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## A NOTE ON A CONJECTURE FOR THE DISTANCE LAPLACIAN MATRIX\*

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**Abstract.** In this note, the graphs of order  $n$  having the largest distance Laplacian eigenvalue of multiplicity  $n - 2$  are characterized. In particular, it is shown that if the largest eigenvalue of the distance Laplacian matrix of a connected graph  $G$  of order  $n$  has multiplicity  $n - 2$ , then  $G \cong S_n$  or  $G \cong K_{p,p}$ , where  $n = 2p$ . This resolves a conjecture proposed by M. Aouchiche and P. Hansen in [M. Aouchiche and P. Hansen. A Laplacian for the distance matrix of a graph. *Czechoslovak Mathematical Journal*, 64(3):751–761, 2014.]. Moreover, it is proved that if  $G$  has  $P_5$  as an induced subgraph then the multiplicity of the largest eigenvalue of the distance Laplacian matrix of  $G$  is less than  $n - 3$ .

**Key words.** Distance Laplacian matrix, Laplacian matrix, Largest eigenvalue, Multiplicity of eigenvalues.

**AMS subject classifications.** 05C12, 05C50, 15A18.

**1. Introduction.** Let  $G = (V, E)$  be a connected graph and the distance (the length of a shortest path) between vertices  $v_i$  and  $v_j$  of  $G$  be denoted by  $d_{i,j}$ . The distance matrix of  $G$ , denoted by  $\mathcal{D}(G)$ , is the  $n \times n$  matrix whose  $(i, j)$ -entry is equal to  $d_{i,j}$ ,  $i, j = 1, 2, \dots, n$ . The transmission  $\text{Tr}(v_i)$  of a vertex  $v_i$  is defined as the sum of the distances from  $v_i$  to all other vertices in  $G$ . For more details about the distance matrix we suggest, for example, [5]. M. Aouchiche and P. Hansen [3] introduced the Laplacian for the distance matrix of a connected graph  $G$  as  $\mathcal{D}^L(G) = \text{Tr}(G) - \mathcal{D}(G)$ , where  $\text{Tr}(G)$  is the diagonal matrix of vertex transmissions. We write  $(\partial_1^L, \partial_2^L, \dots, \partial_n^L = 0)$ , for the distance Laplacian spectrum of a connected graph  $G$ , the  $\mathcal{D}^L$ -spectrum, and assume that the eigenvalues are arranged in a nonincreasing order. The multiplicity of the eigenvalue  $\partial_i^L$  is denoted by  $m(\partial_i^L)$ , for  $1 \leq i \leq n$ . We often use exponents to exhibit the multiplicity of the distance Laplacian eigenvalues when we write the  $\mathcal{D}^L$ -spectrum. The distance Laplacian matrix has been recently

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studied ([2, 4, 6]) and, in [4], M. Aouchiche and P. Hansen proposed some conjectures about it. Among them, we consider in this work the following one:

CONJECTURE 1.1. [4] If  $G$  is a graph on  $n \geq 3$  vertices and  $G \not\cong K_n$ , then  $m(\partial_1^L(G)) \leq n - 2$  with equality if and only if  $G$  is the star  $S_n$  or  $n = 2p$  for the complete bipartite graph  $K_{p,p}$ .

In this paper, we prove the conjecture. In order to obtain this result we analyze how the existence of  $P_4$  as an induced subgraph influences the  $\mathcal{D}^L$ -spectrum of a connected graph. We conclude that, in this case, the largest distance Laplacian eigenvalue has multiplicity less than or equal to  $n - 3$ . This fact motivated us to also investigate the influence of an induced  $P_5$  subgraph in the  $\mathcal{D}^L$ -spectrum of a graph. We prove that if a graph has an induced  $P_5$  subgraph then the largest eigenvalue of its distance Laplacian matrix has multiplicity at most  $n - 4$ . Although we do not make a general approach by characterizing the graphs that have the largest distance Laplacian eigenvalue with multiplicity  $n - 3$ , some considerations on this topic are made.

