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A NOTE ON A CONJECTURE FOR THE DISTANCE LAPLACIAN MATRIX∗

CELSO M. DA SILVA JR.†, MARIA AGUIEIRAS A. DE FREITAS‡, AND RENATA R. DEL-VECCHIO§

Abstract. In this note, the graphs of order \( n \) having the largest distance Laplacian eigenvalue of multiplicity \( n - 2 \) are characterized. In particular, it is shown that if the largest eigenvalue of the distance Laplacian matrix of a connected graph \( G \) of order \( n \) has multiplicity \( n - 2 \), then \( G \cong S_n \) or \( G \cong K_{p,p} \), where \( n = 2p \). This resolves a conjecture proposed by M. Aouchiche and P. Hansen in [M. Aouchiche and P. Hansen. A Laplacian for the distance matrix of a graph. Czechoslovak Mathematical Journal, 64(3):751–761, 2014.]. Moreover, it is proved that if \( G \) has \( P_5 \) as an induced subgraph then the multiplicity of the largest eigenvalue of the distance Laplacian matrix of \( G \) is less than \( n - 3 \).

Key words. Distance Laplacian matrix, Laplacian matrix, Largest eigenvalue, Multiplicity of eigenvalues.

AMS subject classifications. 05C12, 05C50, 15A18.

1. Introduction. Let \( G = (V,E) \) be a connected graph and the distance (the length of a shortest path) between vertices \( v_i \) and \( v_j \) of \( G \) be denoted by \( d_{i,j} \). The distance matrix of \( G \), denoted by \( D(G) \), is the \( n \times n \) matrix whose \((i,j)\)-entry is equal to \( d_{i,j} \), \( i, j = 1, 2, \ldots, n \). The transmission \( Tr(v_i) \) of a vertex \( v_i \) is defined as the sum of the distances from \( v_i \) to all other vertices in \( G \). For more details about the distance matrix we suggest, for example, [5]. M. Aouchiche and P. Hansen [3] introduced the Laplacian for the distance matrix of a connected graph \( G \) as \( D^L(G) = \text{Tr}(G) - D(G) \), where \( \text{Tr}(G) \) is the diagonal matrix of vertex transmissions. We write \((\partial^L_1, \partial^L_2, \ldots, \partial^L_n = 0)\), for the distance Laplacian spectrum of a connected graph \( G \), the \( D^L \)-spectrum, and assume that the eigenvalues are arranged in a nonincreasing order. The multiplicity of the eigenvalue \( \partial^L_i \) is denoted by \( m(\partial^L_i) \), for \( 1 \leq i \leq n \). We often use exponents to exhibit the multiplicity of the distance Laplacian eigenvalues when we write the \( D^L \)-spectrum. The distance Laplacian matrix has been recently

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studied ([2, 4, 6]) and, in [4], M. Aouchiche and P. Hansen proposed some conjectures about it. Among them, we consider in this work the following one:

**Conjecture 1.1.** [4] If $G$ is a graph on $n \geq 3$ vertices and $G \not\cong K_n$, then $m(\partial L_1(G)) \leq n - 2$ with equality if and only if $G$ is the star $S_n$ or $n = 2p$ for the complete bipartite graph $K_{p,p}$.

In this paper, we prove the conjecture. In order to obtain this result we analyze how the existence of $P_4$ as an induced subgraph influences the $D^L$-spectrum of a connected graph. We conclude that, in this case, the largest distance Laplacian eigenvalue has multiplicity less than or equal to $n - 3$. This fact motivated us to also investigate the influence of an induced $P_5$ subgraph in the $D^L$-spectrum of a graph. We prove that if a graph has an induced $P_5$ subgraph then the largest eigenvalue of its distance Laplacian matrix has multiplicity at most $n - 4$. Although we do not make a general approach by characterizing the graphs that have the largest distance Laplacian eigenvalue with multiplicity $n - 3$, some considerations on this topic are made.

