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THE $P^+_0$-MATRIX COMPLETION PROBLEM*

BHABA KUMAR SARMA† AND KALYAN SINHA†

Abstract. A real $n \times n$ matrix $B$ is a $P^+_0$-matrix if for each $k \in \{1, 2, \ldots, n\}$ every $k \times k$ principal minor of $B$ is nonnegative, and at least one $k \times k$ principal minor is positive. A digraph $D$ is said to have $P^+_0$-completion if every partial $P^+_0$-matrix specifying $D$ can be completed to a $P^+_0$-matrix. In this paper, some necessary and sufficient conditions for a digraph to have $P^+_0$-completion are discussed and those digraphs of order at most four that have $P^+_0$-completion are singled out.

Key words. Partial matrix, Matrix completion, $P^+_0$-matrix, $P^+_0$-completion, Digraph.

AMS subject classifications. 15A48.

1. Introduction. A real $n \times n$ matrix $B$ is a

(i) $P$-matrix ($P_0$-matrix) if every principal minor of $B$ is positive (nonnegative).
(ii) $P^+_0$-matrix, if for each $k \in \{1, 2, \ldots, n\}$, every $k \times k$ principal minor of $B$ is nonnegative, and at least one $k \times k$ principal minor is positive.
(iii) $Q$-matrix if for every $k \in \{1, 2, \ldots, n\}$, the sum $S_k(B)$ of all the $k \times k$ principal minors of $B$ is positive.

Clearly, a $P$-matrix is a $P^+_0$-matrix, and a $P^+_0$-matrix is both a $P_0$-matrix and a $Q$-matrix.

A partial matrix $M$ is a rectangular array of numbers in which some entries are specified while others are free to be chosen. A completion of $M$ is a matrix obtained by assigning numbers to the unspecified entries in $M$.

For a class $\Pi$ of matrices (e.g., $P$-, $P_0$- or $Q$-matrices) a partial $\Pi$-matrix is one whose specified entries satisfy the required properties of a $\Pi$-matrix. For example, a partial $P$-matrix ($P_0$-matrix) has all fully specified minors positive (nonnegative), and for a partial $Q$-matrix $M$, $S_k(M) > 0$ for each $k$ for which all $k \times k$ principal submatrices are fully specified. A partial $P^+_0$-matrix $M$ is a partial matrix in which all fully specified principal minors are nonnegative and $S_k(M) > 0$ for every $k = 1, 2, \ldots, n$, whenever all $k \times k$ principal submatrices are fully specified. A $\Pi$-completion of a partial $\Pi$-matrix is a $\Pi$-matrix obtained by some choices of the unspecified entries.

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Graphs and digraphs have played an important role in the study of matrix completion problems. In many cases, the positions of the specified entries determine the existence of completions of partial matrices of a given class.

A pattern for $n \times n$ partial matrices is a subset of $N \times N$, where $N = \{1, 2, \ldots, n\}$. A partial matrix specifies a pattern if its specified entries lie exactly in those positions listed in the pattern. Patterns are usually specified by digraphs. An $n \times n$ partial matrix $M$ is said to specify a digraph $D$ on vertices $\{v_1, \ldots, v_n\}$ if $(v_i, v_j)$ is an arc in $D$ if and only if the entry $m_{ij}$ of $M$ is specified.

For a class $\Pi$ of matrices, the (combinatorial) $\Pi$-matrix completion problem attempts to study the digraphs $D$ having the property that any partial $\Pi$-matrix specifying $D$ has a $\Pi$-completion. Such a digraph $D$ is then said to have $\Pi$-completion. For an exposition in matrix completion problems, see the survey articles [7] and [8]. The completion problems for the classes of $P$- and $P_0$-matrices have been studied by a number of researchers (see [2, 3, 5, 8, 10], for example). DeAlba et al. have discussed the $Q$-matrix completion problem in 2009 in their paper [4].

1.1. Digraphs. In this paper, we will use commonly used graph theoretic terms which can be found in [1], [6] or any other standard book. However, in our discussion, a directed graph or digraph $D$ is a pair $(V, A)$, where $V$ is a finite nonempty set of objects, called vertices, and $A$ a set of ordered pairs of vertices, called arcs or directed edges. The vertex set and the arc set of $D$ are denoted by $V(D)$ and $A(D)$, respectively.

Note that the above definition allows an arc $x = (u, u)$ in the arc set of a digraph $D$, which is called a loop at the vertex $u$. Several works on matrix completion problems used marked digraphs, i.e., digraphs with some vertices marked instead of considering loops at those vertices (see [2, 8]). The current definition is in use in some recent works on matrix completion problems, including [4].

Sometimes, we simply write $v \in D$ (resp. $(u, v) \in D$) to mean $v \in V(D)$ (resp. $(u, v) \in A(D)$). The order of $D$, denoted by $|D|$, is the number of vertices of $D$. If $u \neq v$ and $x = (u, v)$ is an arc in $D$, we say that $x$ is incident with $u$ and $v$; $u$ is adjacent to $v$; and $v$ is adjacent from $u$. The outdegree (resp. indegree) of a vertex $v$ in $D$ is the number of vertices of $D$ adjacent from (resp. to) $v$.

It is customary to represent a digraph by a diagram with nodes representing the vertices and directed line segments (or arcs) representing the arcs of the digraph. A digraph $H$ is a subdigraph of a digraph $D$ if $V(H) \subseteq V(D)$ and $A(H) \subseteq A(D)$. Further, $H$ is an induced subdigraph (induced by $V(H)$) if $A(H) = (V(H) \times V(H)) \cap A(D)$, and is a spanning subdigraph if $V(H) = V(D)$. The complement of $D$ is the digraph $\overline{D}$, where $V(\overline{D}) = V(D)$ and $(v, w) \in A(\overline{D})$ if and only if $(v, w) \notin A(D)$. For
w ∈ V(D) the subdigraph of D induced by V(D) \ \{w\} is denoted by D − w.

Two digraphs \(D_1 = (V_1, A_1)\) and \(D_2 = (V_2, A_2)\) are isomorphic if there is a bijection \(\phi : V_1 \rightarrow V_2\) such that \(A_2 = \{(\phi(u), \phi(v)) : (u, v) \in A_1\}\). An unlabelled digraph is an equivalence class of isomorphic digraphs. Any particular member of an unlabelled digraph is referred as the digraph obtained by a labelling of the unlabelled digraph.

Let \(\hat{D}\) and \(\hat{H}\) be unlabelled digraphs. We say \(\hat{H}\) is an (unlabelled) subdigraph of \(\hat{D}\), if some member of \(\hat{H}\) is a subdigraph of a member of \(\hat{D}\), i.e., if the digraph obtained by a labelling of \(\hat{H}\) is a subdigraph of the digraph obtained by some labelling of \(\hat{D}\).

The digraph \(D\) is symmetric if \((v, w) \in A(D)\) implies \((w, v) \in A(D)\). For a (simple) graph \(G\) with vertex set \(V\) and edge set \(E\), we define the digraph associated to \(G\) to be the symmetric digraph with vertex set \(V\) and the arc set \(A = \{(u, v) : \) either \(u = v\) or \(u\) is adjacent to \(v\) in \(G\}\). Note that the digraph associated to a graph includes all loops. We call the digraph associated to the complete graph \(K_n\) on \(n\) vertices the complete symmetric digraph (or simply the complete digraph) of order \(n\), and denote it by \(K_n^*\). A digraph \(D\) is said to be asymmetric if whenever \((v, w) \in A(D)\) and \(v \neq w\) we have \((w, v) \notin A(D)\). Note that an asymmetric digraph may have loops.

A (directed) cycle \(C\) of length \(k\) (or a \(k\)-cycle) in a digraph \(D\) is a subdigraph with (distinct) vertices \(v_1, v_2, \ldots, v_k\) and with arcs \((v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)\). Note that a 1-cycle is a loop, and a 2-cycle consists of a pair of opposite arcs.

