

2016

## Cyclic Refinements of the Different Versions of Operator Jensen's Inequality

Laszlo Horvath

*University of Pannonia, Egyetem u. 10, 8200 Veszprem, Hungary, lhorvath@almos.uni-pannon.hu*

Khuram A. Khan

*University of Sargodha, Sargodha, Pakistan, khuramsms@gmail.com*

Josip Pecaric

*Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia, pecaric@mahazu.hazu.hr*

Follow this and additional works at: <http://repository.uwyo.edu/ela>

---

### Recommended Citation

Horvath, Laszlo; Khan, Khuram A.; and Pecaric, Josip. (2016), "Cyclic Refinements of the Different Versions of Operator Jensen's Inequality", *Electronic Journal of Linear Algebra*, Volume 31, pp. 125-133.

DOI: <https://doi.org/10.13001/1081-3810.3098>

This Article is brought to you for free and open access by Wyoming Scholars Repository. It has been accepted for inclusion in Electronic Journal of Linear Algebra by an authorized editor of Wyoming Scholars Repository. For more information, please contact [scholcom@uwyo.edu](mailto:scholcom@uwyo.edu).

## CYCLIC REFINEMENTS OF THE DIFFERENT VERSIONS OF OPERATOR JENSEN INEQUALITY\*

LÁSZLÓ HORVÁTH<sup>†</sup>, KHURAM ALI KHAN<sup>‡</sup>, AND JOSIP PEČARIĆ<sup>§</sup>

**Abstract.** Refinements of the operator Jensen inequality for convex and operator convex functions are given by using cyclic refinements of the discrete Jensen inequality. Similar refinements are fairly rare in the literature. Some applications of the results to norm inequalities, to the Hölder-McCarthy inequality and to generalized weighted power means for operators are presented.

**Key words.** Operator Jensen inequality, Operator monotone, Operator mean.

**AMS subject classifications.** 47A63, 26A51.

**1. Introduction.** In this paper,  $(H, \langle \cdot, \cdot \rangle)$  denotes a complex Hilbert space. The  $C^*$ -algebra of all bounded linear operators on  $H$  will be denoted by  $\mathcal{B}(H)$ . We always understand the norm of an operator  $A \in \mathcal{B}(H)$  as

$$\|A\| := \sup_{\|x\| \leq 1} \|Ax\|.$$

The identity operator on  $H$  is denoted by  $I_H$ . The spectrum of an operator  $A \in \mathcal{B}(H)$  is denoted by  $\text{sp}(A)$ . An operator  $A \in \mathcal{B}(H)$  is called positive, if  $\langle Ax, x \rangle \geq 0$  for every  $x \in H$ , or equivalently,  $A$  is self-adjoint and  $\text{sp}(A) \subset [0, \infty[$ . An operator  $A \in \mathcal{B}(H)$  is called strictly positive, if it is positive and invertible. For an interval  $J \subset \mathbb{R}$ ,  $S(J)$  means the class of all self-adjoint operators from  $\mathcal{B}(H)$ , whose spectra are contained in  $J$ .

Let  $J \subset \mathbb{R}$  be an interval, and  $f : J \rightarrow \mathbb{R}$  be a function. The function  $f$  is called convex (on  $J$ ) if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all  $x, y \in J$  and for all  $0 \leq \lambda \leq 1$ . If  $f$  is continuous on  $J$ , and  $A \in S(J)$ , then  $f(A)$  is defined by the continuous functional calculus for self-adjoint operators (see

---

\*Received by the editors on August 27, 2015. Accepted for publication on February 23, 2016.  
Handling Editor: Harm Bart.

<sup>†</sup>Department of Mathematics, University of Pannonia, Egyetem u. 10, 8200 Veszprém, Hungary (lhorvath@almos.uni-pannon.hu).

<sup>‡</sup>Department of Mathematics, University of Sargodha, Sargodha, Pakistan (khuramsms@gmail.com).

<sup>§</sup>Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia (pecaric@mahazu.hazu.hr).

