Further results on generalized inverses in rings with involution

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FURTHER RESULTS ON GENERALIZED INVERSES
IN RINGS WITH INVOLUTION

N. CASTRO-GONZÁLEZ†, JIANLONG CHEN‡, AND LONG WANG‡

Abstract. Let $R$ be a unital ring with an involution. Necessary and sufficient conditions for the existence of the Bott-Duffin inverse of $a \in R$ relative to a pair of self-adjoint idempotents $(e, f)$ are derived. The existence of a $\{1,3\}$-inverse, $\{1,4\}$-inverse, and the Moore-Penrose inverse of a matrix product is characterized, and explicit formulas for their computations are obtained. Some applications to block matrices over a ring are given.

Key words. Ring, outer inverse, Bott-Duffin-inverse, $\{1,3\}$-inverse, $\{1,4\}$-inverse, Moore-Penrose inverse.

AMS subject classifications. 16W10, 15A09.

1. Introduction. Let $R$ be an associative ring with unity $1$. The set of all idempotent elements of $R$ will be denoted by $E(R)$. Let $a \in R$ and $e \in E(R)$ such that $ae + 1 - e$ is invertible. Then the Bott-Duffin $e$-inverse of $a$ (see [3, Chapter 2, Section 10]) is defined as the element $y = e (ae + 1 - e)^{-1}$. It is an outer inverse for $a$, i.e., $yay = y$.

Let $e, f \in E(R)$. Djordjević and Wei introduced a type of outer inverse by prescribing the idempotents $ya$ and $ay$ in [8]: The $(e, f)$-outer generalized inverse of $a$ is the unique element $y \in R$, whenever it exists, satisfying $yay = y, ya = e, ay = 1 - f$.

A characterization of the existence of the $(e, f)$-outer generalized inverse was given in [8, Theorem 2.1].

For $a \in R$, we associate the image and kernel ideals:

$$aR = \{ax : x \in R\}, \quad a^0 = \{x \in R : ax = 0\}.$$
Kantún-Montiel [14] explored the idea of prescribing the image ideal \( yaR \) and the kernel ideal \((ay)^0\) related to the outer inverse: The image-kernel \((e, f)\)-inverse of \( a \) is the unique element \( y \in R \), whenever it exists, satisfying
\[
yay = y, \quad yaR = eR, \quad (ay)^0 = (1 - f)^0.
\]

If \( y \) is the \((e, f)\)-outer generalized inverse of \( a \), then it is the image-kernel \((e, f)\)-inverse of \( a \). The converse part is as follows. If \( y \) is the image-kernel \((e, f)\)-inverse of \( a \), then it is the \((eya, f(1 - ay))\)-outer generalized inverse of \( a \). Elements with equal idempotents related to their image-kernel \((e, f)\) - inverses are characterized in [18]. The representation and approximation for the outer inverse having prescribed range and null space in the setting of complex matrices were given in [23].

Drazin in [9, Definition 3.2] introduced the following generalization of the Bott-Duffin inverse relative of a pair of idempotents: The Bott-Duffin \((e, f)\)-inverse of \( a \) is the unique element \( y \in R \), whenever it exists, such that
\[
y = ey = yf, \quad yae = e, \quad fay = f.
\] (1.1)

We abbreviate Bott-Duffin to B-D. It was showed in [14, Proposition 3.4] that \( y \) is the image-kernel \((e, f)\)-inverse of \( a \) if and only if it is the B-D \((e, 1 - f)\)-inverse of \( a \).

By [9, Theorem 2.2], we know that there exists a B-D \((e, f)\)-inverse of \( a \) if and only if \( e \in Rfae \) and \( f \in faeR \).

On account of the above result, for \( e = f \) the equations in (1.1) have a common solution iff \( e \in Reae \cap eaeR \). This is equivalent to the invertibility of \( ea + 1 - e \), see Lemma [24]. In fact, the element for which (1.1) holds is precisely the classical Bott-Duffin \( e \) - inverse of \( a, y = (ae + 1 - e)^{-1} \).

We ask whether the existence of B-D \((e, f)\)-inverse can be characterized in terms of classical invertibility. We present a result in Section 2 to answer this question in the setting of a ring with an involution under the assumption that both \( e \) and \( f \) are self-adjoint idempotents.

We recall that \( * \) is an involution in \( R \) if it is a map \( * : R \to R \) such that for all \( a, b \in R \): \((a*)* = a, (a + b)* = a* + b*, (ab)* = b*a*\). The set of all idempotent self-adjoint elements of \( R \) \((e = e^2 = e*)\) will be denoted by \( E^e(R) \).

Let \( M_{m \times n}(R) \) denote the set of \( m \times n \) matrices over \( R \) and let \( M_m(R) \) denote the ring of \( m \times m \) matrices over \( R \). For any matrix \( A = (a_{ij}) \in M_{m \times n}(R), A^* \in M_{n \times m}(R) \) stands for \((A)^T\) where \( A = (a^*_{ij}) \).

A matrix \( A \in M_{m \times n}(R) \) is said to be Moore-Penrose invertible with respect to
the involution $\ast$ if the equations

\begin{align*}
1) \quad AXA &= A, \\
2) \quad XAX &= X, \\
3) \quad (AX)^\ast &= AX, \\
4) \quad (XA)^\ast &= XA
\end{align*}

have a unique common solution. Such a solution, when exists, is denoted by $A^\dagger$.

$A$ is called regular if there exists $X$ satisfying (1). Let $A\{1\}$ denote the set of matrices $X \in M_{m \times n}(\mathbb{R})$ which satisfy equation (1).

If $X$ is a solution of both (1) and (3) then it is called a $\{1,3\}$-inverse of $A$. Similarly, if $X$ is a solution of both (1) and (4) then it is called a $\{1,4\}$-inverse of $A$.

We will consider the following sets:

\begin{align*}
A\{1,3\} &= \{X \in A\{1\} : (AX)^\ast = AX\}, \\
A\{1,4\} &= \{X \in A\{1\} : (XA)^\ast = XA\}.
\end{align*}

Necessary and sufficient conditions for the existence of $\{1,3\}$-inverse, $\{1,4\}$-inverse and the Moore-Penrose inverse were presented in [22, Proposition 3.10]. When $A$ is regular, the existence of $A^\dagger$ was characterized by means of classical invertibility, see [20, Remark 3] and [19, Theorem 1.1]:

**Lemma 1.1.** Let $A \in M_{m \times n}(\mathbb{R})$ be regular and let $A^{(1)}$ be an arbitrary element of $A\{1\}$. Then the following conditions are equivalent:

1. $A^\dagger$ exists (with respect to $\ast$).
2. $U = AA^\ast + I_m - AA^{(1)}$ is invertible.
3. $V = A^\ast A + I_n - A^{(1)}A$ is invertible.