2. Preliminaries. In what follows, $G = (V, E)$, or just $G$, denotes a graph with $n$ vertices and $\overline{G}$ denotes its complement. The diameter of a connected graph $G$ is denoted by $\text{diam}(G)$. As usual, we write, respectively, $P_n, C_n, S_n$ and $K_n$, for the path, the cycle, the star and the complete graph, all with $n$ vertices. We denote by $K_{p,p}$ and by $K_{p,p,p}$ the balanced complete bipartite and tripartite graph, respectively. Now, we recall the definitions of some operations with graphs that will be used. For this, let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be vertex disjoint graphs:

- The union of $G_1$ and $G_2$ is the graph $G_1 \cup G_2$ (or $G_1 + G_2$), whose vertex set is $V_1 \cup V_2$ and whose edge set is $E_1 \cup E_2$;
- The complete product or join of graphs $G_1$ and $G_2$ is the graph $G_1 \vee G_2$ obtained from $G_1 \cup G_2$ by joining each vertex of $G_1$ with every vertex of $G_2$.

A graph $G$ is a cograph, also known as a decomposable graph, if no induced subgraph of $G$ is isomorphic to $P_4$ [1]. About the cographs, we also have the following characterizations:

**Theorem 2.1.** [1] Given a graph $G$, the following statements are equivalent:

- $G$ is a cograph.
- The complement of any connected subgraph of $G$ with at least two vertices is disconnected.
- Every connected subgraph of $G$ has diameter less than or equal to 2.

We denote by $(\mu_1, \mu_2, \ldots, \mu_n = 0)$ the $L$-spectrum of $G$, i.e., the spectrum of the Laplacian matrix of $G$, and assume that the eigenvalues are labeled such that
\[ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0. \] It is well known that the multiplicity of the Laplacian eigenvalue 0 is equal to the number of components of \( G \) and that \( \mu_{n-1}(G) = n - \mu_i(G) \), \( \forall 1 \leq i \leq n - 1 \) (see [8] for more details).

The following results regarding the distance Laplacian matrix are already known.

**Theorem 2.2.** [3] Let \( G \) be a connected graph on \( n \) vertices with \( \text{diam}(G) \leq 2 \). Let \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0 \) be the Laplacian spectrum of \( G \). Then the distance Laplacian spectrum of \( G \) is \( 2n - \mu_{n-1} \geq 2n - \mu_{n-2} \geq \cdots \geq 2n - \mu_1 > \partial^L_{n-1} = 0 \). Moreover, for every \( i \in \{1, 2, \ldots, n-1\} \) the eigenspaces corresponding to \( \mu_i \) and to \( 2n - \mu_i \) are the same.

**Theorem 2.3.** [3] Let \( G \) be a connected graph on \( n \) vertices. Then \( \partial^L_{n-1} = n \) if and only if \( \overline{G} \) is disconnected. Moreover, the multiplicity of \( n \) as an eigenvalue of \( \mathcal{D}^L \) is one less than the number of components of \( \overline{G} \).

**Theorem 2.4.** [3] If \( G \) is a connected graph on \( n \geq 2 \) vertices then \( m(\partial^L_1) \leq n-1 \) with equality if and only if \( G \) is the complete graph \( K_n \).

We finish this section enunciating the Cauchy interlacing theorem, that will be necessary for what follows:

**Theorem 2.5.** [7] Let \( A \) be a real symmetric matrix of order \( n \) with eigenvalues \( \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A) \) and let \( M \) be a principal submatrix of \( M \) with order \( m \leq n \) and eigenvalues \( \lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_m(M) \). Then \( \lambda_i(A) \geq \lambda_i(M) \geq \lambda_{i+n-m}(A) \), for all \( 1 \leq i \leq m \).

### 3. Proof of the conjecture.

The next lemmas will be useful to prove the main results of this section:

**Lemma 3.1.** If \( G \) is a connected graph on \( n \geq 2 \) vertices and Laplacian spectrum equal to \( (n, \mu_2, \ldots, \mu_2, \mu_2, 0) \), with \( \mu_2 \neq n \), then \( G \cong S_n \) or \( G \cong K_{p,p} \), where \( n = 2p \).