Rudin [14]). The function  $f$  is said to be operator monotone (increasing on  $J$ ) if  $f$  is continuous on  $J$  and  $A, B \in S(J)$ ,  $A \leq B$  (i.e.  $B - A$  is a positive operator) implies  $f(A) \leq f(B)$ . The function  $f$  is called operator convex (on  $J$ ) if  $f$  is continuous on  $J$  and

$$f(\lambda A + (1 - \lambda) B) \leq \lambda f(A) + (1 - \lambda) f(B)$$

for all  $A, B \in S(J)$  and for all  $\lambda \in [0, 1]$ .

We say that the numbers  $p_1, \dots, p_n$  represent a (positive) discrete probability distribution if  $(p_i > 0) p_i \geq 0$  ( $1 \leq i \leq n$ ) and  $\sum_{i=1}^n p_i = 1$ .

The following well known results are operator versions of Jensen inequality:

**THEOREM 1.1.** (Operator Jensens inequality for convex functions, [3, 11]) *Let  $J \subset \mathbb{R}$  be an interval. Let  $A_i \in S(J)$  and  $x_i \in H$  ( $i = 1, \dots, n$ ) with  $\sum_{i=1}^n \|x_i\|^2 = 1$ . If  $f : J \rightarrow \mathbb{R}$  is continuous and convex, then*

$$(1.1) \quad f\left(\sum_{i=1}^n \langle A_i x_i, x_i \rangle\right) \leq \sum_{i=1}^n \langle f(A_i) x_i, x_i \rangle.$$

**THEOREM 1.2.** (Operator Jensens inequality for operator convex functions, [12]) *Let  $J \subset \mathbb{R}$  be an interval, and  $K$  be a complex Hilbert space. Let  $A_i \in S(J)$  ( $i = 1, \dots, n$ ),  $\Phi_i : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  ( $i = 1, \dots, n$ ) be unital positive linear maps, and let  $p_1, \dots, p_n$  represent a discrete probability distribution. If  $f : J \rightarrow \mathbb{R}$  is operator convex, then*

$$(1.2) \quad f\left(\sum_{i=1}^n p_i \Phi_i(A_i)\right) \leq \sum_{i=1}^n p_i \Phi_i(f(A_i)).$$

A linear map  $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  is positive if  $\Phi(A)$  is positive for all positive  $A \in \mathcal{B}(H)$ , and unital if  $\Phi(I_H) = \Phi(I_K)$ .  $\Phi$  is called strictly positive if  $\Phi(A)$  is strictly positive for all strictly positive  $A \in \mathcal{B}(H)$ .

It is given new cyclic refinements of the discrete Jensen inequality in the papers Brnetić, Khan and Pečarić [2] and Horváth, Khan and Pečarić [5]. Moreover, we refer [1] for numerical inequalities. In this paper, we obtain refinements of (1.1) and (1.2) in the spirit of [5]. Refinements of operator versions of Jensen inequality has been less extensively studied than refinements of the discrete or the integral form of Jensen inequality. For some results, we refer to the papers Khosravi, Aujla, Dragomir and Moslehian [9], Niezgodna [13], Khan and Hanif [7], Kian and Moslehian [8], and the

book Horváth, Khan and Pečarić [4]. Some applications are also given: Refinements of norm inequalities and the Hölder-McCarthy inequality; introduction of some mixed symmetric means for operators and investigation of their monotonicity properties.

**2. Refinements of the operator Jensen inequality for convex functions.**

In the sequel, we shall use the following convention: Let  $2 \leq k \leq n$  be integers,  $i \in \{1, \dots, n\}$  and  $j \in \{0, \dots, k-1\}$ ; if  $i+j > n$ , then  $i+j$  means  $i+j-n$ .