In this case,

\[ A^\dagger = A^\ast (U^\ast)^{-1} = (V^\ast)^{-1} A^\ast. \]

The existence of the Moore-Penrose inverse of a matrix product $PAQ$ was studied in [10, 20]. We recall that if $P$ and $Q$ are both invertible then the Moore-Penrose inverse of $PAQ$ exists if and only if $PA$ has a $\{1,3\}$-inverse and $AQ$ has a $\{1,4\}$-inverse, in which case

\[ (PAQ)^\dagger = (AQ)^{\{1,4\}} A (PA)^{\{1,3\}}, \]  \hspace{1cm} (1.2)

where $(PA)^{\{1,3\}}$ and $(AQ)^{\{1,4\}}$ are arbitrary elements of $(PA)\{1,3\}$ and $(AQ)\{1,4\}$, respectively.

In Section 3, Theorems 3.1, 3.4 and 3.6 provide necessary and sufficient conditions for the existence of a $\{1,3\}$-inverse of $PA$, a $\{1,4\}$-inverse of $AQ$, and the Moore-Penrose inverse of $PAQ$, respectively, under some conditions. We also give explicit
formulas for the computation of these generalized inverses. In Section 4, we consider some applications of our results to block matrices.

For a treatment of generalized inverses of block matrices over a ring we refer the reader to [12].

2. Bott-Duffin inverses in involutory rings. Let $\mathcal{R}$ be a ring with unity 1 and an involution $\ast$. Let $e, f \in E^*(\mathcal{R})$, in this section we derive necessary and sufficient conditions for the existence of the Bott-Duffin $(e, f)$-inverse, as well as an explicit formula for its computation.

It will be convenient to introduce the following sets. For $e \in E(\mathcal{R})$, we consider

$$e\mathcal{R}e + 1 - e = \{exe + 1 - e : x \in \mathcal{R}\},$$

which is a submonoid of $\mathcal{R}$ under multiplication and the group $U_e$ of $e$-units in the subring $e\mathcal{R}e$ (corner ring) given by

$$U_e = \{exe : exe\mathcal{R} = e\mathcal{R}, \mathcal{R}exe = \mathcal{R}e\}.$$

Next known result links invertible elements in $e\mathcal{R}e + 1 - e$ and elements of $U_e$.

**Lemma 2.1.** Let $a \in \mathcal{R}$ and $e \in E(\mathcal{R})$. Then the following conditions are equivalent:

(i) $e \in eae\mathcal{R} \cap \mathcal{R}ae$.

(ii) $eae + 1 - e$ is invertible.

(iii) $ae + 1 - e$ is invertible.

(iv) $eae \in U_e$.

In this case, the $e$-inverse of $eae$ in $U_e$ is given by

$$(eae)^{-1}_{e\mathcal{R}e} = e(eae + 1 - e)^{-1}e.$$  \hspace{1cm} (2.1)

We can now formulate our main result of this section.

**Theorem 2.2.** Let $a \in \mathcal{R}$ and $e, f \in E^*(\mathcal{R})$. Then the following conditions are equivalent:

(a) There exists a Bott-Duffin $(e, f)$-inverse of $a$.

(b) $e \in \mathcal{R}(fae)^{\ast}fae$ and $f \in fae(fae)^{\ast}\mathcal{R}$.

(c) $u = (fae)^{\ast}fae + 1 - e$ is invertible and $faeu^{-1}(fae)^{\ast} = f$.

(d) $v = fae(fae)^{\ast} + 1 - f$ is invertible and $(fae)^{\ast}v^{-1}fae = e$.

(e) Both $u = (fae)^{\ast}fae + 1 - e$ and $v = fae(fae)^{\ast} + 1 - f$ are invertible.

(f) $ea^\ast fae \in U_e$ and $faea^\ast f \in U_f$.
In this case, the B-D \((e,f)\)-inverse of \(a\) is given by
\[
y = u^{-1}(fae)^* = (fae)^*v^{-1} = (ea^*fae)^{-1}\in R^{(a^*,fae)} f = ea^*(faea^*)^{-1}f_{Rf}.
\] (2.2)

Proof. (a) \(\Rightarrow\) (b). Let \(e \in \mathcal{R}\) be a common solution of equations in (1.1). Then \(e = yfae\) and \(f = faey\). By substituting \(e = (fae)^*y^*\) into the last identity we get \(f = fae(fae)^*y^*y\) and thus, \(f \in fae(fae)^*\mathcal{R}\). Similarly, by substituting \(f = y^*(fae)^*\) in \(e = yfaw\) we obtain \(e = yy^*(fae)^*fae\) and \(e \in \mathcal{R}(fae)^*fae\).

(b) \(\Rightarrow\) (c). Suppose that there exists \(s,t \in \mathcal{R}\) such that \(e = s(fae)^*fae\) and \(f = fae(fae)^*t\). Since \(e = e^*\) we have \(e = (fae)^*faes^*\) and it follows that \(se = es^*\). Then
\[
(se + 1 - e)((fae)^*fae + 1 - e) = ((fae)^*fae + 1 - e)(es^* + 1 - e) = 1.
\]
Hence, \(x = se + 1 - e\) is the inverse of \(u = (fae)^*fae + 1 - e\). Further, we have
\[
fae^{-1}(fae)^* = fae^{-1}(fae)^*fae(fae)^*t = fae(fae)^*t = f.
\]

(c) \(\Leftrightarrow\) (d). We prove that (c) implies (d). Let \(v = fae(fae)^* + 1 - f\). Using the relation \(fae^{-1}(fae)^* = f\), we obtain \(v = 1 + fae(1 - u^{-1})(fae)^*\). Hence, \(v\) is invertible if and only if \(1 + (1 - u^{-1})(fae)^*fae\) is invertible. But this last element is equal to \(u^{-1}(fae)^*fae = e\). Now, it is easy to check that \((fae)^*v = u(fae)^*\) and, hence, \((fae)^*v^{-1} = u^{-1}(fae)^*\). Then \((fae)^*v^{-1}fae = u^{-1}(fae)^*fae = e\).

In the same manner, we can see that (d) implies (c).

(d) \(\Rightarrow\) (e). On account of the above equivalence, this implication is immediate.

(e) \(\Leftrightarrow\) (f). It follows by Lemma 2.1.