**Proof.** In this case, the \( L \)-spectrum of \( \overline{G} \) is \( (n - \mu_2, n - \mu_2, \ldots, n - \mu_2, 0, 0) \) and, then, \( \overline{G} \) has exactly 2 components. As each component has no more than two distinct Laplacian eigenvalues, both are isolated vertices or complete graphs. Since the components also have all nonzero eigenvalues equal, we have \( \overline{G} \cong K_1 \sqcup K_{n-1} \) or \( \overline{G} \cong K_p \cup K_p \), where \( n = 2p \). Therefore, \( G \cong S_n \) or \( G \cong K_{p,p} \). On the other hand, it is already known that the \( L \)-spectrum of \( S_n \) and \( K_{p,p} \) are, respectively, \( (n, 1, \ldots, 1, 0) \) and \( (n, p, \ldots, p, 0) \). \( \Box \)

**Lemma 3.2.** Let \( A \) be a real symmetric matrix of order \( n \) with largest eigenvalue \( \lambda \) and \( M \) the \( m \times m \) principal submatrix of \( A \) obtained from \( A \) by excluding the \( n - m \) last rows and columns. If \( M \) also has \( \lambda \) as an eigenvalue, associated with the normalized eigenvector \( \mathbf{x} = (x_1, \ldots, x_m) \), then \( \mathbf{x}^* = (x_1, \ldots, x_m, 0, \ldots, 0) \) is a
corresponding eigenvector to $\lambda$ in $A$.

**Proof.** As $\lambda$ is an eigenvalue of $M$ corresponding to $x$, then $\lambda = \langle Mx, x \rangle$. So, it is enough to see that $\langle Mx, x \rangle = \langle Ax^*, x^* \rangle$. □

A well known result about the Laplacian matrix ([8]) says that, if $G$ is a graph with at least one edge then $\mu_1 \geq \Delta + 1$, where $\Delta$ denotes the maximum degree of $G$. It is possible to get an analogous lower bound for the largest distance Laplacian eigenvalue of a connected graph $G$:

**Theorem 3.3.** If $G$ is a connected graph then $\partial_1^L (G) \geq \max_{i \in V} \Tr(v_i) + 1$. Equality is attained if and only if $G \cong K_n$.

**Proof.** Suppose, without loss of generality, that $\Tr(v_1) = \max_{i \in V} \Tr(v_i) = \Tr_{\text{max}}$ and let $x = \left(1, \frac{1}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)$. Then

$$\partial_1^L (G) = \max_{y \perp 1} \frac{\langle D^L y, y \rangle}{\|y\|^2} \geq \frac{\langle D^L x, x \rangle}{\|x\|^2} = \left(1 + \frac{1}{n-1}\right)^2 \left( \sum_{i=1}^{n} \frac{d_{i,i}}{\|x\|^2} \right) = \frac{n^2 \Tr_{\text{max}}}{(n-1)^2 \|x\|^2}.$$  

Since, $\|x\|^2 = \frac{n}{n-1}$, we obtain

$$\partial_1^L (G) \geq \frac{n}{n-1} \Tr_{\text{max}} = \Tr_{\text{max}} + \frac{\Tr_{\text{max}}}{n-1} \geq \Tr_{\text{max}} + 1.$$  

(3.1)

If the equality is attained for a connected graph $G$ then, from (3.1), we conclude that $\Tr_{\text{max}} = n - 1$. As $G \cong K_n$ is the unique graph with this property and $\partial_1^L (K_n) = n$, the result is proven. □

In order to solve Conjecture 1.1, we first investigate how the existence of $P_3$ as an induced subgraph influences the multiplicity of the largest eigenvalue of the distance Laplacian matrix of a graph:

**Theorem 3.4.** If the connected graph $G$ has at least 4 vertices and it is not a cograph then $m(\partial_1^L) \leq n - 3$.

