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William E. Watkins
California State University - Northridge, bill.watkins@csun.edu

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THE LAPLACIAN QUADRATIC FORM AND
EDGE CONNECTIVITY OF A GRAPH

WILLIAM WATKINS†

Abstract. Let \( G \) be a simple connected graph with associated positive semidefinite integral quadratic form \( Q(x) = \sum (x(i) - x(j))^2 \), where the sum is taken over all edges \( ij \) of \( G \). It is showed that the minimum positive value of \( Q(x) \) for \( x \in \mathbb{Z}^n \) equals the edge connectivity of \( G \). By restricting \( Q(x) \) to \( x \in \mathbb{Z}^{n-1} \times \{0\} \), the quadratic form becomes positive definite. It is also showed that the number of minimal disconnecting sets of edges of \( G \) equals twice the number of vectors \( x \in \mathbb{Z}^{n-1} \times \{0\} \) for which the form \( Q \) attains its minimum positive value.

Key words. Graph, Laplacian matrix, Edge connectivity, Integral quadratic form.

AMS subject classifications. 05C50, 15A63, 15B36.

1. Statement of results. Let \( G \) be a simple connected graph (no loops or multiple edges). The vertex set for \( G \) is \( V(G) = \{1,2,\ldots,n\} \) and the edge set is denoted by \( E(G) \). The Laplacian quadratic form associated with \( G \) is defined by:

\[
Q(x) = \sum_{ij \in E(G)} (x(i) - x(j))^2,
\]

for \( x = (x(1),\ldots,x(n)) \in \mathbb{Z}^n \). The matrix for this quadratic form is the Laplacian matrix \( L(G) \) for the graph. See [2] for a survey of results about the Laplacian matrix.

A set of \( t \) edges \( E = \{i_1j_1,\ldots,i_tj_t\} \) of \( G \) disconnects \( G \) if the graph \( G' = G - E \), obtained by removing these edges from \( G \), is not connected. And the edge connectivity of \( G \) is the fewest number of edges that disconnect \( G \). We call such a set of edges a minimal disconnecting set of edges of \( G \).

Theorem 1.1. Let \( G \) be a simple connected graph. Then the least positive value of \( Q(x) \) for \( x \in \mathbb{Z}^n \) equals the edge connectivity of \( G \).

Let \( k \) be the common value of the least positive value of \( Q(x) \) and the edge connectivity of \( G \). The next theorem compares the number of minimal disconnecting sets of edges
of $G$ with the number of integral vectors $x = (x(1), x(2), \ldots, x(n-1), 0)$ for which $Q(x) = k$.

**Theorem 1.2.** Let $G$ be a simple connected graph and let $k$ be the edge connectivity of $G$. Then the number of vectors $x \in \mathbb{Z}^n$ with $x(n) = 0$ such that $Q(x) = k$ is twice the number of minimal disconnecting sets of edges of $G$.

The restriction of the vectors $x \in \mathbb{Z}^n$ to those with $x(n) = 0$ is necessary because $Q(x)$ is not positive definite. Indeed its null space is spanned by the all-ones vector $e = (1, 1, \ldots, 1)$ and so if $Q(x) = k$ then $Q(x + ze) = k$ for every integer $z$. Thus, there are infinitely many vectors $y$ in $\mathbb{Z}^n$ for which $Q(y) = k$. But the restriction of the quadratic form to $Z = \{x \in \mathbb{Z}^n : x(n) = 0\}$ is positive definite, which implies that there are only finitely many vectors $y \in Z$ such that $Q(y) = k$. Furthermore, the positive integers represented by $Q$ over $\mathbb{Z}^n$ are the same as those represented by $Q$ over $Z$ because $Q(x) = Q(y)$ for $y = x - x(n)e \in Z$.

Before proceeding to the proofs, we insert a few remarks about the relationship between the quadratic form $Q$ and its restriction to $Z$. If we view the restriction as a quadratic form over $(x(1), x(2), \ldots, x(n-1)) \in \mathbb{Z}^{n-1}$, then its matrix is the principal sub matrix of the Laplacian $L(G)$ in rows and columns $1, 2, \ldots, n-1$. The famous matrix tree theorem of Kirchhoff [1, 2] states that the determinant of every $(n-1) \times (n-1)$ sub matrix of $L(G)$ equals plus or minus the number of spanning trees of $G$. In addition, all of the $(n-1) \times (n-1)$ principal sub matrices of $L(G)$ are congruent to each other by a unimodular matrix [3, 4]. So there is nothing special about restricting $Q$ to vectors with $x(n) = 0$. Indeed, if we restrict $Q$ by taking $x \in \mathbb{Z}^n$ with $x(i) = 0$ for some other vertex $i$ instead of $x(n) = 0$, all of the resulting quadratic forms are equivalent to each other.