(e) \(\Rightarrow\) (a). Suppose that both \(u\) and \(v\) are invertible. Now, we will prove that \(y = (fae)^*v^{-1} = u^{-1}(fae)^*\) is a common solution of equations in (1.1). Clearly, \(y = ey = fy\). Now, using this relation,
\[
yae = yfae = (fae)^*v^{-1}fae = u^{-1}(fae)^*fae = e.
\]
In the same manner, we see that \(fay = f\), and thus \(y\) the B-D \((e,f)\)-inverse of \(a\). The last two identities in (2.2) are clear by (2.1) \(\square\).

We observe that if the Moore-Penrose inverse of \(a\) exists, then it is the B-D \((e,f)\)-inverse of \(a\) with \(e = a^ta\) and \(f = aa^t\). In this case, the element \(u\) given in item (c) and \(v\) given in item (d) of the above theorem are of the form \(u = a^ta + 1 - a^ta\) and \(v = aa^t + 1 - aa^t\), which are invertible whenever \(a^t\) exists, see Lemma 2.4. Koliha et al. in [13] Theorem 1] established a relation between Moore-Penrose invertible and well-supported elements in a ring with involution.
Further Results on Generalized Inverses in Rings With Involution

We specialize the preceding theorem to Bott-Duffin e-inverse.

**Corollary 2.3.** Let \( a \in \mathcal{R} \) and \( e \in E^*(\mathcal{R}) \). Then the following conditions are equivalent:

1. There exists a B-D e-inverse of \( a \).
2. \( e \in \mathcal{R}(eae^*eae \cap eae(eae)^*)^* \).
3. \( u = (eae)^*eae + 1 - e \) is invertible and \( eaeu^{-1}(eae)^* = e \).
4. \( v = eae(eae)^* + 1 - e \) is invertible and \( (eae)^*v^{-1}eae = e \).
5. Both \( u = (eae)^*eae + 1 - e \) and \( v = eae(eae)^* + 1 - e \) are invertible.
6. \( ea^*eae \in U_e \) and \( eaea^*e \in U_e \).

In this case, \( y = u^{-1}(eae)^* = (eae)^*v^{-1} \) is the B-D e-inverse of \( a \).

Next, we consider the product \( paq \). We are interested in establishing a relation between the B-D \((e, f)\)-inverse of \( paq \) and certain classes of generalized inverses of \( pa \) and \( aq \).

**Theorem 2.4.** Let \( a, p, q \in \mathcal{R} \) and let \( e, f \in E(\mathcal{R}) \). Then the following conditions are equivalent:

1. \( paq \) has a B-D \((e, f)\)-inverse \( y \).
2. There exist \( x, z \in \mathcal{R} \) such that:
   
   \[
   x = ex, \quad x(aq)e = e, \quad f(paq)x = fp \tag{2.3}
   
   \]
   
   \[
   z = zf, \quad f(pa)z = f, \quad z(pa)qe = qe. \tag{2.4}
   
   \]

In this case, we have \( y = xaz \), where \( x \) and \( z \) are any solution of (2.3) and (2.4), respectively.

**Proof.** First, let \( y \) be the B-D \((e, f)\)-inverse of \( paq \). Then

\[
 y = ey = yf, \quad ypaq = e, \quad fpaqy = f. \tag{2.5}
\]

We will prove that \( x = yp \) satisfies (2.3). From \( y = ey \) it follows \( yp = eyp \), and hence, \( x = ex \). Since \( ypaq = e \) we also have \( xaqe = e \). Using \( fpaqy = f \) it follows that \( fpaqyp = fpaqx = fp \). Analogously, we can prove that \( z = qy \) satisfies (2.4).

Conversely, let \( x \) be any solution of (2.3) and let \( z \) be any solution of (2.4). Define \( y = xaz \). We will prove that \( y \) is a common solution of equations in (2.5). Clearly, \( y = ey = yf \). Now, \( ypaq = xaz(pa)qe = xaqe = e \) and \( fpaqy = fpaqxaz = fpaz = f \).

**Remark 2.5.** Conditions on \( x \) in (2.3) (or conditions on \( z \) in (2.4)) are not sufficient to ensure the uniqueness as we show in the following example. Let \( \mathcal{R} = \mathcal{M}_2(\mathbb{Z}_{12}) \) be the ring of \( 2 \times 2 \) matrices over \( \mathbb{Z}_{12} \), \( a = \begin{pmatrix} 5 & 5 \\ 0 & 0 \end{pmatrix} \), \( p = q = I \), and let
the idempotents $e = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ be given. Then $x = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$ and $\hat{x} = \begin{pmatrix} 5 & 3 \\ 0 & 0 \end{pmatrix}$ are two different solutions of (2.3).

3. Generalized inverses of a matrix product. Theorems 3.1, 3.4 and 3.6 give the existence of $(PA)^{(1,3)}$, $(AQ)^{(1,4)}$ and the Moore-Penrose invertibility of $PAQ$ from the classical invertibility of matrices. These characterization results in the setting of matrices over a ring are news. Formulae (3.1), (3.3) and (3.4) are extensions to matrices over a ring of similar formulae obtained for matrices over the complexes in [5]. We recall that if $*$ is the conjugate transpose of a complex matrix, then a $\{1,3\}$-inverse, a $\{1,4\}$-inverse, and the Moore-Penrose inverse of a complex matrix always exist.

In what follows, $E'$ denotes the matrix $I - E$ for any idempotent matrix $E$.

We begin by giving a characterization for a matrix product $PA$ to have a $\{1,3\}$-inverse when a $\{1,3\}$ exists.

**Theorem 3.1.** Let $A \in \mathcal{M}_{m \times n}(R)$ be such that a $A^{(1,3)}$ exists and let $E = AA^{(1,3)}$, and $P \in \mathcal{M}_m(R)$ be invertible. If $PE' = E'$ then the following conditions are equivalent:

(a) $PA$ has a $\{1,3\}$-inverse.

(b) $E \in \mathcal{M}_m(R)Z \cap Z\mathcal{M}_m(R)$ where $Z = EP^*PE$.

(c) $U = P^*PE + I - E$ is invertible.

(d) $I + R^*R$ is invertible with $R = E'(I - P^{-1})$.

In this case, there exists a $\{1,3\}$-inverse of $PA$ of the form

$$(PA)^{(1,3)} = A^{(1,3)}U^{-1}P^* = A^{(1,3)}P^{-1}(I + R^*R)^{-1}(I + R^*) \tag{3.1}$$

**Proof.** Let us first observe that if $X$ be an arbitrary element of $A\{1,3\}$, then

$X = A^{(1,3)} + (I - A^{(1,3)})Z$ with $Z \in \mathcal{M}_{n \times m}(R)$ and, thus, $AX = AA^{(1,3)}$. Therefore, if $PE' = E'$ we also have $P(I - AX) = I - AX$.

