{vv}^{(i)})\phi_{\hat{H}_i}(x) - \phi_{\hat{G}_i}(x)}{\phi_{\hat{H}_i}(x)} \right),$$

or, equivalently,

$$\phi_{\hat{G}}(x) = \prod_{i=1}^{k} \phi_{\hat{H}_i}(x) \left(x(1-k) + \sum_{i=1}^{k} \frac{\phi_{\hat{G}_i}(x)}{\phi_{\hat{H}_i}(x)} \right).$$

Using (7.4), we get

$$\phi_{\hat{G}}(x) = \prod_{i=1}^{k} \phi_{\hat{H}_i}(x) \left(x - \hat{a}_{vv} - \sum_{i=1}^{k} \sum_{j=1}^{r_i} \frac{\rho_{ij}^2}{x - \mu_{ij}} \right),$$

as required. ∎

Assume that $m_{\hat{H}_i}(\mu) > 0$, for at least one $i \in \{1, \ldots, k\}$. Then, from (7.5), it follows:

$$\phi_{\hat{G}}(x) = \frac{\prod_{i=1}^{k} \phi_{\hat{H}_i}(x)}{x - \mu} \left(-\sum_{i=1}^{k} \sum_{j:\, \mu_{ij} = \mu} \rho_{ij}^2 + (x - \mu) \left(x - \hat{a}_{vv} - \sum_{i=1}^{k} \sum_{j:\, \mu_{ij} \neq \mu} \frac{\rho_{ij}^2}{x - \mu_{ij}} \right) \right).$$

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Setting

$$\rho_\mu^2 = \sum_{i=1}^{k} \sum_{j:\,\mu_{ij}=\mu} \rho_{ij}^2 \,,$$

we immediately arrive at the following result about Parter vertices:

COROLLARY 7.4. *Under the above assumptions on $\hat{G}$ and $\hat{H}_i$'s, the cut vertex v of G is a Parter vertex with respect to μ if and only if for at least one $i \in \{1, \dots, k\}$, v is a Parter vertex in $\hat{G}_i = \hat{H}_i + v$.*

8. Concluding remarks. Some of the results presented in this paper (to be more precise, the ones from Sections 3 and 4) were known in the matrix theory, but not in the spectral graph theory. The proofs given here are deduced (or generalized) from some well-known facts from the spectral graph theory. Also many useful observations typical for the spectral graph theory are provided (in Sections 3–6). On the other hand, some facts not known in the matrix theory are generalized from simple graphs to weighted graphs (or matrices).

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