We should also note that the Laplacian matrices $L(G_1), L(G_2)$ are congruent by a unimodular matrix if and only if the graphs $G_1, G_2$ are cycle isomorphic [3, 4]. Thus, every invariant for unimodular congruence is shared by all graphs in the same cycle-isomorphism class.

**2. Proofs.** Let $G$ be a simple connected graph, $k$ be the edge connectivity of $G$, and $m$ be the minimum positive integer represented by $Q$. The general outline for the proofs is to show that $m = k$ and that if $Q(x) = m$ for $x \in Z$ then all the coordinates of $x$ are either in $\{0, 1\}$ or all are in $\{0, -1\}$. Then we establish a bijection between the minimal disconnecting sets of edges of $G$ and the vectors $x \in \{0, 1\}^{n-1} \times \{0\}$ with $Q(x) = m$. This will prove Theorem 1.2 because if $Q(x) = m$ for some $x \in Z$ then
Q(−x) = m as well. Thus, every pair of vectors ±x with Q(x) = m corresponds to a minimal disconnecting set of edges of G.

2.1. A lemma from graph theory. We need the following lemma about connected graphs:

Lemma 2.1. Let G be a simple connected graph and \( E = \{i_1, j_1, \ldots, i_k, j_k\} \) be a minimal disconnecting set of edges of G. Then the graph \( G' = G - E \) obtained by removing the edges in \( E \) has exactly two connected components.

Proof. Since \( E \) disconnects \( G \), \( G' \) has at least two components. Suppose it has more than two components. The vertices \( i_k, j_k \) are in just one or two of the components leaving a third component whose vertices do not include \( i_k \) or \( j_k \). It follows that this third component is still a component of the subgraph \( G'' = G - \{i_1, j_1, \ldots, i_{k-1}, j_{k-1}\} \). Thus, \( \{i_1, j_1, \ldots, i_{k-1}, j_{k-1}\} \) disconnects \( G \), which contradicts the minimality of \( k \). □

2.2. Notation. We use the following notation: For a positive integer \( l \), let

\[
\mathcal{X}(l) = \{x \in \{0, 1\}^{n-1} \times \{0\} : Q(x) = l\},
\]

\[
\mathcal{E}(l) = \{E \subseteq E(G) : E \text{ disconnects } G \text{ and } |E| = l\}.
\]

Of course, \( \mathcal{X}(l) \) is empty if \( l < m \) and \( \mathcal{E}(l) \) is empty if \( l < k \). Later we will show that \( \mathcal{X}(m) \) is not empty. That is, there is a \( \{0, 1\} \) vector \( x \) with \( Q(x) = m \).

For each \( x \in \{0, 1\}^{n-1} \times \{0\} \), partition the vertices of \( G \) into two sets:

\[
V_0(x) = \{i \in \{1, 2, \ldots, n\} : x(i) = 0\},
\]

\[
V_1(x) = \{i \in \{1, 2, \ldots, n\} : x(i) = 1\},
\]

and the edges of \( G \) into three sets:

\[
E_0(x) = \{ij \in E(G) : x(i) = x(j) = 0\},
\]

\[
E_1(x) = \{ij \in E(G) : x(i) = x(j) = 1\},
\]

\[
E_{01}(x) = \{ij \in E(G) : x(i) = 0 \text{ and } x(j) = 1 \text{, or } x(i) = 1 \text{ and } x(j) = 0\}.
\]

One thing is already clear: If \( x \in \{0, 1\}^{n-1} \times \{0\} \) then

\begin{equation}
|E_{01}(x)| = Q(x).
\end{equation}

Since \( E_0(x), E_1(x), E_{01}(x) \) partition the edges of \( G \), the sum \( \sum(x(i) - x(j))^2 \) over all edges \( ij \) of \( G \) equals the sum of three sums: Over edges in \( E_0(x) \), edges in \( E_1(x) \) and edges in \( E_{01}(x) \). The first and second sums are zero and the third sum equals \( |E_{01}(x)| \).
2.3. The map $\theta : \mathcal{E}(k) \to \mathcal{X}(k)$. Let $k$ be the edge connectivity of $G$ and let $E \in \mathcal{E}(k)$ be a minimal disconnecting set of edges of $G$. By Lemma 2.1, the subgraph $G' = G - E$ has two connected components, $H_0, H_1$. To be definite we take $H_0$ to be the component containing vertex $n$. Define $x_E \in \{0, 1\}^{n-1} \times \{0\}$ by

$$x_E(i) = \begin{cases} 0, & \text{if } i \text{ is a vertex of } H_0, \\ 1, & \text{if } i \text{ is a vertex of } H_1. \end{cases}$$