Let \(\pi\) be a permutation of \(V\). A permutation digraph is a digraph of the form \(D_\pi = (V, A_\pi)\) where \(A_\pi = \{(v, \pi(v)) : v \in V\}\). Clearly, each component of a permutation digraph is a cycle. A permutation subdigraph of order \(k\) of a digraph \(D\) is a permutation digraph that is a subdigraph of \(D\) of order \(k\). A digraph \(D\) of order \(n\) is stratified if \(D\) has a permutation subdigraph of order \(k\) for every \(k = 2, 3, \ldots, n\). A digraph \(D\) is weakly stratified if for each \(k = 2, 3, \ldots, n\), either

(i) \(D\) has a permutation subdigraph of order \(k\), or
(ii) for each \(v \in V(D)\) the digraph \(D − v\) has a permutation subdigraph of order \(k − 1\).

In this paper, we study the (combinatorial) \(P_0^+\)-matrix completion problem. The property of being a \(P_0^+\)-matrix (or a \(Q\)-matrix) is not inherited by principal submatrices, though it is preserved under similarity and transposition. This fact distinguishes the \(P_0^+\)- and \(Q\)-matrix completion problems from the completion problems of several other classes of \(P_0\)-matrices.
2. Partial $P_0^+$-matrices and the $P_0^+$-completion problem. Recall that a partial $P_0^+$-matrix is a partial matrix $M$ in which all fully specified principal minors are nonnegative and $S_k(M) > 0$ for every $k \in \{1, 2, \ldots, n\}$, whenever all $k \times k$ principal submatrices are fully specified.

Let $M$ be a partial $P_0^+$-matrix. If all $1 \times 1$ principal submatrices (i.e., all diagonal entries) in $M$ are specified, then $\text{trace}(M) > 0$. If for some $k \geq 2$ all $k \times k$ principal submatrices are fully specified, then $M$ is fully specified (and therefore $M$ is a $P_0^+$-matrix). Thus, the following proposition which provides a more useful characterization of a partial $P_0^+$-matrix is immediate.

**Proposition 2.1.** A partial matrix $M$ is a partial $P_0^+$-matrix if and only if exactly one of the following holds:

(i) At least one diagonal entry of $M$ is unspecified, and each fully specified principal minor of $M$ is nonnegative.

(ii) All diagonal entries are specified and nonnegative, with at least one of them positive; at least one off-diagonal entry is unspecified and each fully specified principal minor of $M$ is nonnegative.

(iii) All entries of $M$ are specified and $M$ is a $P_0^+$-matrix.

**Definition 2.2.** A partial $P_0^+$-matrix $M$ is said to have a $P_0^+$-completion if there is a completion of $M$ which is a $P_0^+$-matrix. A digraph $D$ is said to have $P_0^+$-completion if every partial $P_0^+$-matrix specifying $D$ has a $P_0^+$-completion. A graph $G$ is said to have $P_0^+$-completion, if the digraph associated to $G$ has $P_0^+$-completion.

The $P_0^+$-matrix completion problem aims to classify all digraphs which have $P_0^+$-completion. To distinguish the $P_0^+$-completion problem from those of $P_0^-$- and $P_-$-matrix classes, we furnish the following example.

**Example 2.3.** For $m, n \geq 2$ consider the graph $G$ obtained by identifying a vertex of $K_m$ with a vertex of $K_n$. Then, $G$ is called a 1-chordal graph with two maximal cliques $K_m$ and $K_n$ [5]. It is known that for any $m$ and $n$, $G$ (i.e., the digraph associated to $G$) has $P$-completion and $P_0$-completion (see [8]). With an appropriate labeling of vertices, a partial matrix specifying $G$ is given by

$$M = \begin{bmatrix} A_{11} & A_{12} & X \\ A_{21} & a_{22} & A_{23} \\ Y & A_{32} & A_{33} \end{bmatrix},$$

where $A_{ii}$ are square, $a_{22}$ is a scalar, and $X$ and $Y$ are fully unspecified. If $M$ is a partial $P_0$-matrix, then a $P_0$-completion, called the zero completion of $M$, is obtained by putting $X = Y = 0$ (see [8]). Similarly, if $M$ is a partial $P$-matrix, then a $P$-completion, called the asymmetric completion of $M$, is obtained by putting $X = A_{12}a_{22}^{-1}A_{23}$ and $Y = 0$ (see [8]). However, these techniques do not work for the
$P_0^+$-completion problem. Note that in case $M$ is a partial $P_0^+$-matrix, $a_{22}$ may be zero. The zero completion of the partial $P_0^+$-matrix

$$M = \begin{bmatrix}
1 & 0 & 1 & ? & ? \\
1 & 0 & 0 & ? & ? \\
-1 & 1 & 0 & 0 & -1 \\
? & ? & 0 & 1 & 0 \\
? & ? & 1 & 1 & 0
\end{bmatrix}$$

specifying the 1-chordal graph with two maximal cliques $K_3$ is not a $P_0^+$-matrix, even though the submatrices of $M$ corresponding to the maximal cliques are $P_0^+$-matrices. However, $M$ has $P_0^+$-completions, e.g.,

$$B_1 = \begin{bmatrix}
1 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0
\end{bmatrix}.$$

It will follow from Corollary 4.9 that for any 1-chordal graph $G$ with two maximal cliques there is a partial $P_0^+$-matrix specifying $G$ which does not have a $P_0^+$-completion. Nevertheless, in case one of the cliques is $K_2$, we have the following positive result in a restrictive situation.

**Theorem 2.4.** Let $M$ be a partial matrix specifying a 1-chordal graph with two maximal cliques $K_n$ and $K_2$. If the principal submatrices of $M$ corresponding to $K_n$ and $K_2$ are $P_0^+$-matrices, then $M$ can be completed to a $P_0^+$-matrix.

**Proof.** Without any loss of generality we assume that $V(K_n) = \{1, \ldots, n\}$ and $V(K_2) = \{n, n+1\}$. Thus, we can write

$$M = \begin{bmatrix}
A_{11} & A_{12} & X \\
A_{21} & a_{22} & a_{23} \\
Y & a_{32} & a_{33}
\end{bmatrix},$$

where $X$ is $(n-1) \times 1$, $Y$ is $1 \times (n-1)$, and they are fully unspecified. Moreover, the two principal submatrices specifying $K_n$ and $K_2$, viz.,

$$M_1 = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & a_{22}
\end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix}
a_{22} & a_{23} \\
a_{32} & a_{33}
\end{bmatrix}$$

are $P_0^+$-matrices. If $a_{22} > 0$, consider the asymmetric completion $\hat{A}$ of $M$, i.e., one obtained by setting $X = A_{12}a_{22}^{-1}a_{23}$ and $Y = 0$. Then

$$\det \hat{A} = \frac{\det M_1 \det M_2}{a_{22}} > 0. \quad (2.1)$$
Let $A$ be any principal submatrix of $\hat{A}$. That $A$ has nonnegative determinant follows from an equation similar to (2.1), if $a_{22}$ is included in $A$, and from the fact that $A$ is block triangular, if $a_{22}$ is excluded. Since $M_1$ has a positive minor of order $k$ for $1 \leq k \leq n$, $\hat{A}$ is a $P^+_0$-matrix.

On the other hand, if $a_{22} = 0$, consider the zero completion $\hat{A}$ of $M$, i.e., one obtained by setting $X = 0 = Y$. Since $M_2$ is a $P^+_0$-matrix, $a_{33}$ must be positive. Then, we have $\det \hat{A} = a_{33} \det M_1 + \det A_{11} \det M_2 > 0$. That a principal minor given by a principal submatrix $A$ of $\hat{A}$ is nonnegative follows from the determinant equality when $n$-th row is present and because the principal submatrix is block diagonal when $n$-th row is absent. Since $M_1$ has a positive minor of order $k$ for $1 \leq k \leq n$, $\hat{A}$ is a $P^+_0$-matrix.