(a)$\Rightarrow$(b). If $PA$ has a $\{1,3\}$-inverse $Y$ then $Y^*A^*P^* = PAY$ and $PA = PAYPA$. Hence, $E = AA^{(1,3)} = AYPE = P^{-1}(PAY)PE = SEP^*PE$, where $S = P^{-1}Y^*A^*$. Since $E^* = E$, then it also follows that $E = EP^*PES^* \in EP^*PE\mathcal{M}_m(R)$ and, (b) holds.

(b)$\Leftrightarrow$(c). This equivalence follows from Lemma 2.1.
(c)⇔(d). Let \( R = E'(I - P^{-1}) \). Using \( R^2 = 0 \), we can write
\[
I + R^*R = I + R^* = R^*P^{-1} = (I + R^*)(I - R^*P^{-1})
\]
\[
= (I + R^*)(PE + (P^{-1})^*E')P^{-1} = (I + R^*)(P^{-1})^*UP^{-1},
\]
where \( U = P^*PE + I - E \). Since \( I + R^* \) is invertible, then \( I + R^*R \) is invertible if and only if \( U \) is invertible. Moreover,
\[
(I + R^*R)^{-1} = P^{-1}P^*(I - R^*).
\]
From this, we also obtain that \( U^{-1}P^* = P^{-1}(I + R^*R)^{-1}(I + R^*) \) whenever (c) holds and, thus, the second equality of (3.1) holds.

(c)⇒(a). Define \( Y = A^{(1,3)}U^{-1}P^* \). We will prove that \( Y \) is a \( \{1, 3\} \)-inverse of \( PA \). Firstly, we see that
\[
YPA = A^{(1,3)}U^{-1}P^*PA = A^{(1,3)}U^{-1}(P^*PE + I - E)A = A^{(1,3)}A. \tag{3.2}
\]
Then \( PAYPA = PA \), and thus, \( Y \) is a \( \{1\} \)-inverse of \( PA \). Since \( U^*E = EU \) it follows that \( (PAY)^* = P(U^{-1})^*EPP^* = PEP^{-1}P^* = PAY \) and so \( Y \in (PA)\{1, 3\} \).

Remark 3.2. If we replace \( A^{(1,3)} \) and \( (PA)^{(1,3)} \) by \( A^{(1,2,3)} \) and \( (PA)^{(1,2,3)} \), respectively, in Theorem 3.1 then we obtain an analogous characterization of the existence of \( \{1, 2, 3\} \)-inverses of the product \( PA \).

Some applications of previous results will be develop in Section 4. Here we include an example using the incidence matrix of a graph.

Example 3.3. Let \( * \) be the conjugate transpose of a complex matrix and let \( A \) be an \( m \times n \) incidence matrix of a connected graph. With an application of formula 3.1 we will derive an expression of a \( \{1, 3\} \)-inverse of \( PA \) when \( P \) is an \( m \times m \) invertible row stochastic matrix.

For any \( A^{(1,3)} \), denote \( E = AA^{(1,3)} \), we have \( E^* = \frac{1}{m}ee^T \) where \( ee^T \) is the \( m \times m \) matrix whose elements are all 1 (see [3] Ex. 109). Since \( P \) is row stochastic, then \( Pee^T = ee^T \) and, thus, \( PE^T = E^* \) holds. Therefore, by (3.1),
\[
(PA)^{(1,3)} = A^{(1,3)}U^{-1}P^* = A^{(1,3)}P^{-1}(I + R^*R)^{-1}(I + R^*),
\]
where \( U = P^*P - (P^*P + I)\frac{1}{m}ee^T \) and \( R = \frac{1}{m}ee^T(I - P^{-1}) \).

Now we state an analogue of the above theorem concerning the \( \{1, 4\} \)-inverse of the matrix product \( AQ \).

Theorem 3.4. Let \( A \in M_{m \times n} (\mathbb{R}) \) be such that \( A \) has a \( \{1, 4\} \)-inverse, let \( F = A^{(1,4)}A \) and let \( Q \in M_{n}(\mathbb{R}) \) be invertible. If \( F'Q = F' \), then the following conditions are equivalent:
In this case, 
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Moore-Penrose inverse of a matrix product $P AQ$

In this case, there exists a 

\begin{equation}
(AQ)^{(1,4)} = Q^* V^{-1} A^{(1,4)} = (I + L^*)(I + L L^*)^{-1} Q^{-1} A^{(1,4)}.
\end{equation}

**Proof.** We first note that $(A^{(1,4)})^*$ is a \{1, 3\}-inverse of $A^*$ and $Q^* (I - A^* (A^{(1,4)})^*) = I - A^* (A^{(1,4)})^*$. An application of Theorem 3.4 to the product $Q^* A^*$ shows that the following conditions are equivalent:

\begin{itemize}
  \item[(a')] $Q^* A^*$ has a \{1, 3\}-inverse.
  \item[(b')] $F \in \mathcal{M}_n(\mathbb{R}) W \cap W \mathcal{M}_n(\mathbb{R})$ where $W = FQQ^* F$.
  \item[(c')] $V = FQQ^* + I - F$ is invertible.
  \item[(d')] $I + LL^*$ is invertible with $L = (I - Q^{-1}) F'$.
\end{itemize}

From these relations, we conclude that (a), (b), (c), and (d) in this theorem are equivalent. Finally, by (3.1) we have

\begin{equation}
Y = (A^{(1,4)})^* U^{-1} Q = (A^{(1,4)})^* (Q^*)^{-1} (I + R^* R)^{-1} (I + R^*).
\end{equation}

Hence, $Y^*$ is a \{1, 4\}-inverse of $AQ$ and, thus, (3.3) holds. \hfill $\square$

**Remark 3.5.** If we replace $A^{(1,4)}$ and $(AQ)^{(1,4)}$ by $A^{(1,2,4)}$ and $(AQ)^{(1,4)}$, respectively, in Theorem 3.4, then we obtain an analogous characterization of the existence of \{1, 2, 4\}-inverses of the product $AQ$.

Based on previous Theorems, we derive a characterization of the existence of the Moore-Penrose inverse of a matrix product $P AQ$ in the case that $A^\dagger$ exists.