The edges of $G$ are partitioned by the edges of $H_0$, the edges of $H_1$, and $E$. Thus, $Q(x_E) = |E| = k$. So, $x_E \in \mathcal{X}(k)$ and the function $E \to x_E$ maps $\mathcal{E}(k)$ into $\mathcal{X}(k)$. It follows from the minimality of $m$ that $m \leq k$.

2.4. $\mathcal{X}(m)$ is not empty. Again let $m$ be the minimum positive integer represented by $Q$, say $Q(x) = m$ for some $x \in \mathcal{Z}$. Define a zero-one vector $y$ by $y(i) = 0$ whenever $x(i)$ is even and $y(i) = 1$ whenever $x(i)$ is odd. Since $x(n) = 0$ is even, $y(n) = 0$. Now $y \neq 0$ because if all the coordinates of $x$ are even, then $x/2 \in \mathcal{Z}$ and $Q(x/2) = m/4$, which contradicts the minimality of $m$. Clearly, $Q(y) \leq Q(x) = m$. Since $y \neq 0$ and $m$ is minimal we have $Q(y) = m$. That is $y \in \mathcal{X}(m)$, which shows that $\mathcal{X}(m)$ is not empty.

2.5. $m = k$. Let $y$ be any vector in $\mathcal{X}(m)$. Then $E_{01}(y)$ is a disconnecting set of edges of $G$ and (by Equation 2.1) $|E_{01}(y)| = Q(y) = m$. From the minimality of $k$, we have $k \leq m$. Therefore, $k = m$ and Theorem 1.1 is proved.

From here on we use $k$ to denote both the minimum positive value of $Q(x)$ and the edge connectivity of $G$.

2.6. $\theta : \mathcal{E}(k) \to \mathcal{X}(k), x \to x_E$ is one-to-one. Let $E, F$ be disconnecting sets of edges in $\mathcal{E}(k)$ with $x_E = x_F$. Then

$$G - E = H_0 + H_1,$$

$$G - F = K_0 + K_1,$$

where $H_0, H_1$ are the components of $G - E$, $K_0, K_1$ are the components of $G - F$, and $n$ is a vertex in $H_0$ and $K_0$. Since $x_E = x_F$ we have $i \in V(H_0)$ if and only if $i \in V(K_0)$. Thus, $V(H_0) = V(K_0)$. The edges of $H_0$ are just the edges $ij$ of $G$ with $i, j \in V(H_0)$. It follows that $E(H_0) = E(K_0)$. Similarly $E(H_1) = E(K_1)$. The edges of $G$ are partitioned in two ways

$$E(G) = E(H_0) \cup E(H_1) \cup E,$$

$$E(G) = E(K_0) \cup E(K_1) \cup F.$$

Thus, $E = F$. 

2.7. \( \theta : \mathcal{E}(k) \to \mathcal{X}(k), x \to x_E \) is onto. Let \( x \in \mathcal{X}(k) \). We must show that there exists \( E \in \mathcal{E}(k) \) such that \( x = x_E \). The obvious, and correct, candidate is \( E = E_{01}(x) \).

Let \( H_i(x) \) be the subgraph of \( G \) with vertices \( V_i(x) \) and edges \( E_i(x) \) for \( i = 1, 2 \). Clearly, \( H_0(x), H_1(x) \) are the components of \( G' = G - E \). So \( x_E(i) = 0 \) if and only if \( i \) is a vertex of \( H_0(x) \). Also \( x(i) = 0 \) if and only if \( i \in V_0(x) = V(H_0) \). So \( x_E = x \) and \( \theta \) maps \( \mathcal{E}(k) \) onto \( \mathcal{X}(k) \).

We have proved that \( |\mathcal{E}(k)| = |\mathcal{X}(k)| \).

2.8. If \( x \in \mathcal{Z} \) and \( Q(x) = k \) then \( x \in \mathcal{X}(k) \) or \( -x \in \mathcal{X}(k) \). In this section, we show that the only vectors \( x \in \mathcal{Z} \) for which \( Q \) achieves the minimum positive value \( k \) are those all of whose coordinates are in \( \{0, 1\} \) or all are in \( \{0, -1\} \).