Unlike several other matrix completion problems, a $P^+_0$-completion of the principal submatrix corresponding to the specified diagonal positions of a partial $P^+_0$-matrix does not provide a $P^+_0$-completion of the partial matrix. For example, consider the partial $P^+_0$-matrix

$$M = \begin{bmatrix} 2 & 1 & u & 1 \\ 1 & 1 & 0 & 1 \\ -1 & v & 0 & x \\ 1 & 1 & y & z \end{bmatrix},$$

with $u, v, x, y, z$ as unspecified entries. The principal submatrix $M[[1, 2, 3]]$ of $M$ induced by the specified diagonal entries has a $P^+_0$-completion, e.g., one obtained by putting $u = 2, v = 0$. However, for any completion of $M$ we have $\det M[[1, 2, 4]] = -1/2$ and hence $M$ cannot be completed to a $P^+_0$-matrix.

3. **Digraphs having $P^+_0$-completion.** We observe that if a digraph $D$ omits all loops, then $D$ has $P^+_0$-completion. Indeed, a completion of a partial $P^+_0$-matrix $M$ specifying $D$ can be obtained by assigning a sufficiently large value to each of the diagonal entries.

A digraph $D$ may not have $P^+_0$-completion even if the subdigraph of $D$ induced by the vertices at which $D$ includes loops has $P^+_0$-completion. For example, the digraph $D_1$ in Figure 3.1 includes a loop at the vertex 1. The subdigraph induced by the vertex 1 has $P^+_0$-completion, being a complete digraph. However, the partial $P^+_0$-matrix

$$M = \begin{bmatrix} 0 & 0 \\ 0 & z \end{bmatrix}$$

specifying $D_1$ does not have a $P^+_0$-completion.

The following example shows that a digraph may not have $P^+_0$-completion even
if each of its components has $P_0^+$-completion.

**Example 3.1.** Consider the digraph $D_2 = K^*_2 \cup K^*_3$. The components $K^*_2$ and $K^*_3$ of $D_2$ have $P_0^+$-completion. However, the partial $P_0^+$-matrix

$$M = \begin{bmatrix}
1 & 0 & \? & \? & \? \\
0 & 1 & \? & \? & \? \\
\? & \? & 0 & 0 & 0 \\
\? & \? & 0 & 0 & 0 \\
\? & \? & 0 & 0 & 0
\end{bmatrix}$$

specifying $D_2$ does not have a $P_0^+$-completion, because for any completion of $M$ the last three rows are linearly dependent.

The following example shows that the property of having $P_0^+$-completion is not inherited by the induced subdigraphs.

**Example 3.2.** The digraph $D_3$ in Figure 3.2 has $P_0^+$-completion whereas its subdigraph $D_0$ induced by the vertices 1 and 2 does not have. To see this consider a partial $P_0^+$-matrix

$$M = \begin{bmatrix}
d_1 & a_{12} & u \\
v & d_2 & -s \\
a_{31} & s & d_4
\end{bmatrix}$$

specifying $D_3$, where $u, v, \pm s$ are the unspecified entries. Now, for $t > 0$ choose a completion $B(t)$ of $M$ with $|u| = |v| = |s| = t$ and with appropriate signs for $u, v$ and $s$ so that $a_{12}v \leq 0$, $a_{31}u \leq 0$ and $uvs > 0$. Then, all $2 \times 2$ principal minors of $B$ are
nonnegative and $B(2, 3) > 0$. Further,

$$\det B = t^3 + p(t),$$

where $p(t)$ is a polynomial in $t$ of degree at most 2. Thus, $B(t)$ is a $P^+_0$-matrix for large values of $t$. On the other hand, the partial $P^+_0$-matrix

$$M_0 = \begin{bmatrix} 0 & 0 \\ ? & 1 \end{bmatrix}$$

specifying $D_0$ does not have a $P^+_0$-completion.

**Theorem 3.3.** Suppose $D \neq K^*_n$ is a digraph having $P^+_0$-completion and $\widehat{D}$ is a spanning subdigraph of $D$. If $\widehat{D}$ has $P_0$-completion, then $\widehat{D}$ has $P^+_0$-completion.

**Proof.** Let $\widehat{M} = [\widehat{m}_{ij}]$ be a partial $P^+_0$-matrix specifying $\widehat{D}$. Then, being a partial $P_0$-matrix, $\widehat{M}$ has a $P_0$-completion $\widehat{B} = [\widehat{a}_{ij}]$. Let $M = [m_{ij}]$ be the partial matrix specifying $D$ defined by

$$m_{ij} = \begin{cases} 
\widehat{m}_{ij}, & \text{if } (i, j) \in A(\widehat{D}), \\
\widehat{a}_{ij}, & \text{if } (i, j) \in A(D) \setminus A(\widehat{D}), i \neq j, \\
\max\{1, \widehat{a}_{ij}\}, & \text{if } (i, j) \in A(D) \setminus A(\widehat{D}), i = j.
\end{cases}$$

Then, $M$ is a partial $P^+_0$-matrix specifying $D$, and therefore has a $P^+_0$-completion $B$. Clearly, $B$ is a $P^+_0$-completion of $\widehat{M}$. \[ \square \]

For all digraphs $D$ that we have examined, including the digraphs of order at most 4, the following has been observed: if $D$ has $P_0$-completion and its complement $\overline{D}$ is stratified, then $D$ has $P^+_0$-completion. However, we do not know whether the result is true in general.

**4. Necessary conditions for $P^+_0$-completion.** In this section, we present some necessary conditions for a digraph to have $P^+_0$-completion.

**Theorem 4.1.** Let $D \neq K^*_n$ be a digraph having $P^+_0$-completion. Then, every proper induced subdigraph of $D$ has $P_0$-completion.

**Proof.** Let $D$ be of order $n$ and $\alpha$ be a proper subset of $\{1, 2, \ldots, n\}$. Consider the subdigraph $D_\alpha$ of $D$ induced by $\alpha$ and let $M_\alpha$ be a partial $P_0$-matrix specifying $D_\alpha$. We extend $M_\alpha$ to a partial matrix $M$ specifying $D$ by setting all remaining specified off-diagonal entries as 0 and the remaining specified diagonal entries, if any, as 1. Then, all fully specified principal minors of $M$ are nonnegative, and $M$ is a partial $P^+_0$-matrix. Now, since $D$ has $P^+_0$-completion, $M$ can be completed to a $P^+_0$-matrix $\widehat{B}$. Clearly, the principal submatrix of $\widehat{B}$ induced by $\alpha$ is a $P_0$-completion of $M_\alpha$. \[ \square \]
In [2], Choi et al. showed that any digraph which contains one of the (unlabelled) digraphs in Figure 4.1 as an induced subdigraph does not have \( P_0 \)-completion. In particular, these digraphs do not have \( P_0 \)-completion. Therefore, an immediate implication of Theorem 4.1 is the following.

**Corollary 4.2.** Any digraph which contains one of the (unlabelled) digraphs in Figure 4.1 as a proper induced subdigraph does not have \( P_0^+ \)-completion.

That the converse of the Theorem 4.1 is not true can be seen from the following example.

**Example 4.3.** Consider the digraph \( D_4 \) in Figure 4.2. Each of the strongly connected components of \( D_4 \), being complete, has \( P_0 \)-completion. It follows from [8, Theorem 5.8] that \( D_4 \) has \( P_0 \)-completion. Consequently, each of the induced subdigraphs of \( D_4 \) has \( P_0 \)-completion. However, \( D_4 \) does not have \( P_0^+ \)-completion (see Remark 4.6).