**Theorem 3.6.** Let $A \in \mathcal{M}_{m \times n}(\mathbb{R})$ be such that $A^\dagger$ exists, let $E = AA^\dagger$, $F = A^\dagger A$ and let $P \in \mathcal{M}_m(\mathbb{R})$ and $Q \in \mathcal{M}_n(\mathbb{R})$ be invertible matrices. If $PE' = E'$ and $F'Q = F'$, then the following are equivalent:

\begin{itemize}
  \item[(a)] $(PAQ)^\dagger$ exists.
  \item[(b)] $E \in \mathcal{M}_n(\mathbb{R}) Z \cap Z \mathcal{M}_n(\mathbb{R})$ and $F \in \mathcal{M}_n(\mathbb{R}) W \cap W \mathcal{M}_n(\mathbb{R})$, where $Z = EP^* PE$ and $W = FQQ^* F$.
  \item[(c)] $U = P^* PE + I - E$ and $V = FQQ^* + I - F$ are invertible.
  \item[(d)] $I + R^* R$ and $I + LL^*$ are invertible with $R = E'(I - P^{-1})$ and $L = (I - Q^{-1}) F'$.
\end{itemize}

In this case,

\begin{equation}
(PAQ)^\dagger = Q^* V^{-1} A^\dagger U^{-1} P^*
\end{equation}

\begin{equation}
= (I + L^*)(I + L L^*)^{-1} Q^{-1} A^\dagger P^{-1} (I + R^* R)^{-1} (I + R^*).
\end{equation}
Further Results on Generalized Inverses in Rings With Involution

Proof. We know that the Moore-Penrose inverse of \( PAQ \) exists if and only if \( PA \) has a \( \{1,3\} \)-inverse and \( AQ \) has a \( \{1,4\} \)-inverse, in which case

\[
(PAQ)^\dagger = (AQ)^{(1,4)}A(PA)^{(1,3)}.
\]

Now, the proof of the theorem is a consequence of Theorems 3.1 and 3.4.

4. Applications. Several authors described generalized inverses of block matrices and their properties \([1, 2, 6, 7, 11, 12, 16, 17, 21]\).

In this section, some applications of Theorems 3.1, 3.4 and 3.6 are indicated.

First, we characterize the existence of a \( \{1,3\} \)-inverse of a \( 2 \times 2 \) block matrix \( M \) over \( \mathbb{R} \) of the form

\[
M = \begin{pmatrix} a & c \\ b & d \end{pmatrix},
\]

where \( a \in M_m(\mathbb{R}) \) is invertible, \( b, c \) and \( d \) are matrices over \( \mathbb{R} \) of orders \( k \times m, m \times l \) and \( k \times l \), respectively. We denote by either \( I \) or \( 1_m \) the identity matrix in \( M_m(\mathbb{R}) \).

Consider the factorization

\[
M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1_m & 0 \\ ba^{-1} & 1_f \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} 1_m & a^{-1}c \\ 0 & 1_f \end{pmatrix} = PAQ.
\]

(4.2)

Theorem 4.1. Let \( M \) as in (4.1) and let \( s = d - ba^{-1}c \). Assume that \( s \{1,3\} \neq \emptyset \) and let \( s^{(1,3)} \in s \{1,3\} \) and \( e = 1_k - ss^{(1,3)} \). Then \( M \{1,3\} \neq \emptyset \) if and only if \( u = 1_m + (ba^{-1})^*eba^{-1} \) is invertible. In this case, a \{1,3\}-inverse of \( M \) is given by

\[
M^{(1,3)} = \begin{pmatrix} \alpha u^{-1} & \alpha u^{-1}(ba^{-1})^*e - a^{-1}cs^{(1,3)} \\ -s^{(1,3)}ba^{-1}u^{-1} & s^{(1,3)}(1_k - ba^{-1}u^{-1}(ba^{-1})^*) \end{pmatrix}
\]

(4.3)

where \( \alpha = (1_m + a^{-1}cs^{(1,3)})a^{-1} \).

Proof. Let \( P, A \) and \( Q \) as in (4.2). It is easy to check that a \( \{1,3\} \)-inverse of \( A \) is of the form \( A^{(1,3)} = \begin{pmatrix} a^{-1} & 0 \\ 0 & s^{(1,3)} \end{pmatrix} \). Then

\[
E' = I - AA^{(1,3)} = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix},
\]

where \( e = 1_k - ss^{(1,3)} \). Hence, \( PE' = E' \) holds. Set

\[
R = E'(I - P^{-1}) = \begin{pmatrix} 0 & 0 \\ eba^{-1} & 0 \end{pmatrix}, \quad I + R^*R = \begin{pmatrix} u & 0 \\ 0 & 1_k \end{pmatrix},
\]

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where $u = 1_m + (ba^{-1})^e ba^{-1}$. By Theorem 3.1 there exists a $\{1,3\}$-inverse of $PA$ if and only if $u$ is invertible in the ring $M_m(R)$. By (3.1), a $\{1,3\}$-inverse of $PA$ of the form $(PA)^{1,3} = A^{1,3} P^{-1}(I + R^* R)^{-1}(I + R^*)$. Therefore,

$$(PA)^{1,3} = \begin{pmatrix} a^{-1} & 0 \\ 0 & s^{1,3} \end{pmatrix} \begin{pmatrix} 1_m & 0 \\ {-ba^{-1}} & 1_l \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1_k \end{pmatrix} \begin{pmatrix} 1_m (ba^{-1})^e \\ 0 \\ 1_k \end{pmatrix} = \begin{pmatrix} a^{-1} u^{-1} \\ -s^{1,3} ba^{-1} u^{-1} \end{pmatrix} \begin{pmatrix} a^{-1} u^{-1} (ba^{-1})^e \\ s^{1,3} (1_k - ba^{-1} u^{-1} (ba^{-1})^e) \end{pmatrix}.$$

(4.4)

Now, since $Q$ is invertible then $PAQ\{1,3\} \neq \emptyset$ if and only if $PA\{1,3\} \neq \emptyset$ in which case $(PAQ)^{1,3} = Q^{-1}(PA)^{1,3}$. Pre-multiplying (4.4) by $Q^{-1}$, (4.3) is proved.

We can now state the analogue of previous theorem for the characterization of the existence of a $\{1,4\}$-inverse of matrix $M$.