Our first result a new refinement of the operator Jensen inequality for convex functions:

**THEOREM 2.1.** *Let  $2 \leq k \leq n$  be integers, let  $\mathbf{x} := (x_1, \dots, x_n) \in H^n$  such that  $x_i \neq 0$  ( $i = 1, \dots, n$ ) and  $\sum_{i=1}^n \|x_i\|^2 = 1$ , and let  $\lambda := (\lambda_1, \dots, \lambda_k)$  represent a positive discrete probability distribution. Let  $J \subset \mathbb{R}$  be an interval,  $A_i \in S(J)$  ( $i = 1, \dots, n$ ) and  $\mathbf{A} := (A_1, \dots, A_n)$ . If  $f : J \rightarrow \mathbb{R}$  is continuous and convex, then*

$$\begin{aligned} f\left(\sum_{i=1}^n \langle A_i x_i, x_i \rangle\right) &\leq D_c = D_c(f, \mathbf{A}, \mathbf{x}, \lambda) \\ &= \sum_{i=1}^n \left(\sum_{j=0}^{k-1} \lambda_{j+1} \|x_{i+j}\|^2\right) f\left(\frac{1}{\sum_{j=0}^{k-1} \lambda_{j+1} \|x_{i+j}\|^2} \sum_{j=0}^{k-1} \lambda_{j+1} \langle A_{i+j} x_{i+j}, x_{i+j} \rangle\right) \\ &\leq \sum_{i=1}^n \langle f(A_i) x_i, x_i \rangle. \end{aligned}$$

*Proof.* Since

$$\sum_{j=0}^{k-1} \left\| \frac{\sqrt{\lambda_{j+1}} x_{i+j}}{\left(\sum_{j=0}^{k-1} \lambda_{j+1} \|x_{i+j}\|^2\right)^{1/2}} \right\|^2 = 1,$$

the operator Jensen inequality for convex functions yields

$$D_c = \sum_{i=1}^n \left(\sum_{j=0}^{k-1} \lambda_{j+1} \|x_{i+j}\|^2\right)$$

$$\begin{aligned} & \cdot f \left( \sum_{j=0}^{k-1} \left\langle A_{i+j} \frac{\sqrt{\lambda_{j+1}} x_{i+j}}{\left( \sum_{j=0}^{k-1} \lambda_{j+1} \|x_{i+j}\|^2 \right)^{1/2}}, \frac{\sqrt{\lambda_{j+1}} x_{i+j}}{\left( \sum_{j=0}^{k-1} \lambda_{j+1} \|x_{i+j}\|^2 \right)^{1/2}} \right\rangle \right) \\ & \leq \sum_{i=1}^n \sum_{j=0}^{k-1} \lambda_{j+1} \langle f(A_{i+j}) x_{i+j}, x_{i+j} \rangle = \left( \sum_{i=1}^n \langle f(A_i) x_i, x_i \rangle \right) \left( \sum_{j=1}^k \lambda_j \right) \\ & = \sum_{i=1}^n \langle f(A_i) x_i, x_i \rangle. \end{aligned}$$

Conversely, it is easy to check that

$$\sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} \|x_{i+j}\|^2 \right) = 1,$$

and therefore, the convexity of  $f$  implies

$$\begin{aligned} D_c & \geq f \left( \sum_{i=1}^n \sum_{j=0}^{k-1} \lambda_{j+1} \langle A_{i+j} x_{i+j}, x_{i+j} \rangle \right) = f \left( \left( \sum_{i=1}^n \langle A_i x_i, x_i \rangle \right) \left( \sum_{j=1}^k \lambda_j \right) \right) \\ & = f \left( \sum_{i=1}^n \langle A_i x_i, x_i \rangle \right). \quad \square \end{aligned}$$

The following particular case is interesting.

**COROLLARY 2.2.** *Let  $2 \leq k \leq n$  be integers, let  $x \in H$  with  $\|x\| = 1$ , and let  $\lambda_1, \dots, \lambda_k$  and  $p_1, \dots, p_n$  represent positive discrete probability distributions. Let  $J \subset \mathbb{R}$  be an interval, and  $A_i \in S(J)$  ( $i = 1, \dots, n$ ). If  $f : J \rightarrow \mathbb{R}$  is continuous and convex, then:*

(a)

$$\begin{aligned} f \left( \left\langle \sum_{i=1}^n p_i A_i x, x \right\rangle \right) & \leq \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right) f \left( \frac{1}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} \left\langle \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} A_{i+j} x, x \right\rangle \right) \\ & \leq \left\langle \sum_{i=1}^n p_i f(A_i) x, x \right\rangle. \end{aligned}$$

(b) In case of  $A := A_1 = \dots = A_n$ ,

$$\begin{aligned} f(\langle Ax, x \rangle) &\leq \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right) f \left( \frac{1}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} \left\langle \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} Ax, x \right\rangle \right) \\ &\leq \langle f(A)x, x \rangle. \end{aligned}$$

*Proof.* (a) Theorem 2.1 can be applied to the vectors  $x_i := \sqrt{p_i}x$  ( $i = 1, \dots, n$ ).