**2. Preliminaries.** In what follows,  $G = (V, E)$ , or just  $G$ , denotes a graph with  $n$  vertices and  $\overline{G}$  denotes its complement. The diameter of a connected graph  $G$  is denoted by  $diam(G)$ . As usual, we write, respectively,  $P_n$ ,  $C_n$ ,  $S_n$  and  $K_n$ , for the path, the cycle, the star and the complete graph, all with  $n$  vertices. We denote by  $K_{p,p}$  and by  $K_{p,p,p}$  the balanced complete bipartite and tripartite graph, respectively. Now, we recall the definitions of some operations with graphs that will be used. For this, let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be vertex disjoint graphs:

- The *union* of  $G_1$  and  $G_2$  is the graph  $G_1 \cup G_2$  (or  $G_1 + G_2$ ), whose vertex set is  $V_1 \cup V_2$  and whose edge set is  $E_1 \cup E_2$ ;
- The *complete product* or *join* of graphs  $G_1$  and  $G_2$  is the graph  $G_1 \vee G_2$  obtained from  $G_1 \cup G_2$  by joining each vertex of  $G_1$  with every vertex of  $G_2$ .

A graph  $G$  is a cograph, also known as a decomposable graph, if no induced subgraph of  $G$  is isomorphic to  $P_4$  [1]. About the cographs, we also have the following characterizations:

THEOREM 2.1. [1] *Given a graph  $G$ , the following statements are equivalent:*

- $G$  is a cograph.
- The complement of any connected subgraph of  $G$  with at least two vertices is disconnected.
- Every connected subgraph of  $G$  has diameter less than or equal to 2.

We denote by  $(\mu_1, \mu_2, \dots, \mu_n = 0)$  the  $L$ -spectrum of  $G$ , i.e., the spectrum of the Laplacian matrix of  $G$ , and assume that the eigenvalues are labeled such that

$\mu_1 \geq \mu_2 \geq \dots \geq \mu_n = 0$ . It is well known that the multiplicity of the Laplacian eigenvalue 0 is equal to the number of components of  $G$  and that  $\mu_{n-i}(\overline{G}) = n - \mu_i(G)$ ,  $\forall 1 \leq i \leq n - 1$  (see [8] for more details).

The following results regarding the distance Laplacian matrix are already known.

**THEOREM 2.2.** [3] *Let  $G$  be a connected graph on  $n$  vertices with  $\text{diam}(G) \leq 2$ . Let  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1} > \mu_n = 0$  be the Laplacian spectrum of  $G$ . Then the distance Laplacian spectrum of  $G$  is  $2n - \mu_{n-1} \geq 2n - \mu_{n-2} \geq \dots \geq 2n - \mu_1 > \partial_n^L = 0$ . Moreover, for every  $i \in \{1, 2, \dots, n - 1\}$  the eigenspaces corresponding to  $\mu_i$  and to  $2n - \mu_i$  are the same.*

**THEOREM 2.3.** [3] *Let  $G$  be a connected graph on  $n$  vertices. Then  $\partial_{n-1}^L = n$  if and only if  $\overline{G}$  is disconnected. Moreover, the multiplicity of  $n$  as an eigenvalue of  $\mathcal{D}^L$  is one less than the number of components of  $\overline{G}$ .*

**THEOREM 2.4.** [3] *If  $G$  is a connected graph on  $n \geq 2$  vertices then  $m(\partial_1^L) \leq n - 1$  with equality if and only if  $G$  is the complete graph  $K_n$ .*

We finish this section enunciating the Cauchy interlacing theorem, that will be necessary for what follows::