**Theorem 4.2.** Let $M$ as in (4.1) and let $s = d - ba^{-1}c$. Assume that $s\{1,4\} \neq \emptyset$ and let $s^{1,4} \in s\{1,4\}$ and $f = 1_l - s^{1,4} s$. Then $M\{1,4\} \neq \emptyset$ if and only if $v = 1_m + a^{-1} cf(a^{-1} c)^*$ is invertible. In this case, a $\{1,4\}$-inverse of $M$ is given by

$$M^{1,4} = \begin{pmatrix} v^{-1} \beta \\ f(a^{-1} c)^* v^{-1} \beta - s^{1,4} ba^{-1} \\ 1_l - f(a^{-1} c)^* v^{-1} a^{-1} c s^{1,4} \end{pmatrix},$$

(4.5)

where $\beta = a^{-1}(1_m + cs^{1,4} ba^{-1})$.

**Proof.** We use the factorization (4.2). Choose $A^{1,4} = \begin{pmatrix} a^{-1} & 0 \\ 0 & s^{1,4} \end{pmatrix}$, which gives $F' = I - A^{1,4} A = \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix}$, where $f = 1_l - s^{1,4} s$. Now, set

$$L = (I - Q^{-1}) F' = \begin{pmatrix} 0 & a^{-1} cf \\ 0 & 0 \end{pmatrix}, \quad I + LL^* = \begin{pmatrix} v & 0 \\ 0 & 1_l \end{pmatrix},$$

where $v = 1_m + a^{-1} cf(a^{-1} c)^*$. With an application of Theorem 3.3 we obtain $A Q\{1,4\} \neq \emptyset$ iff $v$ is invertible and

$$(AQ)^{1,4} = \begin{pmatrix} v^{-1} a^{-1} \\ f(a^{-1} c)^* v^{-1} a^{-1} \\ 1_l - f(a^{-1} c)^* v^{-1} a^{-1} c s^{1,4} \end{pmatrix}.$$  

(4.6)

Since $(PAQ)^{1,4} = (AQ)^{1,4} P^{-1}$. Pre-multiplying (4.6) by $P^{-1}$, (4.5) is proved.

If $s = d - ba^{-1} c$ is regular, by Lemma 1.3 we have that $s^\dagger$ exists iff $s^* s + 1_l - s^{1,4} s$ is invertible. In this case, using previous results we can characterize the existence of the Moore-Penrose inverse of matrix $M$.

**Theorem 4.3.** Let $M$ as in (4.1) and let $s = d - ba^{-1} c$. Assume that $s^\dagger$ exists and set $e = 1_k - ss^\dagger$ and $f = 1_l - s^\dagger s$. Then $M^\dagger$ exists if and only if both
Further Results on Generalized Inverses in Rings With Involution

$u = 1_m + (ba^{-1})^*eba^{-1}$ and $v = 1_m + a^{-1}cf(a^{-1}c)^*$ are invertible. In this case,

$$M^\dagger = \begin{pmatrix} f(a^{-1}c)^* \gamma & \gamma(ba^{-1})^*e - v^{-1}a^{-1}cs^\dagger \\ \delta & \delta \end{pmatrix}, \quad (4.7)$$

where

$$\gamma = v^{-1}(a^{-1} + a^{-1}cs^\dagger ba^{-1})u^{-1},$$
$$\delta = s^\dagger + f(a^{-1}c)^*\gamma(ba^{-1})^*e - f(a^{-1}c)^*v^{-1}a^{-1}cs^\dagger - s^\dagger ba^{-1}u^{-1}(ba^{-1})^*e.$$

**Proof.** In view of the factorization (4.2), we have that $M$ is Moore-Penrose invertible iff both $u$ and $v$ defined as in the statement of this theorem are invertible. Using expressions (4.4) and (4.6), we compute $M^\dagger = (AQ)^{(1,4)}A(PA)^{(1,3)}$ and we obtain the expression (4.7).

In what sequel, let $T$ be a matrix over $\mathcal{R}$ of the form

$$T = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}, \quad (4.8)$$

where $a$, $b$ and $d$ are matrices over $\mathcal{R}$ of orders $m \times n$, $k \times n$, and $k \times l$, respectively.

We will characterize the existence of a $\{1,3\}$-inverse of $T$ when $a\{1,2,3\} \neq \emptyset$ and $d\{1,3\} \neq \emptyset$. Set

$$e = 1_k - dd^{(1,3)}, \quad f = 1_n - a^{(1,2,3)}a, \quad c = ebf. \quad (4.9)$$

Consider the factorization

$$T = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} = \begin{pmatrix} 1_m & 0 \\ eba^{(1,2,3)} & 1_k \end{pmatrix} \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} 1_n \\ d^{(1,3)}b & 1_l \end{pmatrix} = PAQ. \quad (4.10)$$

**Theorem 4.4.** Let $e$, $f$, and $c$ as in (4.2) and $T$ as in (4.10). Assume that $c\{1,3\} \neq \emptyset$ and let $c^{(1,3)} \in c\{1,3\}$. Then $T\{1,3\} \neq \emptyset$ if and only if $u = 1_m + (ba^{(1,2,3)})^*eba^{(1,2,3)}$ is invertible, where $g = 1_k - cc^{(1,3)}$. In this case,

$$T^{(1,3)} = \begin{pmatrix} 1_n \\ -da^{(1,3)}b & 1_l \end{pmatrix} \begin{pmatrix} \sigma & \sigma(ba^{(1,2,3)})^*eg + f^{(1,3)} \\ -\eta & d^{(1,3)} - \eta(ba^{(1,2,3)})^*eg \end{pmatrix}, \quad (4.11)$$

where $\sigma = (1 - f^{(1,3)}eb)a^{(1,2,3)}u^{-1}$ and $\eta = d^{(1,3)}eba^{(1,2,3)}u^{-1}$.

**Proof.** Let $P$, $A$, and $Q$ as in (4.10). We observe that $cc^{(1,3)} = cc^{(1,3)}e$. Using this, it is easy to check that a $\{1,3\}$-inverse of $A$ is given by

$$A^{(1,3)} = \begin{pmatrix} a^{(1,2,3)} & f^{(1,3)} \\ 0 & d^{(1,3)} \end{pmatrix}.$$
Then

\[ E' = I - AA^{(1,3)} = \begin{pmatrix} 1_m - aa^{(1,2,3)} & 0 \\ 0 & eg \end{pmatrix}, \]

where \( g = 1_k - cc^{(1,3)} \). We can see that \( PE' = E' \). Then we can apply Theorem 3.1 to the product \( PA \). With the notation \( R = E'(I - P^{-1}) \), using that \( egc = gc \), we have

\[ R = \begin{pmatrix} 0 & (g_{ba(1,2,3)})^*eg \end{pmatrix}, \quad R^* = \begin{pmatrix} 0 & (ba^{(1,2,3)})^*eg \\ 0 & 0 \end{pmatrix}, \quad I + R^*R = \begin{pmatrix} u & 0 \\ 0 & 1_k \end{pmatrix}, \]

where \( u = 1_m + (ba^{(1,2,3)})^*egba^{(1,2,3)} \). By Theorem 3.1 \( PA \) has a \( \{1,3\} \)-inverse if and only if \( I + R^*R \) is invertible, or equivalently, \( u \) is invertible. In this case, a \( \{1,3\} \)-inverse of \( PA \) has the form

\[ (PA)^{(1,3)} = A^{(1,3)}P^{-1}(I + R^*R)^{-1}(I + R^*). \]