(b) It is a special case of (a).  $\square$

Some norm inequalities can be obtained from Corollary 2.2 (a).

**COROLLARY 2.3.** *Let  $2 \leq k \leq n$  be integers, and let  $\lambda_1, \dots, \lambda_k$  and  $p_1, \dots, p_n$  represent positive discrete probability distributions. Let  $J \subset [0, \infty[$  be an interval, and  $A_i \in S(J)$  ( $i = 1, \dots, n$ ). If  $f : J \rightarrow \mathbb{R}$  is nonnegative, continuous, increasing and convex, then*

$$\begin{aligned} f \left( \left\| \sum_{i=1}^n p_i A_i \right\| \right) &\leq \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right) f \left( \frac{1}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} \left\| \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} A_{i+j} \right\| \right) \\ &\leq \left\| \sum_{i=1}^n p_i f(A_i) \right\|. \end{aligned}$$

*Proof.* If  $A \in \mathcal{B}(H)$  is a positive operator, then  $\|A\| = \sup_{\|x\|=1} \langle Ax, x \rangle$ . By using this, the continuity and the increase of  $f$ , and Corollary 2.2 (a), we have

$$\begin{aligned} f \left( \left\| \sum_{i=1}^n p_i A_i \right\| \right) &= f \left( \sup_{\|x\|=1} \left\langle \sum_{i=1}^n p_i A_i x, x \right\rangle \right) = \sup_{\|x\|=1} f \left( \left\langle \sum_{i=1}^n p_i A_i x, x \right\rangle \right) \\ &\leq \sup_{\|x\|=1} \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right) f \left( \frac{1}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} \left\langle \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} A_{i+j} x, x \right\rangle \right) \\ &\leq \sup_{\|x\|=1} \left\langle \sum_{i=1}^n p_i f(A_i) x, x \right\rangle = \left\| \sum_{i=1}^n p_i f(A_i) \right\|. \quad \square \end{aligned}$$

REMARK 2.4. We consider now some special cases of Corollary 2.3. Let  $2 \leq k \leq n$  be integers, and let  $\lambda_1, \dots, \lambda_k$  and  $p_1, \dots, p_n$  represent positive discrete probability distributions. Let  $J \subset [0, \infty[$  be an interval, and  $A_i \in S(J)$  ( $i = 1, \dots, n$ ).

(a) For  $\alpha \geq 1$ ,

$$(2.1) \quad \left\| \sum_{i=1}^n p_i A_i \right\|^\alpha \leq \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right)^{1-\alpha} \left\| \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} A_{i+j} \right\|^\alpha \leq \left\| \sum_{i=1}^n p_i A_i^\alpha \right\|,$$

and for  $0 < \alpha < 1$  the reverse inequalities hold. If the operators are strictly positive, (2.1) is also true for  $\alpha < 0$ .

(b) By choosing  $f = \exp$ , we have

$$\begin{aligned} \exp \left( \left\| \sum_{i=1}^n p_i A_i \right\| \right) &\leq \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right) \exp \left( \frac{1}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} \left\| \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} A_{i+j} \right\| \right) \\ &\leq \left\| \sum_{i=1}^n p_i \exp(A_i) \right\|. \end{aligned}$$

From Corollary 2.2, (b) a refinement of the Hölder-McCarthy inequality (see [6]) is derived.