**THEOREM 2.5.** [7] *Let  $A$  be a real symmetric matrix of order  $n$  with eigenvalues  $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$  and let  $M$  be a principal submatrix of  $M$  with order  $m \leq n$  and eigenvalues  $\lambda_1(M) \geq \lambda_2(M) \geq \dots \geq \lambda_m(M)$ . Then  $\lambda_i(A) \geq \lambda_i(M) \geq \lambda_{i+n-m}(A)$ , for all  $1 \leq i \leq m$ .*

**3. Proof of the conjecture.** The next lemmas will be useful to prove the main results of this section:

**LEMMA 3.1.** *If  $G$  is a connected graph on  $n \geq 2$  vertices and Laplacian spectrum equal to  $(n, \mu_2, \dots, \mu_2, \mu_2, 0)$ , with  $\mu_2 \neq n$ , then  $G \cong S_n$  or  $G \cong K_{p,p}$ , where  $n = 2p$ .*

*Proof.* In this case, the  $L$ -spectrum of  $\overline{G}$  is  $(n - \mu_2, n - \mu_2, \dots, n - \mu_2, 0, 0)$  and, then,  $\overline{G}$  has exactly 2 components. As each component has no more than two distinct Laplacian eigenvalues, both are isolated vertices or complete graphs. Since the two components also have all nonzero eigenvalues equal, we have  $\overline{G} \cong K_1 \cup K_{n-1}$  or  $\overline{G} \cong K_p \cup K_p$ , where  $n = 2p$ . Therefore,  $G \cong S_n$  or  $G \cong K_{p,p}$ . On the other hand, it is already known that the  $L$ -spectrum of  $S_n$  and  $K_{p,p}$  are, respectively,  $(n, 1, \dots, 1, 0)$  and  $(n, p, \dots, p, 0)$ .  $\square$

**LEMMA 3.2.** *Let  $A$  be a real symmetric matrix of order  $n$  with largest eigenvalue  $\lambda$  and  $M$  the  $m \times m$  principal submatrix of  $A$  obtained from  $A$  by excluding the  $n - m$  last rows and columns. If  $M$  also has  $\lambda$  as an eigenvalue, associated with the normalized eigenvector  $\mathbf{x} = (x_1, \dots, x_m)$ , then  $\mathbf{x}^* = (x_1, \dots, x_m, 0, \dots, 0)$  is a*

corresponding eigenvector to  $\lambda$  in  $A$ .

*Proof.* As  $\lambda$  is an eigenvalue of  $M$  corresponding to  $\mathbf{x}$ , then  $\lambda = \langle M\mathbf{x}, \mathbf{x} \rangle$ . So, it is enough to see that  $\langle M\mathbf{x}, \mathbf{x} \rangle = \langle A\mathbf{x}^*, \mathbf{x}^* \rangle$ .  $\square$

A well known result about the Laplacian matrix ([8]) says that, if  $G$  is a graph with at least one edge then  $\mu_1 \geq \Delta + 1$ , where  $\Delta$  denotes the maximum degree of  $G$ . It is possible to get an analogous lower bound for the largest distance Laplacian eigenvalue of a connected graph  $G$ :

**THEOREM 3.3.** *If  $G$  is a connected graph then  $\partial_1^L(G) \geq \max_{i \in V} \text{Tr}(v_i) + 1$ . Equality is attained if and only if  $G \cong K_n$ .*

*Proof.* Suppose, without loss of generality, that  $\text{Tr}(v_1) = \max_{i \in V} \text{Tr}(v_i) = \text{Tr}_{\max}$  and let  $\mathbf{x} = \left(1, \frac{-1}{n-1}, \frac{-1}{n-1}, \dots, \frac{-1}{n-1}\right)$ . Then

$$\partial_1^L(G) = \max_{\mathbf{y} \perp \mathbf{1}} \frac{\langle D^L \mathbf{y}, \mathbf{y} \rangle}{\|\mathbf{y}\|^2} \geq \frac{\langle D^L \mathbf{x}, \mathbf{x} \rangle}{\|\mathbf{x}\|^2} = \left(1 + \frac{1}{n-1}\right)^2 \left(\frac{\sum_{i=1}^n d_{1,i}}{\|\mathbf{x}\|^2}\right) = \frac{n^2 \text{Tr}_{\max}}{(n-1)^2 \|\mathbf{x}\|^2}.$$