Substituting into this expression the matrix products

\[ A^{(1,3)}P^{-1} = \begin{pmatrix} a^{(1,2,3)} & \sigma \\ 0 & d^{(1,3)} \end{pmatrix} \begin{pmatrix} 1_m & 0 \\ 0 & 1_k \end{pmatrix} = \begin{pmatrix} (1 - fc^{(1,3)})_{ba^{(1,2,3)}} & fc^{(1,3)} \\ -d^{(1,3)}_{ba^{(1,2,3)}} & d^{(1,3)} \end{pmatrix} \]

and

\[ (I + R^*R)^{-1}(I + R^*) = \begin{pmatrix} u^{-1} & u^{-1}(ba^{(1,2,3)})^*eg \\ 0 & 1 \end{pmatrix}, \]

we obtain, after an easy computation,

\[ (PA)^{(1,3)} = \begin{pmatrix} \sigma & \sigma(ba^{(1,2,3)})^*eg + fc^{(1,3)} \\ -\eta & -\eta(ba^{(1,2,3)})^*eg \end{pmatrix}, \]

where \( \sigma = (1 - fc^{(1,3)})_{ba^{(1,2,3)}}u^{-1} \) and \( \eta = d^{(1,3)}_{ba^{(1,2,3)}}u^{-1} \). Now, since \( Q \) is invertible then \( PAQ\{1,3\} \neq \emptyset \) if and only if \( PA\{1,3\} \neq \emptyset \) in which case \( (PAQ)^{(1,3)} = Q^{-1}(PA)^{(1,3)} \). The latter establishes the formula (4.11).

In order to characterize the existence of a \( \{1,4\} \)-inverse of \( T \) when \( a\{1,4\} \neq \emptyset \) and \( d\{1,2,4\} \neq \emptyset \), set

\[ e = 1_k - dd^{(1,2,4)}, \quad f = 1_n - a^{(1,4)}a, \quad c = ef \]

and consider the factorization

\[ T = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} = \begin{pmatrix} 1_m & 0 \\ 0 & 1_k \end{pmatrix} \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ d^{(1,2,4)}_{ba^{(1,4)}} & 1_l \end{pmatrix} = \hat{P}\hat{A}\hat{Q}. \]
Similarly, with an application Theorem 3.4, we derive the analogue of previous theorem.

**Theorem 4.5.** Let \(e, f,\) and \(c\) as in (4.12) and \(T\) as in (4.15). Assume that \(c\{1, 4\} \neq \emptyset\) and let \(c^{(1, 4)} \in c\{1, 4\}.\) Then \(T\{1, 4\} \neq \emptyset\) if and only if \(v = 1_t + d^{(1, 2, 4)}bf(d^{(1, 2, 4)}b)^*\) is invertible, where \(h = 1_n - c^{(1, 4)}c.\) In this case,

\[
T^{(1, 4)} = \begin{pmatrix}
a^{(1, 4)} - hf(d^{(1, 2, 4)}b)^* & h f (d^{(1, 2, 4)}b)^* \rho + c^{(1, 4)}e \\
-\mu & \rho \\
\end{pmatrix} \begin{pmatrix}
1_m \\
-ba^{(1, 4)} \\
1_k \\
\end{pmatrix}
\]

where \(\rho = v^{-1}d^{(1, 2, 4)}(1 - bf c^{(1, 4)}e)\) and \(\mu = v^{-1}d^{(1, 2, 4)}bf a^{(1, 4)}.\)

We will characterize the Moore-Penrose invertibility of \(T\) when there exist \(a^\dagger\) and \(d^\dagger.\) Set

\[
e = 1_k - dd^\dagger, \quad f = 1_n - a^\dagger a, \quad c = ef.
\]

We begin by finding conditions for the existence of \(c^\dagger.\)

**Proposition 4.6.** Let \(e, f\) and \(c\) as in (4.15). If any of the following conditions hold, then \(c^\dagger\) exists.

(i) \(w = cc^* + dd^*\) is invertible.

(ii) \(z = c^*c + a^*a\) is invertible.

**Proof.** (i) First, we prove that if \(w\) is invertible then \(c\) is regular. Let \(x = c^*w^{-1}\). Since \(ew = cc^*\) we also have \(e = cc^*w^{-1} = cx.\) Using this, we get \(cx = ec = c.\) Hence, \(x\) is a 1-inverse of \(c.\)

Now, choose \(c^{(1)} = c^*w^{-1}.\) Then \(cc^{(1)} = e\) and by Lemma 1.1 \(c^\dagger\) exists if and only if \(v = cc^* + 1 - e\) is invertible. We only need to show that \(v\) is invertible. Since \((cc^* + 1 - e)(dd^* + e) = cc^* + dd^*\) we have that \(cc^* + 1 - e\) is invertible because both \(w\) and \(dd^* + e\) are invertible, the last one due to the fact that \(d^\dagger\) exists.

(ii) The proof is similar to the case (i). \(\square\)

**Theorem 4.7.** Let \(T\) as in (4.8) and let \(e, f,\) and \(c\) as in (4.15). If \(c^\dagger\) exists, then \(T^\dagger\) exists if and only if both \(u = 1_m + (ba^\dagger)^*eg ba^\dagger\) and \(v = 1_t + d^\dagger bhf(d^\dagger b)^*\) are invertible, where \(g = 1_k - cc^\dagger\) and \(h = 1_n - c^\dagger c.\) In this case,

\[
\begin{pmatrix}
a & 0 & \dagger \\
b & d
\end{pmatrix} = \begin{pmatrix}
(1 - hf(d^\dagger b)^*v^{-1}d^\dagger b)\sigma \\
-\rho ba^\dagger u^{-1}
\end{pmatrix} \begin{pmatrix}
\gamma \\
\rho(1_t - ba^\dagger u^{-1}(ba^\dagger)^*eg)
\end{pmatrix},
\]

where \(\rho = v^{-1}d^\dagger (1_t - bc^\dagger),\) \(\sigma = (1_k - c^\dagger b)a^\dagger u^{-1}\) and