Suppose \( x \in \mathcal{Z} \) and \( Q(x) = k \). Define a vector \( y \in \{0, 1\}^{n-1} \times \{0\} \) by

\[
y(i) = \begin{cases} 
0 & \text{if } x(i) \text{ is even,} \\
1 & \text{if } x(i) \text{ is odd.}
\end{cases}
\]

Arguing as in Section 2.4, we get \( y \neq 0 \). Now partition the edges of \( G \) into three sets, \( E_0(y), E_1(y), \) and \( E_{01}(y) \). It is clear that \((y(i) - y(j))^2 \leq (x(i) - x(j))^2\), for all \( i, j \).

Therefore, we have the following inequalities for the sums:

\[
0 = \sum_{ij \in E_0(y)} (y(i) - y(j))^2 \leq \sum_{ij \in E_0(y)} (x(i) - x(j))^2 \\
0 = \sum_{ij \in E_1(y)} (y(i) - y(j))^2 \leq \sum_{ij \in E_1(y)} (x(i) - x(j))^2 \\
k = \sum_{ij \in E_{01}(y)} (y(i) - y(j))^2 \leq \sum_{ij \in E_{01}(y)} (x(i) - x(j))^2.
\]

But \( Q(x) \), which is the sum of the three sums above on the right, equals \( k \). Therefore, \( Q(y) = k \) and \( y \in \mathcal{X}(k) \). In addition, we have equality for each of the three inequalities. This shows that \( x(i) = x(j) \) for all \( ij \in E_0(y) \), \( x(i) = x(j) \) for all \( ij \in E_1(y) \), and \( |x(i) - x(j)| = 1 \) for all \( ij \in E_{01}(y) \).

We now show that there is an integer \( a \) such that \( x(i) = a \) for all \( i \in V_0(y) \) and an integer \( b \) such that \( x(i) = b \) for all \( i \in V_1(y) \). The set of edges \( E_{01}(y) \) disconnects \( G \) and it is a minimal disconnecting set \( (|E_{01}(y)| = k) \). Lemma 2.1 applies so \( G' = G - E_{01}(y) = H_0 + H_1 \) where \( H_0, H_1 \) are the connected components of \( G' \) and \( n \) is a vertex of \( H_0 \). It is clear that \( V(H_i) = V_i(y) \) and \( E(H_i) = E_i(y) \) for \( i = 1, 2 \).

Because \( H_0 \) is connected, there is a path joining any two vertices in \( H_0 \). But \( x(i) = x(j) \) for any edge \( ij \) in \( E_0(y) = E(H_0) \). It follows that there is an integer \( a \) such that \( x(i) = a \) for all \( i \in V(H_0) = V_0(y) \). Likewise there is an integer \( b \) such that \( x(i) = b \).
for all \( i \in V(H_1) = V_1(y) \). Now \( x(n) = 0 \) and \( n \in V(H_0) \), so \( a = 0 \). There is at least one edge \( ij \) in \( E_01(y) \) or else \( G \) is not connected. By adjusting the notation we may suppose that \( i \) is a vertex in \( H_0 \) and \( j \) a vertex in \( H_1 \) for this edge in \( E_01(y) \). Therefore, \( 1 = |x(i) - x(j)| = |0 - b| = 1 \). It follows that \( b = \pm 1 \) and therefore either \( x \in \mathcal{X}(k) \) or \( -x \in \mathcal{X}(k) \).

2.9. Conclusion. The preceding arguments show that for every \( x \in \mathcal{Z} \) with \( Q(x) = k \), either \( x \in \mathcal{X}(k) \) or \( -x \in \mathcal{X}(k) \). And that the number of minimal disconnecting sets for \( G \) equals the number of \( x \in \mathcal{X}(k) \) for which \( Q(x) = k \). Thus, the number of vectors \( x \in \mathcal{Z} \) such that \( Q(x) = k \) is twice the number of minimal disconnecting sets of edges of \( G \). The proof of Theorem 1.2 is complete.

2.10. A combinatorial observation. The author wishes to thank the referee for this observation: If the vertices of a connected graph \( G \) are colored with two colors, 0 and 1, then the number of two-colored edges is at least the edge connectivity of \( G \) with equality if and only if the set of two-colored edges is a minimal disconnecting set of edges, \( E \). Indeed, the number of two-colored edges is just \( E_{01}(x_E) \) for the 0, 1 coloring vector \( x_E \).

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