COROLLARY 2.5. Let  $2 \leq k \leq n$  be integers, let  $x \in H$  be a unit vector, and let  $\lambda_1, \dots, \lambda_k$  and  $p_1, \dots, p_n$  represent positive discrete probability distributions. Let  $A \in \mathcal{B}(H)$  be a positive operator. Then the following hold:

(a) For every  $\alpha \geq 1$ ,

$$(2.2) \quad \langle Ax, x \rangle^\alpha \leq \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right)^{1-\alpha} \left\langle \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} A_{i+j} x, x \right\rangle^\alpha \leq \langle A^\alpha x, x \rangle.$$

(b) For every  $0 < \alpha < 1$ ,

$$\langle Ax, x \rangle^\alpha \geq \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right)^{1-\alpha} \left\langle \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} A_{i+j} x, x \right\rangle^\alpha \geq \langle A^\alpha x, x \rangle.$$

(c) If  $A$  is strictly positive and  $\alpha < 0$ , then (2.2) also holds.

**3. Refinements of the operator Jensen inequality for operator convex functions.** In the next result, we obtain a new refinement for operator Jensen inequality for operator convex functions.

**THEOREM 3.1.** *Let  $2 \leq k \leq n$  be integers, and let  $\lambda := (\lambda_1, \dots, \lambda_k)$  and  $\mathbf{p} := (p_1, \dots, p_n)$  represent positive discrete probability distributions. Let  $J \subset \mathbb{R}$  be an interval,  $A_i \in S(J)$  ( $i = 1, \dots, n$ ) and  $\mathbf{A} := (A_1, \dots, A_n)$ . Let  $K$  be a complex Hilbert space,  $\Phi_i : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  ( $i = 1, \dots, n$ ) be unital positive linear maps, and  $\Phi := (\Phi_1, \dots, \Phi_n)$ . If  $f : J \rightarrow \mathbb{R}$  is operator convex, then*

$$\begin{aligned} f\left(\sum_{i=1}^n p_i \Phi_i(A_i)\right) &\leq D_{oc} = D_{oc}(f, \mathbf{A}, \Phi, \mathbf{p}, \lambda) \\ &:= \sum_{i=1}^n \left(\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}\right) f\left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \Phi_{i+j}(A_{i+j})}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}}\right) \\ &\leq \sum_{i=1}^n p_i \Phi_i(f(A_i)). \end{aligned}$$

*Proof.* The operator Jensen inequality for operator convex functions shows that

$$\begin{aligned} D_{oc} &\leq \sum_{i=1}^n \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \Phi_{i+j}(f(A_{i+j})) \\ &= \left(\sum_{i=1}^n p_i \Phi_i(f(A_i))\right) \left(\sum_{j=1}^k \lambda_j\right) = \sum_{i=1}^n p_i \Phi_i(f(A_i)). \end{aligned}$$

Since

$$\sum_{i=1}^n \left(\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}\right) = 1,$$

we can apply the operator Jensen inequality for operator convex functions again, and have

$$D_{oc} \geq f\left(\sum_{i=1}^n \left(\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \Phi_{i+j}(A_{i+j})\right)\right) = f\left(\sum_{i=1}^n p_i \Phi_i(A_i)\right). \quad \square$$

In the following variant of the previous result, the maps  $\Phi_1, \dots, \Phi_n$  are defined directly in terms of unitary operators.



**COROLLARY 3.2.** *Let  $2 \leq k \leq n$  be integers, and let  $\lambda := (\lambda_1, \dots, \lambda_k)$  and  $\mathbf{p} := (p_1, \dots, p_n)$  represent positive discrete probability distributions. Let  $J \subset \mathbb{R}$  be an interval,  $A_i \in S(J)$  ( $i = 1, \dots, n$ ) and  $\mathbf{A} := (A_1, \dots, A_n)$ . Let  $C_i \in \mathcal{B}(H)$  ( $i = 1, \dots, n$ ) be unitary operators. If  $f : J \rightarrow \mathbb{R}$  is operator convex, then*

$$f \left( \sum_{i=1}^n p_i C_i^* A_i C_i \right) \leq \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right) f \left( \frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} C_{i+j}^* A_{i+j} C_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} \right) \\ \leq \sum_{i=1}^n p_i C_i^* f(A_i) C_i.$$

*Proof.* For every  $i = 1, \dots, n$ , the map  $\Phi_i : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$  defined by

$$\Phi_i(A) = C_i^* A C_i$$

is a unital positive linear map, and hence Theorem 3.1 can be applied.  $\square$

As an application, we present some monotonicity results for operator means.