\[
\gamma = c^\dagger + hf (d^\dagger b)^* \rho(1_t - ba^\dagger u^{-1}(ba^\dagger)^*eg) + \sigma (ba^\dagger)^*eg.
\]
Proof. We can apply Theorem 4.4 with \( a^{(1,2,3)} = a^\dagger \), \( d^{(1,3)} = d^\dagger \), and \( c^{(1,3)} = c^\dagger \) to obtain that \( T^{(1,3)} \) exists iff \( u = 1_m + (ba^\dagger)^*e_{g\,ba^\dagger} \) is invertible and, using that \( d^\dagger e = 0 \), a \( \{1,3\} \)-inverse of \( T \) is of the form

\[
T^{(1,3)} = \begin{pmatrix}
1_n & 0 \\
-d^\dagger b & 1_l
\end{pmatrix}
\begin{pmatrix}
\sigma & \sigma(ba^\dagger)^*eg + fc^\dagger \\
0 & d^\dagger
\end{pmatrix} = \tilde{Q}X,
\]

where \( \sigma = (1 - c^\dagger b)a^\dagger u^{-1} \). Similarly, we apply Theorem 4.5 to derive that \( T^{(1,4)} \) exists iff \( v = 1_l + d^\dagger bh(f(d^\dagger b)^*) \) is invertible, and a \( \{1,4\} \)-inverse of \( T \) is of the form

\[
T^{(1,4)} = \begin{pmatrix}
a^\dagger & hf(d^\dagger b)^*c\dagger + c^\dagger \\
0 & \rho
\end{pmatrix}
\begin{pmatrix}
1_m & 0 \\
-ba^\dagger & 1_k
\end{pmatrix} = Y\tilde{P},
\]

where \( \rho = v^{-1} d^\dagger (1_l - bc^\dagger) \).

We now compute \( T^\dagger = T^{(1,4)}TT^{(1,3)} = Y\tilde{P}T\tilde{Q}X \). One sees that

\[
\tilde{P}T\tilde{Q} = \begin{pmatrix}
a & 0 \\
bf - dd^\dagger b & d
\end{pmatrix}.
\]

Using that \( \rho(bf - dd^\dagger b) = -v^{-1}d^\dagger b(a^\dagger a + c^\dagger c) \) and \( \rho d = v^{-1}d^\dagger d \), we have

\[
Y\tilde{P}T\tilde{Q} = \begin{pmatrix}
(1 - hf(d^\dagger b)^*v^{-1}d^\dagger b)(a^\dagger a + c^\dagger c) & hf(d^\dagger b)^*v^{-1}d^\dagger d \\
-v^{-1}d^\dagger b(a^\dagger a + c^\dagger c) & v^{-1}d^\dagger d
\end{pmatrix}.
\]

Using \( (a^\dagger a + c^\dagger c)\sigma = \sigma \) we obtain

\[
T^\dagger = Y\tilde{P}T\tilde{Q}X = \begin{pmatrix}
(1 - hf(d^\dagger b)^*v^{-1}d^\dagger b)\gamma & \gamma \\
-v^{-1}d^\dagger b (a^\dagger a)^*eg + \rho & -v^{-1}d^\dagger b (ba^\dagger)^*eg + c^\dagger
\end{pmatrix},
\]

where \( \gamma = (1 + hf(d^\dagger b)^*v^{-1}d^\dagger b)(a^\dagger a)^*eg + c^\dagger) + hf(d^\dagger b)^*v^{-1}d^\dagger \).

Finally, the formula in this theorem is proved by taking into account that \( v^{-1}d^\dagger b d = \rho ba^\dagger u^{-1} \).

We recall that a ring \( \mathcal{R} \) with involution has the Gelfand-Naimark property (GN-property) if \( 1 + x^* x \) is invertible for all \( x \in \mathcal{R} \).

We can rewrite \( u \) and \( v \) in Theorem 4.4 as \( u = 1_m + (egba^\dagger)^*(egba^\dagger) \) and \( v = 1_l + d^\dagger bh(f(d^\dagger bhf)^*) \). On account of this, we obtain the next corollary.

**Corollary 4.8.** Let \( \mathcal{R} \) be a ring with involution such that it has the GN-property. Consider \( T = \begin{pmatrix}
a & 0 \\
b & d
\end{pmatrix} \) with \( a, b, d \in \mathcal{R} \). Let \( e, f, \) and \( c \) as in (4.15). If \( a^\dagger, d^\dagger \) and \( c^\dagger \) exists then \( T^\dagger \) exists, and it is given by (4.16)–(4.17).
We find, after some computations, that our formula (4.16)–(4.17) is the same as the expression given by (10)–(19) in [14, Section 2.2] for the Moore-Penrose inverse of a $2 \times 2$ lower triangular matrix. Here, we conclude that Corollary 4.8 generalizes the main result of [13, Section 2.2] to more general conditions that those assumed therein.

**Corollary 4.9.** Let $T = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$ be such that $a^\dagger$ exists and $d$ is an invertible matrix of order $k \times k$. Then $T^\dagger$ exists if and only if $v = 1_k + d^{-1}bf(d^{-1}b)^*$ is invertible. In this case

$$\begin{pmatrix} a & 0 \\ b & d \end{pmatrix}^\dagger = \begin{pmatrix} (1_n - f(d^{-1}b)^*v^{-1}d^{-1})a^\dagger & f(d^{-1}b)^*v^{-1}d^{-1} \\ -v^{-1}d^{-1}ba^\dagger & v^{-1}d^{-1} \end{pmatrix}.$$

**Proof.** Follows from previous theorem with $d^\dagger = d^{-1}$. Then $e = 0$, $c = 0$, $g = 1_k$, and $h = 1_n$ and (4.16) reduces to the formula in this corollary.

For the special case of a companion matrix of the form

$$\begin{pmatrix} 0 & a \\ 1_k & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 1_k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1_k & 0 \end{pmatrix} = TU,$$

where $a \in \mathcal{R}$ is Moore-Penrose invertible and $b \in \mathcal{M}_{k \times 1}(\mathcal{R})$, using that $(TU)^\dagger = U^*T^\dagger$ we obtain from previous corollary that $(TU)^\dagger$ exists if and only if $v = 1_k + b(1 - a^\dagger a)b^*$ is invertible. In this case,

$$(TU)^\dagger = \begin{pmatrix} -v^{-1}ba^\dagger \\ (1 - f(b^*v^{-1}b)a^\dagger & v^{-1}f(b^*v^{-1}) \end{pmatrix},$$

which is the result in [19] Theorem 2.1].

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