Since,  $\|\mathbf{x}\|^2 = \frac{n}{n-1}$ , we obtain

$$\partial_1^L(G) \geq \frac{n}{n-1} \text{Tr}_{\max} = \text{Tr}_{\max} + \frac{\text{Tr}_{\max}}{n-1} \geq \text{Tr}_{\max} + 1. \quad (3.1)$$

If the equality is attained for a connected graph  $G$  then, from (3.1), we conclude that  $\text{Tr}_{\max} = n - 1$ . As  $G \cong K_n$  is the unique graph with this property and  $\partial_1^L(K_n) = n$ , the result is proven.  $\square$

In order to solve Conjecture 1.1, we first investigate how the existence of  $P_4$  as an induced subgraph influences the multiplicity of the largest eigenvalue of the distance Laplacian matrix of a graph:

**THEOREM 3.4.** *If the connected graph  $G$  has at least 4 vertices and it is not a cograph then  $m(\partial_1^L) \leq n - 3$ .*

*Proof.* Let  $G$  be a connected graph on  $n \geq 4$  vertices which is not a cograph. Then  $G$  has  $P_4$  as an induced subgraph. Let  $M$  be the principal submatrix of  $\mathcal{D}^L(G)$  associated with this induced subgraph and denote the eigenvalues of  $M$  by  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ . Suppose that  $m(\partial_1^L) \geq n - 2$ . By Cauchy interlacing (Theorem 2.5) is easy to check that  $\lambda_1 = \lambda_2 = \partial_1^L$ . By Lemma 3.2, if  $\mathbf{x} = (x_1, x_2, x_3, x_4)$  and  $\mathbf{y} = (y_1, y_2, y_3, y_4)$  are eigenvectors associated to  $\partial_1^L$  in  $M$ , then  $\mathbf{x}^* = (x_1, x_2, x_3, x_4, 0, \dots, 0)$  and  $\mathbf{y}^* = (y_1, y_2, y_3, y_4, 0, \dots, 0)$  are eigenvectors associated to  $\partial_1^L$  in  $\mathcal{D}^L(G)$ . As  $\mathbf{x}^*, \mathbf{y}^* \perp \mathbf{1}$ , with a linear combination of this vectors, is possible to get  $z^* = (z_1, z_2, 0, z_4, 0, \dots, 0)$

such that  $\mathbf{z}^* \perp \mathbf{1}$  and it is still an eigenvector for  $\mathcal{D}^L(G)$  associated to  $\partial_1^L$ . Thus,  $\mathbf{z} = (z_1, z_2, 0, z_4)$  is an eigenvector for  $M$  such that  $z_1 + z_2 + z_4 = 0$ .

Now, we observe that there are just two options for the matrix  $M$ :

$$M_1 = \begin{bmatrix} t_1 & -1 & -2 & -3 \\ -1 & t_2 & -1 & -2 \\ -2 & -1 & t_3 & -1 \\ -3 & -2 & -1 & t_4 \end{bmatrix} \quad \text{or} \quad M_2 = \begin{bmatrix} t_1 & -1 & -2 & -2 \\ -1 & t_2 & -1 & -2 \\ -2 & -1 & t_3 & -1 \\ -2 & -2 & -1 & t_4 \end{bmatrix},$$

where  $t_1, t_2, t_3, t_4$  denote the transmissions of the vertices that induce  $P_4$  in  $\mathcal{D}^L(G)$ .