**Theorem 4.4.** Let \( D \) be a digraph of order \( n \) that omits at least one loop. If \( D \) has \( P_0^+ \)-completion, then \( \overline{D} \) is stratified.

**Proof.** Suppose \( D \) has \( P_0^+ \)-completion. Let \( k \geq 2 \), and assume \( \overline{D} \) has no per-
The $P_0^+$-matrix completion problem

mutation subdigraph of order $k$. If $M$ is the partial matrix that specifies $D$ with all specified entries zero, and $B$ is a completion of $M$, then all $k \times k$ principal minors of $B$ are zero, so $B$ is not a $P_0^+$-matrix. This implies that $\overline{D}$ must be stratified.

To see that the converse of Theorem 4.4 is not true, consider the digraph $D_5$ in Figure 4.3, which omits loop at the vertex 4. Though $\overline{D_5}$ is stratified, $D_5$ does not have $P_0^+$-completion, in view of Corollary 4.2, since the subdigraph induced by the vertices \{1, 2, 3\} does not have $P_0$-completion (see Figure 4.1).

**Theorem 4.5.** Let $D \neq K_n^*$ be a digraph of order $n \geq 2$ that includes all loops and has $P_0^+$-completion. Then $\overline{D}$ is weakly stratified.

**Proof.** Suppose for some $k$ ($2 \leq k \leq n$) $\overline{D}$ has no permutation subdigraph of order $k$ and there is a vertex $v$ in $D$ such that $\overline{D} - v$ does not have a permutation subdigraph of order $k - 1$. Let $M = [m_{ij}]$ be the partial matrix specifying $D$ with $m_{vv} = 1$ and all other specified entries zero. Then for any completion $B$ of $M$ all $k \times k$ principal minors of $B$ are zero, and therefore, $B$ is not a $P_0^+$-matrix.

**Remark 4.6.** The digraph $D_4$ in Figure 4.2 includes all loops, and $\overline{D_4}$ is not weakly stratified. Thus, $D_4$ does not have $P_0^+$-completion.

The converse of Theorem 4.5 is not true which can be seen from the following example.

**Example 4.7.** Consider the symmetric 4-cycle $C_4$ (Figure 4.4). It is easy to see that $\overline{C_4}$ is weakly stratified. To see that $C_4$ does not have $P_0^+$-completion, consider...
the partial $P_0^+$-matrix

$$M = \begin{bmatrix}
0 & 0 & x_{13} & 0 \\
1 & 0 & 0 & x_{24} \\
x_{31} & -1 & 1 & 1 \\
1 & x_{42} & 0 & 0
\end{bmatrix}$$

specifying $C_4$. For a completion $B$ of $M$ the $3 \times 3$ principal minor $B(1, 2, 3)$ of $B$ indexed by $\{1, 2, 3\}$ is $-x_{13}$. Similarly, $B(1, 3, 4) = x_{13}$. Therefore, for $B$ to be a $P_0^+$-matrix, $x_{13} = 0$. However, this yields $\det B = 0$. Thus, $B$ cannot be a $P_0^+$-matrix.

**Corollary 4.8.** If a digraph $D$ of order $n \geq 2$ contains a vertex $v$ with indegree or outdegree $n$, then $D$ does not have $P_0^+$-completion.

**Proof.** Clearly, $v$ has either indegree or outdegree zero in $\overline{D}$, and therefore $v$ does not lie on any cycle in $\overline{D}$. Consequently, $\overline{D}$ does not have a spanning permutation subdigraph, i.e., $\overline{D}$ is not stratified. Further, for any vertex $u \neq v$ in $D$ the digraph $\overline{D} - u$ does not have a spanning permutation subdigraph. Hence, $\overline{D}$ is not weakly stratified. $\square$

**Corollary 4.9.** A 1-chordal graph $G$ with two maximal cliques and with all loops does not have $P_0^+$-completion.

**Proof.** Let $G$ be of order $n$ and $v$ be the vertex in $G$ common to the maximal cliques. Then, $v$ has indegree as well as outdegree $n$ in the digraph associated to $G$, and the result follows from Corollary 4.8. $\square$

**Corollary 4.10.** Let $D$ be a digraph of order $n$ that includes all loops and has $P_0^+$-completion. Then, $\overline{D}$ has a 2-cycle.

**Proof.** Since $D$ has $P_0^+$-completion and $D$ includes all loops, $\overline{D}$ must be weakly stratified. Because $\overline{D}$ does not contain any loop, $\overline{D}$ has a permutation subdigraph of order 2 only if it has a 2-cycle. $\square$

**Remark 4.11.** It is known that any asymmetric digraph has $P_0$-completion; see for example [2, Theorem 2.2]. In contrast, a maximal asymmetric digraph with all loops does not have $P_0^+$-completion, since the complement of such a digraph does not have a 2-cycle.

5. **Classification of small digraphs as to $P_0^+$-completion.** In this section, we apply the results in the previous sections to classify the digraphs of order at most four that include all loops as to $P_0^+$-completion.

Any matrix which is permutation similar to a $P_0^+$-matrix is a $P_0^+$-matrix. Therefore, if a digraph $D$ has $P_0^+$-completion, then any digraph which is isomorphic to $D$
The $P_0^+$-matrix completion problem

has $P_0^+$-completion, that is, any digraph obtained by a labelling of the unlabelled digraph associated to $D$ has $P_0^+$-completion.

The nomenclature of the digraphs considered in the sequel is as per their order in the atlas in [6, Appendix, pp. 233]. Here, $D_p(q, n)$ is the one obtained by attaching a loop at each of the vertices to the $n$-th member in the list of digraphs with $p$ vertices and $q$ (non-loop) arcs in the atlas. The classification is broken up into a series of lemmas.

**Lemma 5.1.** The digraphs $D_p(q, n)$ which are listed below do not have $P_0^+$-completion.

\[
\begin{align*}
p &= 2; & q &= 1 \\
p &= 3; & q &= 3; & n &= 2, 3 \\
& & q &= 4; & n &= 2, 3, 4 \\
& & q &= 5 \\
p &= 4; & q &= 6; & n &= 45–48 \\
& & q &= 7; & n &= 29–38 \\
& & q &= 8; & n &= 16–27 \\
& & q &= 9; & n &= 4–13 \\
& & q &= 10; & n &= 2–5 \\
& & q &= 11.
\end{align*}
\]

*Proof.* Each of the digraphs listed contains all loops but its complement does not contain a 2-cycle. Hence, by Corollary 4.10, the digraph does not have $P_0^+$-completion. 

**Lemma 5.2.** The digraphs $D_p(q, n)$ which are listed below do not have $P_0^+$-completion.

\[
\begin{align*}
p &= 3; & q &= 2; & n &= 1, 3, 4 \\
& & q &= 3; & n &= 1, 4 \\
& & q &= 4; & n &= 1 \\
p &= 4; & q &= 3; & n &= 8, 11 \\
& & q &= 4; & n &= 10, 12, 14, 15, 21, 27 \\
& & q &= 5; & n &= 4–6, 11, 14–17, 19, 21–24, 26, 28, 29, 31, 34, 36, 37 \\
& & q &= 6; & n &= 1, 2, 9–23, 26, 27, 29, 30, 32–41, 43, 44 \\
& & q &= 7; & n &= 1, 3–28 \\
& & q &= 8; & n &= 1, 3–15 \\
& & q &= 9; & n &= 1–3 \\
& & q &= 10; & n &= 1.
\end{align*}
\]
Proof. Each of the digraphs listed contains all loops but its complement is not weakly stratified. Thus, by Theorem 4.5, the digraphs do not have $P_0^+$-completion. □