**Proof.** Let $G$ be a connected graph on $n \geq 4$ vertices which is not a cograph. Then $G$ has $P_4$ as an induced subgraph. Let $M$ be the principal submatrix of $D^L (G)$ associated with this induced subgraph and denote the eigenvalues of $M$ by $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. Suppose that $m(\partial_1^L) \geq n - 3$. By Cauchy interlacing (Theorem 2.5) is easy to check that $\lambda_1 = \lambda_2 = \partial_4^L$. By Lemma 3.2, if $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ are eigenvectors associated to $\partial_4^L$ in $M$, then $x^* = (x_1, x_2, x_3, x_4, 0, \ldots, 0)$ and $y^* = (y_1, y_2, y_3, y_4, 0, \ldots, 0)$ are eigenvectors associated to $\partial_4^L$ in $D^L (G)$. As $x^*, y^* \perp 1$, with a linear combination of this vectors, is possible to get $z^* = (z_1, z_2, 0, z_4, 0, \ldots, 0)$
such that $z^* \perp 1$ and it is still an eigenvector for $D^L(G)$ associated to $\partial^L_1$. Thus, $z = (z_1, z_2, 0, z_4)$ is an eigenvector for $M$ such that $z_1 + z_2 + z_4 = 0$.

Now, we observe that there are just two options for the matrix $M$:

$$
M_1 = \begin{bmatrix}
t_1 & -1 & -2 & -3 \\
-1 & t_2 & -1 & -2 \\
-2 & -1 & t_3 & -1 \\
-3 & -2 & -1 & t_4
\end{bmatrix}
\text{ or } M_2 = \begin{bmatrix}
t_1 & -1 & -2 & -2 \\
-1 & t_2 & -1 & -2 \\
-2 & -1 & t_3 & -1 \\
-2 & -2 & -1 & t_4
\end{bmatrix},
$$

where $t_1, t_2, t_3, t_4$ denote the transmissions of the vertices that induce $P_4$ in $D^L(G)$.

From the third entry of $M_1z = \lambda_1z$ it follows that $-2z_1 - z_2 - z_4 = 0$. This, together with the fact that $z_1 + z_2 + z_4 = 0$, allows us to conclude that $(0, 1, 0, -1)$ is an eigenvector corresponding to $\partial^L_2$ in $M_1$. From the first entry of $M_1z = \lambda_1z$, we have a contradiction. Similarly we have a contradiction, considering $M_2$ instead of $M_1$. \(\square\)

The next theorem resolves the Conjecture 1.1:

\textbf{Theorem 3.5.} If $G$ is a graph on $n \geq 3$ vertices and $G \not\cong K_n$, then $m(\partial^L_1(G)) \leq n - 2$ with equality if and only if $G$ is the star $S_n$ or the complete bipartite graph $K_{p,p}$, if $n = 2p$.

\textit{Proof.} As $G \not\cong K_n$, we already know that $m(\partial^L_1(G)) \leq n - 2$ (Theorem 2.4). Therefore, it remains to check for which graphs we have $m(\partial^L_1(G)) = n - 2$. Let $G$ be a connected graph satisfying this property. Thus, $m(\partial^L_{n-1}(G)) = 1$. We consider two cases, when $\partial^L_{n-1}(G) = n$ and when $\partial^L_{n-1}(G) \neq n$:

- If $\partial^L_{n-1}(G) = n$, the $D^L$-spectrum of $G$ is $(\partial^L_1, \partial^L_2, \ldots, \partial^L_n, n, 0)$, with $\partial^L_1(G) \neq n$. By Theorem 2.3, $G$ is disconnected and has exactly two components. Furthermore, as $G$ is connected and $\overline{G}$ is disconnected, diam$(G) \leq 2$. So, by Theorem 2.2, the $L$-spectrum of $G$ is $(n, 2n - \partial^L_1, \ldots, 2n - \partial^L_1, 2n - \partial^L_1, 0)$ and, from Lemma 3.1, $G \cong S_n$ or $G \cong K_{p,p}$;

- If $\partial^L_{n-1}(G) \neq n$, the $D^L$-spectrum of $G$ is $(\partial^L_1, \partial^L_2, \ldots, \partial^L_{n-1}, 0)$ with $\partial^L_1 \neq \partial^L_{n-1}$ and $\partial^L_{n-1} \neq n$. We claim there is no graph with this property. Indeed, by Theorem 2.3, as $\partial^L_{n-1} \neq n$, $\overline{G}$ is also connected. By Theorem 2.1, $G$ has $P_4$ as an induced subgraph and, therefore, by Theorem 3.4, $G$ cannot have a distance Laplacian eigenvalue with multiplicity $n - 2$.