From the third entry of  $M_1 \mathbf{z} = \lambda_1 \mathbf{z}$  it follows that  $-2z_1 - z_2 - z_4 = 0$ . This, together with the fact that  $z_1 + z_2 + z_4 = 0$ , allow us to conclude that  $(0, 1, 0, -1)$  is an eigenvector corresponding to  $\partial_1^L$  in  $M_1$ . From the first entry of  $M_1 \mathbf{z} = \lambda_1 \mathbf{z}$ , we have a contradiction. Similarly we have a contradiction, considering  $M_2$  instead of  $M_1$ .  $\square$

The next theorem resolves the Conjecture 1.1:

**THEOREM 3.5.** *If  $G$  is a graph on  $n \geq 3$  vertices and  $G \not\cong K_n$ , then  $m(\partial_1^L(G)) \leq n - 2$  with equality if and only if  $G$  is the star  $S_n$  or the complete bipartite graph  $K_{p,p}$ , if  $n = 2p$ .*

*Proof.* As  $G \not\cong K_n$ , we already know that  $m(\partial_1^L(G)) \leq n - 2$  (Theorem 2.4). Therefore, it remains to check for which graphs we have  $m(\partial_1^L(G)) = n - 2$ . Let  $G$  be a connected graph satisfying this property. Thus,  $m(\partial_{n-1}^L(G)) = 1$ . We consider two cases, when  $\partial_{n-1}^L(G) = n$  and when  $\partial_{n-1}^L(G) \neq n$ :

- If  $\partial_{n-1}^L(G) = n$ , the  $\mathcal{D}^L$ -spectrum of  $G$  is  $(\partial_1^L, \partial_1^L, \dots, \partial_1^L, n, 0)$ , with  $\partial_1^L(G) \neq n$ . By Theorem 2.3,  $\overline{G}$  is disconnected and has exactly two components. Furthermore, as  $G$  is connected and  $\overline{G}$  is disconnected,  $\text{diam}(G) \leq 2$ . So, by Theorem 2.2, the  $L$ -spectrum of  $G$  is  $(n, 2n - \partial_1^L, \dots, 2n - \partial_1^L, 2n - \partial_1^L, 0)$  and, from Lemma 3.1,  $G \cong S_n$  or  $G \cong K_{p,p}$ ;
- If  $\partial_{n-1}^L(G) \neq n$ , the  $\mathcal{D}^L$ -spectrum of  $G$  is  $(\partial_1^L, \partial_1^L, \dots, \partial_1^L, \partial_{n-1}^L, 0)$  with  $\partial_1^L \neq \partial_{n-1}^L$  and  $\partial_{n-1}^L \neq n$ . We claim there is no graph with this property. Indeed, by Theorem 2.3, as  $\partial_{n-1}^L \neq n$ ,  $\overline{G}$  is also connected. By Theorem 2.1,  $G$  has  $P_4$  as an induced subgraph and, therefore, by Theorem 3.4,  $G$  cannot have a distance Laplacian eigenvalue with multiplicity  $n - 2$ .

It is already known [4] the  $\mathcal{D}^L$ -spectra of the star and the complete bipartite graph, and this complete the proof:

- $\mathcal{D}^L$ -spectrum of  $S_n : ((2n - 1)^{(n-2)}, n, 0)$ ;
- $\mathcal{D}^L$ -spectrum of  $K_{p,p} : ((3p)^{(n-2)}, n, 0)$ .  $\square$

**4. Graphs with  $P_5$  as forbidden subgraph.** In the previous section, we established a relationship between the  $\mathcal{D}^L$ -spectrum of a connected graph and the existence of a  $P_4$  induced subgraph. Then, it is natural to think how the existence of a  $P_5$  induced subgraph could influence its  $\mathcal{D}^L$ -spectrum. In this case, we prove the following theorem, regarding the largest distance Laplacian eigenvalue:

**THEOREM 4.1.** *If  $G$  is a connected graph on  $n \geq 5$  vertices and  $m(\partial_1^L(G)) = n - 3$  then  $G$  does not have a  $P_5$  as induced subgraph.*