**Lemma 5.3.** The digraphs $D_4(q, n)$ which are listed below do not have $P_0^+$-completion.

- $q = 4; \ n = 13$
- $q = 5; \ n = 12, 13, 18, 20$
- $q = 6; \ n = 24, 25, 28, 31, 42.$

Proof. Each of the digraphs listed has an induced subdigraph isomorphic to one of the digraphs in Figure 4.1. Hence, by Corollary 4.2, the digraphs do not have $P_0^+$-completion. □

**Lemma 5.4.** The digraphs $D_4(q, n)$ which are listed below do not have $P_0^+$-completion.

- $q = 6; \ n = 6, 8$
- $q = 7; \ n = 2$
- $q = 8; \ n = 2.$

Proof. The digraph $D_4(8, 2)$ is the symmetric 4-cycle $C_4$, and that it does not have $P_0^+$-completion has been seen in Example 4.7. Next, for any completion $B_1$ of the partial $P_0^+$-matrix

$$M_1 = \begin{bmatrix} 0 & 0 & x_{13} & 0 \\ 1 & 0 & x_{23} & x_{24} \\ x_{31} & -1 & 1 & 1 \\ 1 & x_{42} & 0 & 0 \end{bmatrix}$$

specifying $D_4(7, 2)$ we have $B(1, 2, 3) = -x_{13}$ and $B(1, 3, 4) = x_{13}$. Therefore, for $B_1$ to be a $P_0^+$-matrix, $x_{13} = 0$. However, this yields det $B = 0$. Thus, $B_1$ cannot be a $P_0^+$-matrix. Similarly, the partial $P_0^+$-matrices

$$M_2 = \begin{bmatrix} 0 & 1 & x_{13} & 1 \\ 0 & 0 & -1 & x_{24} \\ x_{31} & x_{32} & 1 & x_{34} \\ 0 & x_{42} & 1 & 0 \end{bmatrix} \quad \text{and} \quad M_3 = \begin{bmatrix} 0 & 0 & x_{13} & 0 \\ 1 & 0 & x_{23} & x_{24} \\ x_{31} & -1 & 1 & 1 \\ 1 & x_{42} & x_{43} & 0 \end{bmatrix}$$
specifying the digraphs $D_4(6, 6)$ and $D_4(6, 8)$, respectively, do not have $P_0^+$-completions. Indeed, for any completion $B_2$ of $M_2$, we have $B_2(1, 2, 3) = -x_{31}$ and $B_2(1, 3, 4) = x_{31}$. Further, $M_3$ is the transpose of $M_2$. ∎

The following result can be easily verified.

**Lemma 5.5.** For real numbers $a, b, c, d$, the inequalities
\[
ax + by \geq 0 \\
px + dy \geq 0 \\
xy < 0
\]
do not have a solution for $x$ and $y$ only if one of the following holds:

(i) $a = 0, c = 0, \quad bd < 0 \quad$ or $\quad b = 0, d = 0, \quad ac < 0$;

(ii) $b = 0, c = 0, \quad ad > 0 \quad$ or $\quad a = 0, d = 0, \quad bc > 0$;

(iii) $c > 0, d > 0, \quad \frac{a}{c} = \frac{b}{d} < 0$.

**Lemma 5.6.** The digraphs $D_4(6, n), \quad n = 3, 4, 5, 7$, have $P_0^+$ completion.

![Fig. 5.2. Digraphs $D_4(q, n)$ have $P_0^+$-completion.](image)

**Proof.** For each partial matrix considered below we denote the specified diagonal entries by $d_i$ and the specified off-diagonal entries by $a_{ij}$. First, let $M$ be a partial $P_0^+$-matrix specifying the digraph $D_4(6, 3)$. We show that for some choices of $s$ and $t$ the completion
\[
B = \begin{bmatrix}
d_1 & a_{12} & s & 0 \\
a_{21} & d_2 & a_{23} & t \\
-s & a_{32} & d_3 & a_{34} \\
0 & -t & a_{43} & d_4
\end{bmatrix}
\]
of $M$ is a $P_0^+$-matrix. Clearly, for any choices of $s$ and $t$, all $2 \times 2$ principal minors of $B$ are nonnegative. Moreover, $B(1, 3)$ and $B(2, 4)$ are positive for nonzero $s$ and $t$. 


Now, the $3 \times 3$ principal minors of $B$ are

\begin{align*}
B(1,2,3) &= d_1 B(2,3) + s (a_{21} a_{32} - a_{12} a_{23}) + (d_2 s^2 - d_3 a_{12} a_{21}) \quad (5.1) \\
B(1,2,4) &= d_1 t^2 + d_4 B(1,2) \quad (5.2) \\
B(1,3,4) &= d_1 B(3,4) + d_4 s^2 \quad (5.3) \\
B(2,3,4) &= d_2 B(3,4) + t (a_{32} a_{43} - a_{23} a_{34}) + (d_3 t^2 - d_4 a_{23} a_{32}) \quad (5.4)
\end{align*}

Since $B(1,2) \geq 0$, it follows that $a_{12} a_{21} \leq 0$ if $d_2 = 0$. Similarly, $a_{23} a_{32} \leq 0$ if $d_3 = 0$. Consequently, for $s$ and $t$ with large magnitude so that $d_2 s^2 - d_3 a_{12} a_{21}$ and $d_3 t^2 - d_4 a_{23} a_{32}$ are nonnegative and with appropriate signs so that the term $s(a_{21} a_{32} - a_{12} a_{23})$ in (5.1) and the term $t (a_{32} a_{43} - a_{23} a_{34})$ in (5.4) are nonnegative, we get all $3 \times 3$ principal minors of $B$ nonnegative. Moreover, at least one of them can be made positive, because $d_i > 0$ for some $i$. Further, choosing $|t| = |s|$ we get

\[ \det B = s^4 + p(s), \]

where $p(s)$ is a polynomial in $s$ of degree at most 3. Hence, for large values of $|s|$, $\det B > 0$, and $B$ is a $P_0^+$-matrix.

Next, let $M$ be a partial $P_0^+$-matrix specifying the digraph $D_4(6,4)$. We show that for the following choices of the unspecified entries and some suitable choice of $t$ and $s$, the completion

\[ B = \begin{bmatrix}
  d_1 & a_{12} & t & -a_{41} \\
  a_{21} & d_2 & a_{23} & s \\
  -t & -a_{23} & d_3 & a_{34} \\
  a_{41} & -s & a_{43} & d_4
\end{bmatrix} \]

of $M$ is a $P_0^+$-matrix. Clearly, for any nonzero choices of $t$ and $s$, all $2 \times 2$ principal minors of $B$ are nonnegative and $B(1,3) > 0$, $B(2,4) > 0$. Now, $3 \times 3$ principal minors of $B$ are

\begin{align*}
B(1,2,3) &= d_3 B(1,2) + d_1 a_{23}^2 + d_4 t^2 - t(a_{12} + a_{21})a_{23} \quad (5.5) \\
B(1,2,4) &= d_4 B(1,2) + d_2 a_{41}^2 + d_1 s^2 + s(a_{12} + a_{21})a_{41} \quad (5.6) \\
B(1,3,4) &= d_1 B(3,4) + d_3 a_{41}^2 + d_4 t^2 + t(a_{34} + a_{43})a_{41} \quad (5.7) \\
B(2,3,4) &= d_2 B(3,4) + d_4 a_{23}^2 + d_3 s^2 - s(a_{34} + a_{43})a_{23} \quad (5.8)
\end{align*}

**Case 1:** $a_{23} = 0$ or $a_{41} = 0$. If $a_{23} = 0$, then choose signs for $s$ and $t$ such that $s(a_{12} + a_{21})a_{41} \geq 0$ and $t(a_{34} + a_{43})a_{41} \geq 0$. Then, all $3 \times 3$ principal minors are nonnegative, and since $d_i > 0$ for some $i$, at least one of these minors is positive. Finally, for large values of $|s|$ and $|t|$, $\det B > 0$. The case when $a_{41} = 0$ is similar.
The $P^+_0$-matrix completion problem

**Case 2:** $a_{23} \neq 0$ and $a_{41} \neq 0$. We note that one or more of the diagonal entries of $M$ are positive, since $M$ is a partial $P^+_0$-matrix. If $d_1 > 0$, then we choose $t$ such that

$$t(a_{34} + a_{43})a_{41} \geq 0$$

$$d_1a_{23}^2 - t(a_{12} + a_{21})a_{23} > 0.$$  

Further, we choose appropriate sign for $s$ such that $s(a_{34} + a_{43})a_{23} \leq 0$. Then for large values of $|s|$ all $3 \times 3$ principal minors of $B$ are positive, since $M$ and $zw$ are nonpositive and at least one of them negative, we have all $2 \times 2$ principal minors of $B$ are positive. We get similar results if one of $d_2, d_3$ and $d_4$ is positive instead of $d_1$.