It is already known [4] the $D^L$-spectra of the star and the complete bipartite graph, and this complete the proof:

- $D^L$-spectrum of $S_n$: $((2n - 1)(n-2), n, 0)$;
- $D^L$-spectrum of $K_{p,p}$: $((3p)(n-2), n, 0)$. \(\square\)
4. Graphs with $P_5$ as forbidden subgraph. In the previous section, we established a relationship between the $D^L$-spectrum of a connected graph and the existence of a $P_4$ induced subgraph. Then, it is natural to think how the existence of a $P_5$ induced subgraph could influence its $D^L$-spectrum. In this case, we prove the following theorem, regarding the largest distance Laplacian eigenvalue:

**Theorem 4.1.** If $G$ is a connected graph on $n \geq 5$ vertices and $m(\partial^L(G)) = n - 3$ then $G$ does not have a $P_5$ as induced subgraph.

**Proof.** Suppose that $G$ has a $P_5$ as an induced subgraph and let $M$ be the principal submatrix of $D^L(G)$ corresponding to the vertices in this $P_5$. Denote the eigenvalues of $M$ by $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5$. If $m(\partial^L(G)) = n - 3$, by Cauchy interlacing theorem it follows that $\lambda_1 = \lambda_2 = \partial^L_1$. By Lemma 3.2, if $x = (x_1, x_2, x_3, x_4, x_5)$ and $y = (y_1, y_2, y_3, y_4, y_5)$ are eigenvectors associated to $\partial^L_1$ for $M$, then $x^* = (x_1, x_2, x_3, x_4, x_5, 0, \ldots, 0)$ and $y^* = (y_1, y_2, y_3, y_4, y_5, 0, \ldots, 0)$ are eigenvectors for $D^L(G)$, associated to $\partial^L_5$. As $x^*, y^* \perp 1$, with a linear combination of this vectors, is possible to get $z^* = (z_1, z_2, z_3, z_4, 0, \ldots, 0)$ such that $z^* \perp 1$, and it is still an eigenvector for $D^L(G)$ associated to $\partial^L_5$. Then, $z = (z_1, z_2, z_3, z_4, 0)$ is an eigenvector for $M$ such that $z_1 + z_2 + z_3 + z_4 = 0$.

Now, we observe that the matrix $M$ can be written as

$$
M = \begin{bmatrix}
t_1 & -1 & -2 & -d_{1,4} & -d_{1,5} \\
-1 & t_2 & -1 & -2 & -d_{2,5} \\
-2 & -1 & t_3 & -1 & -2 \\
-d_{1,4} & -2 & -1 & t_4 & -1 \\
-d_{5,1} & -d_{5,2} & -2 & -1 & t_5
\end{bmatrix},
$$

(4.1)

where $t_1, t_2, t_3, t_4, t_5$ denote the transmissions of the vertices that induce $P_5$ in $D^L(G)$, $d_{1,5} = 2, 3$ or 4, $d_{2,5} = 2$ or 3 and $d_{1,4} = 2$ or 3. As $P_5$ is an induced subgraph, it is easy to check that if $d_{1,4} = 4$ then $d_{2,5} = 3$ and $d_{1,4} = 3$. Considering the following cases, we see that all possibilities lead to a contradiction:

- $d_{1,5} = 2$ and $d_{2,5} = 2$:
  