*Proof.* Suppose that  $G$  has a  $P_5$  as an induced subgraph and let  $M$  be the principal submatrix of  $\mathcal{D}^L(G)$  corresponding to the vertices in this  $P_5$ . Denote the eigenvalues of  $M$  by  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5$ . If  $m(\partial_1^L) = n - 3$ , by Cauchy interlacing theorem it follows that  $\lambda_1 = \lambda_2 = \partial_1^L$ . By Lemma 3.2, if  $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$  and  $\mathbf{y} = (y_1, y_2, y_3, y_4, y_5)$  are eigenvectors associated to  $\partial_1^L$  for  $M$ , then  $\mathbf{x}^* = (x_1, x_2, x_3, x_4, x_5, 0, \dots, 0)$  and  $\mathbf{y}^* = (y_1, y_2, y_3, y_4, y_5, 0, \dots, 0)$  are eigenvectors for  $\mathcal{D}^L(G)$ , associated to  $\partial_1^L$ . As  $\mathbf{x}^*, \mathbf{y}^* \perp \mathbf{1}$ , with a linear combination of this vectors, is possible to get  $\mathbf{z}^* = (z_1, z_2, z_3, z_4, 0, \dots, 0)$  such that  $\mathbf{z}^* \perp \mathbf{1}$ . and it is still an eigenvector for  $\mathcal{D}^L(G)$  associated to  $\partial_1^L$ . Then,  $\mathbf{z} = (z_1, z_2, z_3, z_4, 0)$  is an eigenvector for  $M$  such that  $z_1 + z_2 + z_3 + z_4 = 0$ .

Now, we observe that the matrix  $M$  can be written as

$$M = \begin{bmatrix} t_1 & -1 & -2 & -d_{1,4} & -d_{1,5} \\ -1 & t_2 & -1 & -2 & -d_{2,5} \\ -2 & -1 & t_3 & -1 & -2 \\ -d_{1,4} & -2 & -1 & t_4 & -1 \\ -d_{5,1} & -d_{5,2} & -2 & -1 & t_5 \end{bmatrix}, \quad (4.1)$$

where  $t_1, t_2, t_3, t_4, t_5$  denote the transmissions of the vertices that induce  $P_5$  in  $\mathcal{D}^L(G)$ ,  $d_{1,5} = 2, 3$  or  $4$ ,  $d_{2,5} = 2$  or  $3$  and  $d_{1,4} = 2$  or  $3$ . As  $P_5$  is an induced subgraph, it is easy to check that if  $d_{1,5} = 4$  then  $d_{2,5} = 3$  and  $d_{1,4} = 3$ . Considering the following cases, we see that all possibilities lead to a contradiction:

- $d_{1,5} = 2$  and  $d_{2,5} = 2$ :

As  $\mathbf{z} \perp \mathbf{1}$ , from the fifth entry of  $M\mathbf{z} = \partial_1^L \mathbf{z}$ , it follows that  $z_4 = 0$ . So, using also the fourth entry of this equation, we have

$$\begin{cases} -d_{1,4}z_1 - 2z_2 - z_3 = 0, \\ z_1 + z_2 + z_3 = 0. \end{cases}$$

If  $d_{1,4} = 2$ , then  $z_3 = 0$  and  $z_1 = -z_2$ . So, we can assume that  $\mathbf{z} = (-1, 1, 0, 0, 0)$  is an eigenvector of  $M$ , which is a contradiction according to the third entry of the equation. If  $d_{1,4} = 3$ , then  $z_3 = z_1$  and  $z_2 = -2z_1$ . So, we can assume that  $\mathbf{z} = (1, -2, 1, 0, 0)$  is an eigenvector of  $M$ . From the third

entry of the equation, we conclude that  $t_3 = \partial_1^L$ , which is a contradiction (Theorem 3.3).