Similarly, with suitable values of $t$ and $s$, a partial $P^+_0$-matrix $M$ specifying the digraph $D_4(6,5)$ can be completed to a $P^+_0$-matrix

$$B = \begin{bmatrix}
d_1 & a_{12} & t & -a_{41} \\
a_{21} & d_2 & -a_{32} & s \\
-t & a_{32} & d_3 & a_{34} \\
a_{41} & -s & a_{43} & d_4
\end{bmatrix}.$$  

Finally, let $M$ be a partial $P^+_0$-matrix specifying the digraph $D_4(6,7)$. We show that for some choices of $s, t$ and $x, y, z, w$ the completion

$$B = \begin{bmatrix}
d_1 & a_{12} & x & s \\
t & d_2 & a_{23} & z \\
y & a_{32} & d_3 & a_{34} \\
a_{41} & w & a_{43} & d_4
\end{bmatrix}$$

of $M$ is a $P^+_0$-matrix. Clearly, for any choices of $s, t, x, y, z, w$ such that $a_{12}t, a_{41}s, xy$ and $zw$ are nonpositive and at least one of them negative, we have all $2 \times 2$ principal minors of $B$ are nonnegative and at least one of them is positive. Now, the $3 \times 3$ principal minors of $B$ are

$$B(1,2,3) = (d_1B(2,3) - d_3a_{12}t - d_2xy) + a_{32}tx + a_{12}a_{23}y$$  

$$B(1,3,4) = (d_1B(3,4) - d_3a_{41}s - d_4xy) + a_{34}a_{41}x + a_{43}sy$$  

$$B(1,2,4) = (d_4B(1,2) - d_2a_{41}s - d_1zw) + a_{41}a_{12}z + stw$$  

$$B(2,3,4) = (d_2B(3,4) - d_4a_{23}a_{32} - d_3zw) + a_{32}a_{43}z + a_{23}a_{34}w.$$  

Since the terms in the parentheses in the right sides of the above equations are all nonnegative under our choices of the unspecified entries, the $3 \times 3$ principal minors are nonnegative if

$$a_{32}tx + a_{12}a_{23}y \geq 0$$  

$$a_{34}a_{41}x + a_{43}sy \geq 0$$  

$$a_{41}a_{12}z + stw \geq 0$$  

$$a_{32}a_{43}z + a_{23}a_{34}w \geq 0.$$
In view of Lemma 5.5, the above equations have solutions with \( s = t = 0 \), and arbitrary large values of \(-xy\) and \(-zw\) in all cases except when \( a_{12}a_{23}a_{34}a_{41} > 0 \) and \( a_{32}a_{43} = 0 \). In the latter case, putting \( s = t = 0 \), we have

\[
B = \begin{bmatrix}
  d_1 & a_{12} & x & 0 \\
  0 & d_2 & a_{23} & z \\
  y & 0 & d_3 & a_{34} \\
  a_{41} & w & 0 & d_4
\end{bmatrix}.
\]

If \( d_1 > 0 \), then a \( P_0^+ \)-completion of \( M \) is obtained by setting \( x = y = 0 \) and choosing \(-z = w\) such that \( a_{23}a_{34}(d_1w - a_{12}a_{41}) > 0 \) in \( B \). A \( P_0^+ \)-completion of \( M \) can be obtained in a similar way in case some other \( d_i \), instead of \( d_1 \), is positive. Note that the structure of the above matrix \( B \) is invariant under a cyclic permutation. This completes the proof. \( \square \)

**Lemma 5.7.** The digraphs \( D_4(5, 25) \) and \( D_4(5, 27) \) have \( P_0^+ \)-completion.

**Proof.** Let \( M \) be a partial \( P_0^+ \)-matrix specifying the digraph \( D_4(5, 25) \). We show that for some choices of \( r, s, t, x, y, z \) and \( w \) the completion

\[
B = \begin{bmatrix}
  d_1 & r & a_{13} & a_{14} \\
  a_{21} & d_2 & s & x \\
  t & a_{32} & d_3 & z \\
  a_{41} & y & w & d_4
\end{bmatrix}
\]

of \( M \) is a \( P_0^+ \)-matrix.

**Case 1:** \( a_{13} \neq 0 \) or \( a_{21} \neq 0 \). We put

\[
r = -a_{21}, \quad s = -a_{32}, \quad t = -a_{13}, \quad y = -x \quad \text{and} \quad w = -z.
\]

Then, all \( 2 \times 2 \) principal minors of \( B \) are nonnegative and one of \( B(1, 2) \) and \( B(1, 3) \) is positive. Further, the \( 3 \times 3 \) principal minors of \( B \) are

\[
\begin{align*}
B(1, 2, 3) &= d_1B(2, 3) + d_3a_{21}^2 + d_2a_{13}^2 \\
B(1, 2, 4) &= d_2B(1, 4) + d_4a_{21}^2 + d_1x^2 - a_{21}(a_{14} + a_{41})x \\
B(1, 3, 4) &= d_3B(1, 4) + d_4a_{32}^2 + d_1z^2 + a_{32}(a_{14} + a_{41})z \\
B(2, 3, 4) &= d_4B(2, 3) + d_3x^2 + d_2z^2.
\end{align*}
\]
For nonzero $x$ and $z$ satisfying

$$a_{21}(a_{14} + a_{41})x \leq 0 \quad \text{and} \quad a_{13}(a_{14} + a_{41})z \geq 0,$$  \hspace{1cm} (5.21)

all $3 \times 3$ principal minors of $B$ are nonnegative, and since $d_i > 0$ for some $i$, at least one of these minors is positive. Now, breaking the determinant along the first column of $B$ we get

$$\det B = \det \begin{bmatrix} d_1 & -a_{21} & a_{13} & a_{14} \\ a_{21} & d_2 & -a_{32} & x \\ -a_{13} & a_{32} & d_3 & z \\ -a_{14} & -x & -z & d_4 \end{bmatrix} + \det \begin{bmatrix} 0 & -a_{21} & a_{13} & a_{14} \\ 0 & d_2 & -a_{32} & x \\ 0 & a_{32} & d_3 & z \\ a_{41} + a_{14} & -x & -z & d_4 \end{bmatrix}.$$  

The terms in the first determinant not involving the diagonal entries $d_i$ are given by the determinant of the skew-symmetric matrix

$$\begin{bmatrix} 0 & -a_{21} & a_{13} & a_{14} \\ a_{21} & 0 & -a_{32} & x \\ -a_{13} & a_{32} & 0 & z \\ -a_{14} & -x & -z & 0 \end{bmatrix},$$

which equals $(a_{13}x + a_{21}z + a_{14}a_{32})^2$. Further, the only terms of total degree more than 1 in $x$ and $z$ involving $d_i$ in the first determinant are $d_1d_2z^2$ and $d_1d_3x^2$. The second determinant does not contribute any term with total degree more than 1 in $x$ and $z$. Therefore, we get

$$\det B = (a_{13}x + a_{21}z + a_{14}a_{32})^2 + d_1d_2z^2 + d_1d_3x^2 + p(x, z),$$

where $p(x, z)$ is a polynomial in $x$ and $z$ with total degree at most 1. Since either $a_{13} \neq 0$ or $a_{21} \neq 0$, for $x$ and $z$ with large magnitudes we have $\det B > 0$.