  As $z.1$, from the fifth entry of $Mz = \partial^L_5z$, it follows that $z_4 = 0$. So, using also the fourth entry of this equation, we have

$$
\begin{align*}
-d_{1,4}z_1 + 2z_2 - z_3 &= 0, \\
z_1 + z_2 + z_3 &= 0.
\end{align*}
$$

If $d_{1,4} = 2$, then $z_3 = 0$ and $z_1 = -z_2$. So, we can assume that $z = (-1, 1, 0, 0, 0)$ is an eigenvector of $M$, which is a contradiction according to the third entry of the equation. If $d_{1,4} = 3$, then $z_3 = z_1$ and $z_2 = -2z_1$. So, we can assume that $z = (1, -2, 1, 0, 0)$ is an eigenvector of $M$. From the third
entry of the equation, we conclude that $t_3 = \partial_1^L$, which is a contradiction (Theorem 3.3).

- $d_{1,5} = 2$ and $d_{2,5} = 3$:
  As $z \perp 1$, from the fifth entry of $Mz = \partial_1^Lz$, it follows that $z_2 = z_4 = 1$ and $z_1 + z_3 = -2$. So, we can consider $z = (z_1, 1, -2 - z_1, 1, 0)$, and from the second entry of the same equation, we conclude that $t_2 = \partial_2^L$.

- If $d_{1,5} = 3$ and $d_{2,5} = 2$:
  As $z \perp 1$, from the fifth entry of $Mz = \partial_1^Lz$, it follows that $z_1 = z_4 = 1$ and $z_2 + z_3 = -2$. So, we can consider $z = (1, -2 - z_3, z_3, 1, 0)$, and we have
  \[
  \begin{cases}
  t_1 + 2 - z_3 - d_{1,4} = \partial_1^L, \\
  -d_{1,4} + 4 + z_3 + t_4 = \partial_1^L.
  \end{cases}
  \]
  If $d_{1,4} = 2$, by Theorem 3.3 we have
  \[
  \begin{cases}
  z_3 = t_1 - \partial_1^L \leq -1, \\
  z_3 = \partial_1^L - t_4 \geq 1.
  \end{cases}
  \]
  If $d_{1,4} = 3$, again by Theorem 3.3, we have
  \[
  \begin{cases}
  z_3 = t_1 - \partial_1^L - 1 \leq -2, \\
  z_3 = \partial_1^L - t_4 - 1 \geq 0.
  \end{cases}
  \]

- If $d_{1,5} = d_{2,5} = 3$:
  As $z \perp 1$, from the fifth entry of $Mz = \partial_1^Lz$, it follows that $z_3 = -2z_4$ and $z_1 + z_2 = 1$. So, we can consider $z = (z_1, 1 - z_1, -2, +1, 0)$, and we have
  \[
  \begin{cases}
  -z_1 - 2t_3 - 2 = -2\partial_1^L, \\
  (2 - d_{1,4})z_1 + t_4 = \partial_1^L.
  \end{cases}
  \]
  If $d_{1,4} = 2$, then $t_4 = \partial_1^L$, which is a contradiction. If $d_{1,4} = 3$, then
  \[
  \begin{cases}
  z_1 = 2(\partial_1^L - t_3 - 1), \\
  z_1 = t_4 - \partial_1^L,
  \end{cases}
  \]
  which is a contradiction, since Theorem 3.3 implies $z_1 < 0$ and $z_1 > 0$.

- $d_{1,5} = 4$, $d_{2,5} = 3$ and $d_{1,4} = 3$:
  As $z \perp 1$, from the fifth entry of $Mz = \partial_1^Lz$, it follows that $-3z_1 - 2z_2 - z_3 = 0$. From this fact and the fourth entry of this equation, we obtain $t_4z_4 = z_4\partial_1^L$.
  If $z_4 \neq 0$, we get a contradiction. If $z_4 = 0$, we conclude that $-2z_1 - z_2 = 0$. So, we can consider $z = (1, -2, 1, 0, 0)$, which implies in $t_1 = \partial_1^L$, a contradiction. □
Although by this theorem we cannot completely describe the graphs that have largest distance Laplacian eigenvalue with multiplicity \( n - 3 \), it is possible to obtain a partial characterization and some remarks about this issue.