- $d_{1,5} = 2$  and  $d_{2,5} = 3$ :

As  $\mathbf{z} \perp \mathbf{1}$ , from the fifth entry of  $M\mathbf{z} = \partial_1^L \mathbf{z}$ , it follows that  $z_2 = z_4 = 1$  and  $z_1 + z_3 = -2$ . So, we can consider  $\mathbf{z} = (z_1, 1, -2 - z_1, 1, 0)$ , and from the second entry of the same equation, we conclude that  $t_2 = \partial_1^L$ .

- If  $d_{1,5} = 3$  and  $d_{2,5} = 2$ :

As  $\mathbf{z} \perp \mathbf{1}$ , from the fifth entry of  $M\mathbf{z} = \partial_1^L \mathbf{z}$ , it follows that  $z_1 = z_4 = 1$  and  $z_2 + z_3 = -2$ . So, we can consider  $\mathbf{z} = (1, -2 - z_3, z_3, 1, 0)$ , and we have

$$\begin{cases} t_1 + 2 - z_3 - d_{1,4} = \partial_1^L, \\ -d_{1,4} + 4 + z_3 + t_4 = \partial_1^L. \end{cases}$$

If  $d_{1,4} = 2$ , by Theorem 3.3 we have

$$\begin{cases} z_3 = t_1 - \partial_1^L \leq -1, \\ z_3 = \partial_1^L - t_4 \geq 1. \end{cases}$$

If  $d_{1,4} = 3$ , again by Theorem 3.3, we have

$$\begin{cases} z_3 = t_1 - \partial_1^L - 1 \leq -2, \\ z_3 = \partial_1^L - t_4 - 1 \geq 0. \end{cases}$$

- If  $d_{1,5} = d_{2,5} = 3$ :

As  $\mathbf{z} \perp \mathbf{1}$ , from the fifth entry of  $M\mathbf{z} = \partial_1^L \mathbf{z}$ , it follows that  $z_3 = -2z_4$  and  $z_1 + z_2 = 1$ . So, we can consider  $\mathbf{z} = (z_1, 1 - z_1, -2, +1, 0)$ , and we have

$$\begin{cases} -z_1 - 2t_3 - 2 = -2\partial_1^L, \\ (2 - d_{1,4})z_1 + t_4 = \partial_1^L. \end{cases}$$

If  $d_{1,4} = 2$ , then  $t_4 = \partial_1^L$ , which is a contradiction. If  $d_{1,4} = 3$ , then

$$\begin{cases} z_1 = 2(\partial_1^L - t_3 - 1), \\ z_1 = t_4 - \partial_1^L, \end{cases}$$

which is a contradiction, since Theorem 3.3 implies  $z_1 < 0$  and  $z_1 > 0$ .

- $d_{1,5} = 4$ ,  $d_{2,5} = 3$  and  $d_{1,4} = 3$ :

As  $\mathbf{z} \perp \mathbf{1}$ , from the fifth entry of  $M\mathbf{z} = \partial_1^L \mathbf{z}$ , it follows that  $-3z_1 - 2z_2 - z_3 = 0$ . From this fact and the fourth entry of this equation, we obtain  $t_4 z_4 = z_4 \partial_1^L$ . If  $z_4 \neq 0$ , we get a contradiction. If  $z_4 = 0$ , we conclude that  $-2z_1 - z_2 = 0$ . So, we can consider  $\mathbf{z} = (1, -2, 1, 0, 0)$ , which implies in  $t_1 = \partial_1^L$ , a contradiction.  $\square$



Although by this theorem we cannot completely describe the graphs that have largest distance Laplacian eigenvalue with multiplicity  $n - 3$ , it is possible to obtain a partial characterization and some remarks about this issue.