**Case 2:** $a_{13} = a_{21} = 0$. In this case, the $3 \times 3$ principal minors of

$$B = \begin{bmatrix} d_1 & r & 0 & a_{14} \\ 0 & d_2 & s & x \\ t & a_{32} & d_3 & z \\ a_{41} & y & w & d_4 \end{bmatrix}$$

are given by

$$B(1, 2, 3) = d_1B(2, 3) + rst$$  \hspace{1cm} (5.22)
$$B(1, 2, 4) = d_2B(1, 4) + x(a_{41}r - d_1y)$$  \hspace{1cm} (5.23)
$$B(1, 3, 4) = d_3B(1, 4) + w(a_{14}t - d_1z)$$  \hspace{1cm} (5.24)
$$B(2, 3, 4) = d_4B(2, 3) - d_3xy - d_2zw + a_{32}xw + syz.$$  \hspace{1cm} (5.25)
If $a_{14} \neq 0$ then we put
\[ r = x = z = 0, \quad t = s, \quad w = \frac{a_{14}s}{|a_{14}|} \quad y = -|s|, \]
where the nonzero $s$ is to be chosen such that $a_{32}s \leq 0$. Then, all $3 \times 3$ principal minors are nonnegative and $B(1, 3, 4) > 0$. Since $\det B$ is a monic polynomial in $|s|$ of degree 3, $B$ is a $P_1^+$-matrix for sufficiently large values of $|s|$. If $a_{41} \neq 0$, then we put
\[ t = y = w = 0, \quad r = z, \quad x = \frac{a_{41}r}{|a_{41}|} \quad s = \frac{-|r|}{a_{41}}, \]
where the nonzero $s$ is to be chosen such that $a_{32}s \leq 0$. Then, $B$ is a $P_0^+$-matrix for sufficiently large values of $|r|$. Finally, if $a_{14} = a_{41} = 0$, then we put $y = z = 0, x = t, \ r = st, \ w = -st$ with $s \neq 0$ such that $a_{32}s \leq 0$. Then, $B$ is a $P_0^+$-matrix for large values of $|s|$.

Similarly, with suitable values of $r, x, y, w, z, t$ and $s$, any partial $P_0^+$-matrix
\[
M = \begin{bmatrix}
  d_1 & r & a_{13} & a_{14} \\
  a_{23} & d_2 & a_{23} & x \\
  t & s & d_3 & z \\
  a_{41} & y & w & d_4
\end{bmatrix}
\]
specifying the digraph $D_4(5, 27)$ can be completed to a $P_0^+$-matrix. \qed

**Lemma 5.8.** The asymmetric digraphs $D_4(5, n), n = 30, 32, 33, 35, 38$, have $P_0^+$-completion.

**Proof.** We prove the result for $D_4(5, 38)$; the proofs for the other digraphs are similar. Consider a partial $P_0^+$-matrix
\[
M = \begin{bmatrix}
  d_1 & a_{12} & x & a_{14} \\
  y & d_2 & r & a_{24} \\
  s & a_{32} & d_3 & z \\
  u & w & a_{43} & d_4
\end{bmatrix}
\]
The \( P_0^+ \)-matrix completion problem

specifying the digraph \( D_4(5, 38) \), where \( x, y, r, s, z, u, w \) are unspecified entries. Now, for a completion \( B \) of \( M \) the \( 3 \times 3 \) principal minors of \( B \) are given by

\[
B(1, 2, 3) = d_1B(2, 3) + a_{12}sr - d_3ya_{12} + xy a_{32} - d_2xs \\
B(1, 2, 4) = d_1B(2, 4) - a_{12}yd_4 + a_{14}yw + a_{12}a_{24}u - d_2 a_{14}u \\
B(1, 3, 4) = d_1B(3, 4) + xzu - sx d_4 + a_{14}sa_{43} - a_{14}ud_3 \\
B(2, 3, 4) = d_2B(3, 4) + rzw - ra_{32}d_4 + a_{24}a_{32}a_{43} - a_{24}wd_4.
\]

**Case 1:** \( a_{24} \neq 0 \) or \( a_{12}a_{43} + a_{14}a_{32} \neq 0 \). In that case, we put

\[
y = -a_{12}, \quad r = -a_{32}, \quad u = -a_{14}, \quad w = -a_{24}, \quad z = -a_{43}, \quad x = -s = t.
\]

Then, all \( 2 \times 2 \) principal minors are nonnegative and for \( t \neq 0, B(1, 3) > 0 \). Further, it can be easily seen that each principal minor of order \( 3 \times 3 \) is nonnegative. Now, \( M \) being a partial \( P_0^+ \)-matrix, at least one of the \( d_i \) is positive. Moreover, in case \( (a_{14}a_{32} + a_{43}a_{12}) \neq 0 \), at least one of the \( a_{12}a_{43} \) and \( a_{14}a_{32} \) is nonzero. It can be easily verified that in each of the three cases, viz. \( a_{24} \neq 0, a_{12} \neq a_{43} \) and \( a_{14} \neq a_{32} \), at least one \( 3 \times 3 \) principal minor of \( B \) is positive, depending on \( i \) with \( d_i > 0 \). Further, from the condition \( a_{24} \neq 0 \) or \( a_{12}a_{43} + a_{14}a_{32} \neq 0 \), for any nonzero \( t \) we have

\[
\det B = (a_{14}a_{32} + a_{43}a_{12} + ta_{24})^2 + d_1 d_2 a_{43}^2 + d_1 d_3 a_{24}^2 + d_1 d_4 a_{52}^2 \\
+ d_2 d_3 a_{14}^2 + d_2 d_4 t^2 + d_3 d_4 a_{12}^2 + d_1 d_2 d_3 d_4 > 0.
\]

**Case 2:** \( a_{24} = 0, a_{12}a_{43} + a_{14}a_{32} = 0 \). This case is divided into the following subcases.

**Subcase I:** \( (a_{32} \neq 0 \text{ or } a_{14} \neq 0) \) and \( (a_{12} \neq 0 \text{ or } a_{43} \neq 0) \). In this case, we consider a completion \( B \) of \( M \) by putting \( r = s = u = 0 \) in \( M \). We can choose nonzero \( y, w \) and \( z \) with appropriate signs such that \( a_{12}y \leq 0, a_{43}z \leq 0 \) and \( a_{14}yw \geq 0 \). Then, each of the \( 2 \times 2 \) principal minors of \( B \) becomes nonnegative. Since at least one of \( a_{12} \) and \( a_{43} \) is nonzero, we have either \( B(1, 2) > 0 \) or \( B(3, 4) > 0 \). Now, the equation \( a_{12}a_{43} + a_{14}a_{32} = 0 \) implies that the sign of each nonzero term in the equation is opposite of the sign of the product of the other three, if the product is nonzero. So, we can always choose an appropriate sign for \( x \) such that \( ya_{32}x \geq 0 \) and \( -xyzw > 0 \). Then, each of the \( 3 \times 3 \) principal minors is nonnegative. Further, at least one of \( a_{14}yw \) and \( ya_{32}x \) is positive, since \( a_{14} \) or \( a_{32} \) is nonzero, which yields either \( B(1, 2, 3) \) or \( B(1, 2, 4) \) positive. Finally, we choose \( |x| = |y| = |z| = |w| = t > 0 \). Then, for sufficiently large values of \( t \) we have \( \det B = t^4 + p(t) > 0 \), where \( p(t) \) is a polynomial of degree at most 3.