**Proposition 4.2.** Let \( G \) be a connected graph with order \( n \geq 4 \) such that \( m(\partial^n L) = n - 3 \). If \( \partial^n L_{n-1} = n \) is an eigenvalue with multiplicity 2 then \( G \cong K_{\frac{n-1}{2}, \frac{n}{2}, \frac{n}{2}} \), or \( G \cong K_{\frac{n-1}{2}, \frac{n}{2}, \frac{n}{2}} \cup K_1 \), or \( G \cong K_{n-2} \cup K_2 \).

**Proof.** As \( \partial^n L_{n-1} = n \), \( \overline{G} \) is disconnected and \( \text{diam}(G) = 2 \). Moreover, by Theorem 2.2, the \( L \)-spectrum of \( \overline{G} \) is

\[
(n - \partial^n L_1, \ldots, n - \partial^n L_1, 0, 0, 0),
\]

that is, \( \overline{G} \) has three components, all of them with the same nonzero eigenvalue. So, the three components are isolated vertices or complete graphs with the same order, that is, \( \overline{G} \cong K_{\frac{n-1}{2}} \cup K_{\frac{n}{2}} \cup K_{\frac{n}{2}} \), if 3 \( \mid n \), \( \overline{G} \cong K_{\frac{n-1}{2}} \cup K_{\frac{n}{2}} \cup K_1 \), if 2 \( \mid (n - 1) \), or \( \overline{G} \cong K_{n-2} \cup K_1 \cup K_1 \).

Finally, as the graphs we have cited above have diameter 2, by Theorem 2.2, its known to its \( L \)-spectrum to write the \( D^L \)-spectrum:

- \( D^L \)-spectrum of \( K_{\frac{n-1}{2}, \frac{n}{2}, \frac{n}{2}} \) : \( \left( \frac{4n}{3}, 2, 0 \right) \);
- \( D^L \)-spectrum of \( K_{\frac{n-1}{2}} \cup K_{\frac{n}{2}, \frac{n}{2}} \cup K_1 \) : \( \left( \frac{3n - 1}{2}, 2, 0 \right) \);
- \( D^L \)-spectrum of \( K_{n-2} \cup K_2 \) : \( \left( 2 + (n - 1), 2 \right) \).

To finish the characterization of the graphs whose largest eigenvalue of the distance Laplacian matrix has multiplicity \( n - 3 \) we should analyze two situations:

- If \( \partial^n L_{n-1} = n \) is an eigenvalue with multiplicity one;
- If \( \partial^n L_{n-1} \neq n \).

Although we have not characterized precisely these two cases, proceeding similarly to the last proposition, we can conclude in the first case that if the \( D^L \)-spectrum of a connected graph \( G \) is \( (\partial^n L_1, \ldots, \partial^n L_1, \partial^n L_{n-2}, n, 0) \) then the \( L \)-spectrum of \( \overline{G} \) is written as \( (\partial^n L_1 - n, \ldots, \partial^n L_1 - n, \partial^n L_{n-2} - n, 0, 0) \). So, \( \overline{G} \) is a graph with 2 components such that the largest Laplacian eigenvalue has multiplicity \( n - 3 \). For example, the graph \( G \cong K_{2, n-2} \) has this property since the \( D^L \)-spectrum is equal to \( (2n - 2)^{(n-3)}, n + 2, n, 0) \).

In the last case, as \( \partial^n L_{n-1} \neq n \), then \( \overline{G} \) is a connected graph. So, \( G \) has \( P_4 \) as an induced subgraph. On the other hand, from Theorem 4.1, \( G \) does not have \( P_5 \) as an induced subgraph. For example, \( C_5 \) satisfies this condition, since its \( D^L \)-spectrum is \( \left( \frac{15 + \sqrt{3}}{2}, \frac{15 + \sqrt{3}}{2}, \frac{15 - \sqrt{3}}{2}, \frac{15 - \sqrt{3}}{2}, 0 \right) \).
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