PROPOSITION 4.2. *Let  $G$  be a connected graph with order  $n \geq 4$  such that  $m(\partial_1^L) = n - 3$ . If  $\partial_{n-1}^L = n$  is an eigenvalue with multiplicity 2 then  $G \cong K_{\frac{n}{3}, \frac{n}{3}, \frac{n}{3}}$ , or  $G \cong K_{\frac{n-1}{2}, \frac{n-1}{2}} \vee K_1$ , or  $G \cong \overline{K}_{n-2} \vee K_2$ .*

*Proof.* As  $\partial_{n-1}^L = n$ ,  $\overline{G}$  is disconnected and  $\text{diam}(G) = 2$ . Moreover, by Theorem 2.2, the  $L$ -spectrum of  $\overline{G}$  is

$$(n - \partial_1^L, \dots, n - \partial_1^L, 0, 0, 0),$$

that is,  $\overline{G}$  has three components, all of them with the same nonzero eigenvalue. So, the three components are isolated vertices or complete graphs with the same order, that is,  $\overline{G} \cong K_{\frac{n}{3}} \cup K_{\frac{n}{3}} \cup K_{\frac{n}{3}}$ , if  $3 \mid n$ ,  $\overline{G} \cong K_{\frac{n-1}{2}} \cup K_{\frac{n-1}{2}} \cup K_1$ , if  $2 \mid (n - 1)$ , or  $\overline{G} \cong K_{n-2} \cup K_1 \cup K_1$ .

Finally, as the graphs we have cited above have diameter 2, by Theorem 2.2, its enough to know its  $L$ -spectrum to write the  $\mathcal{D}^L$ -spectrum:

- $\mathcal{D}^L$ -spectrum of  $K_{\frac{n}{3}, \frac{n}{3}, \frac{n}{3}}$  :  $\left( \left( \frac{4n}{3} \right)^{(n-3)}, n^{(2)}, 0 \right)$ ;
- $\mathcal{D}^L$ -spectrum of  $K_{\frac{n-1}{2}, \frac{n-1}{2}} \vee K_1$  :  $\left( \left( \frac{3n-1}{2} \right)^{(n-3)}, n^{(2)}, 0 \right)$ ;
- $\mathcal{D}^L$ -spectrum of  $\overline{K}_{n-2} \vee K_2$  :  $((2(n-1))^{(n-3)}, n^{(2)}, 0)$ .  $\square$

To finish the characterization of the graphs whose largest eigenvalue of the distance Laplacian matrix has multiplicity  $n - 3$  we should analyze two situations:

- If  $\partial_{n-1}^L = n$  is an eigenvalue with multiplicity one;
- If  $\partial_{n-1}^L \neq n$ .

Although we have not characterized precisely these two cases, proceeding similarly to the last proposition, we can conclude in the first case that if the  $\mathcal{D}^L$ -spectrum of a connected graph  $G$  is  $(\partial_1^L, \dots, \partial_1^L, \partial_{n-2}^L, n, 0)$  then the  $L$ -spectrum of  $\overline{G}$  is written as  $(\partial_1^L - n, \dots, \partial_1^L - n, \partial_{n-2}^L - n, 0, 0)$ . So,  $\overline{G}$  is a graph with 2 components such that the largest Laplacian eigenvalue has multiplicity  $n - 3$ . For example, the graph  $G \cong K_{2, n-2}$  has this property since the  $\mathcal{D}^L$ -spectrum is equal to  $((2n - 2)^{(n-3)}, n + 2, n, 0)$ .

In the last case, as  $\partial_{n-1}^L \neq n$ , then  $\overline{G}$  is a connected graph. So,  $G$  has  $P_4$  as an induced subgraph. On the other hand, from Theorem 4.1,  $G$  does not have  $P_5$  as an induced subgraph. For example,  $C_5$  satisfies this condition, since its  $\mathcal{D}^L$ -spectrum is  $\left( \frac{15+\sqrt{5}}{2}, \frac{15+\sqrt{5}}{2}, \frac{15-\sqrt{5}}{2}, \frac{15-\sqrt{5}}{2}, 0 \right)$ .

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