**Subcase II:** \( (a_{32} = 0 \text{ and } a_{14} = 0) \) and \( (a_{12} \neq 0 \text{ or } a_{43} \neq 0) \). In this case, consider a completion \( B \) of \( M \) obtained by putting \( s = u = 0 \) in \( M \). Now, we choose appropriate signs for \( y, z \) and \( r \) such that \( a_{12}y \leq 0, a_{43}z \leq 0 \) and \( rzw > 0 \). Since at least one of \( a_{12} \) and \( a_{43} \) is nonzero, each \( 2 \times 2 \) principal minor of \( B \) is nonnegative, and either
B(1, 2) > 0 or B(3, 4) > 0. Moreover, each of the 3 × 3 principal minors of B is nonnegative and B(2, 3, 4) > 0. Finally, we choose appropriate sign for x and put |x| = |y| = |z| = |w| = t > 0. Then, for large values of t we have det B = t^4 + p(t) > 0, where p(t) is a polynomial of degree at most 3.

Subcase III: a_{24} = 0; (a_{32} ≠ 0 or a_{14} ≠ 0) and (a_{12} = 0 and a_{43} = 0). Consider a completion B of M obtained by putting, u = r = 0 and s = −x = t. Then, all 2 × 2 principal minors of B are nonnegative and for large values of t, B(1, 3) > 0. Next, we choose appropriate signs for t and w such that −tya_{32} ≥ 0 and a_{14}yw ≥ 0. Since a_{32} ≠ 0 or a_{14} ≠ 0, each of the 3 × 3 principal minors is nonnegative, and either B(1, 2, 3) or B(1, 2, 4) is positive. Finally, we choose appropriate sign for z and put |x| = |y| = |z| = |w| = t > 0. Then, det B > 0 for large values of t.

Subcase IV: We are left with the case when all specified off-diagonal entries are zero. In this case, a P_0^+ -completion of M is obtained by putting x = r = z = w = −y = −s = t > 0.

Combining all cases, we see that the partial P_0^+ -matrix M can be completed to a P_0^+ -matrix, and therefore, the digraph D_4(5, 38) has P_0^+ -completion. □

**Theorem 5.9.** For 1 ≤ p ≤ 4, the digraph D_p(q, n) has P_0^+ -completion if and only if it is one of the digraphs listed below:

- p = 1
- p = 2; q = 0, 2
- p = 3; q = 0, 1, 6
  - q = 2; n = 2
- p = 4; q = 0, 1, 12
  - q = 2; n = 1–5
  - q = 3; n = 1–7, 9, 10, 12, 13
  - q = 4; n = 1–9, 11, 16–20, 22–26
  - q = 5; n = 1–3, 7–10, 25, 27, 30, 32, 33, 35, 38
  - q = 6; n = 3–5, 7.

**Proof.** It is clear that D_p(q, n) has P_0^+ -completion if q = 0 or it is a complete digraph.

**Case:** p = 3. It follows from Example 3.2 that D_3(2, 2) (i.e., D_3 in Figure 3.2) has P_0^+ -completion. Further, D_3(1, 1) is a spanning subdigraph of D_3(2, 2) and has P_0 -completion. Therefore, in view of Theorem 3.3, D_3(1, 1) has P_0^+ -completion. Rest of the digraphs of order 3 appear in the lists in Lemma 5.1 and Lemma 5.2 and do not
The \( P_0^{+} \)-matrix completion problem

Fig. 5.5. The digraphs have \( P_0^{+} \)-completion.

have \( P_0^{+} \)-completion.

Case: \( p = 4 \). The digraphs \( D_4(q,n), q = 5, n = 1 - 3; q = 4, n = 1 - 9; q = 3, n = 1 - 7; q = 2, n = 1 - 5; q = 1, \) have \( P_0 \)-completion (see [2]) and each of them is a spanning subdigraph of the digraph \( D_4(6,3) \). Therefore, in view of Theorem 3.3, they have \( P_0^{+} \)-completion, since \( D_4(6,3) \) has \( P_0^{+} \)-completion (see Lemma 5.6).

The digraphs \( D_4(q,n), q = 5, n = 7, 9; q = 4, n = 16 - 18, \) have \( P_0 \)-completion and each of them is a spanning subdigraph of \( D_4(6,4) \). Therefore, they have \( P_0^{+} \)-completion, since \( D_4(6,4) \) has \( P_0^{+} \)-completion (see Lemma 5.6).

The digraphs \( D_4(q,n), q = 5, n = 8, 10; q = 4, n = 19, \) have \( P_0 \)-completion and each of them is a spanning subdigraph of \( D_4(6,5) \). Therefore, they have \( P_0^{+} \)-completion, since \( D_4(6,5) \) has \( P_0^{+} \)-completion (see Lemma 5.6).

The digraphs \( D_4(q,n), q = 4, n = 11, 20, 24; q = 3, n = 9, 10, 12, \) have \( P_0 \)-completion and each of them is a spanning subdigraph of \( D_4(5,25) \). Therefore, they have \( P_0^{+} \)-completion, since \( D_4(5,25) \) has \( P_0^{+} \)-completion (see Lemma 5.7).

Finally, the digraphs \( D_4(q,n), q = 4, n = 22, 26; q = 3, n = 13, \) have \( P_0 \)-completion and each of them is a spanning subdigraph of \( D_4(5,27) \). Therefore, they have \( P_0^{+} \)-completion, since \( D_4(5,27) \) has \( P_0^{+} \)-completion (see Lemma 5.7).

The proof is complete, as the rest of the digraphs of order 4 appear in the lists in Lemmas 5.1–5.8.

6. Comparison with \( P_0^{+} \), \( P_0^{-} \) and \( Q \)-matrix completion problems. Although a \( P_0^{+} \)-matrix is a \( Q \)-matrix as well as a \( P_0 \)-matrix, a digraph which has both \( Q \)-completion and \( P_0 \)-completion may not have \( P_0^{+} \)-completion. The digraph \( D_4(7,3) \) in Figure 6.1 is an example of a digraph having \( Q \)-completion as well as \( P_0 \)-completions, but not \( P_0^{+} \)-completion.

We have observed that for all digraphs we have considered including the digraphs of order at most 4, if a digraph \( D \) has \( P_0^{+} \)-completion, then \( D \) has both \( P_0 \)-completion and \( Q \)-completion. However, we do not know whether the result is true in general.
The following result shows that the completion problems for the classes of $P$- and $P_0^+$-matrices are related. The effective argument used in the proof was applied for comparing a large number of pairs of completion problems by L. Hogben [9].

**Theorem 6.1.** Any digraph that has $P_0^+$-completion also has $P$-completion.

**Proof.** Let $D$ be a digraph which has $P_0^+$-completion and $M$ be a partial $P$-matrix specifying $D$. Then, all fully specified principal minors of $M$ are positive. Since determinant of a matrix is a continuous function of its entries, there is $\epsilon > 0$ such that the partial matrix $M_0$ obtained from $M$ by decreasing the specified diagonal entries by $\epsilon$ is a partial $P$-matrix. Since a partial $P$-matrix is a partial $P_0^+$-matrix, $M_0$ is a partial $P_0^+$-matrix specifying $D$. Consequently, $M_0$ has a $P_0^+$-completion $B_0$. We now have a $P$-completion of $M$, namely, $B = B_0 + \epsilon I$, where $I$ is the identity matrix. \[\square\]

The converse of Theorem 6.1 is not true. For example, the digraph $D_3(2, 1)$ does not have $P_0^+$-completion (see Lemma 5.2). However, $D_3(2, 1)$ has $P$-completion, since any digraph of order 3 has $P$-completion (see [10]).

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The $P^+_0$-matrix completion problem