Let  $A_i \in \mathcal{B}(H)$  ( $i = 1, \dots, n$ ) be strictly positive operators,  $\mathbf{A} := (A_1, \dots, A_n)$ , and let  $\mathbf{p} := (p_1, \dots, p_n)$  represent a positive discrete probability distribution. Let  $K$  be a complex Hilbert space,  $\Phi_i : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  ( $i = 1, \dots, n$ ) be unital strictly positive linear maps, and  $\Phi := (\Phi_1, \dots, \Phi_n)$ . The generalized weighted power mean of the operators  $A_i$  ( $i = 1, \dots, n$ ) is defined by (see [10])

$$M_n^{[\alpha]}(\mathbf{A}, \Phi, \mathbf{p}) = M_n^{[\alpha]}(A_1, \dots, A_n; \Phi_1, \dots, \Phi_n; p_1, \dots, p_n) \\ := \left( \sum_{i=1}^n p_i \Phi_i(A_i^\alpha) \right)^{1/\alpha}, \quad \alpha \in \mathbb{R} \setminus \{0\}.$$

**THEOREM 3.3.** *Let  $2 \leq k \leq n$  be integers, and let  $\lambda := (\lambda_1, \dots, \lambda_k)$  and  $\mathbf{p} := (p_1, \dots, p_n)$  represent positive discrete probability distributions. Let  $A_i \in \mathcal{B}(H)$  ( $i = 1, \dots, n$ ) be strictly positive operators,  $\mathbf{A} := (A_1, \dots, A_n)$ . Let  $K$  be a complex Hilbert space,  $\Phi_i : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  ( $i = 1, \dots, n$ ) be unital strictly positive linear maps, and  $\Phi := (\Phi_1, \dots, \Phi_n)$ . Then*

$$(3.1) \quad \left( \sum_{i=1}^n p_i \Phi_i(A_i^\alpha) \right)^{1/\alpha} \leq M_n^{[\alpha, \beta]} = M_n^{[\alpha, \beta]}(\mathbf{A}, \Phi, \mathbf{p}, \lambda) \\ := \left( \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right) \left( \frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \Phi_{i+j}(A_{i+j}^\alpha)}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} \right)^{\beta/\alpha} \right)^{1/\beta}$$

$$\leq \left( \sum_{i=1}^n p_i \Phi_i \left( A_i^\beta \right) \right)^{1/\beta},$$

if either  $\alpha \leq \beta \leq -1$  or  $1 \leq \beta \leq -\alpha$  or  $1 \leq \beta$  and  $\alpha \leq \beta \leq 2\alpha$ .

The reverse inequalities hold in (3.1) if either  $1 \leq \beta \leq \alpha$  or  $-\alpha \leq \beta \leq -1$  or  $\beta \leq -1$  and  $2\alpha \leq \beta \leq \alpha$ .

*Proof.* The following properties of the function  $g : ]0, \infty[ \rightarrow \mathbb{R}$ ,  $g(x) = x^r$  are well known (see [3]): It is operator convex if either  $1 \leq r \leq 2$  or  $-1 \leq r \leq 0$ , and  $-g$  is operator convex if  $0 \leq r \leq 1$ ;  $g$  is operator monotone increasing if  $0 \leq r \leq 1$  and operator monotone decreasing if  $-1 \leq r \leq 0$ .

By using these properties, Theorem 3.1 can be applied to the function  $f : ]0, \infty[ \rightarrow \mathbb{R}$ ,  $f(x) = x^{\beta/\alpha}$  and the operators  $A_i^\alpha$  ( $i = 1, \dots, n$ ).  $\square$

REMARK 3.4.  $M_n^{[\alpha, \beta]}$  can be considered as the mixed symmetric mean corresponding to  $D_{oc}$  in Theorem 3.1.

#### REFERENCES

- [1] S. Abramovich. Convexity, subadditivity and generalized Jensen's inequality. *Ann. Funct. Anal.*, 4:183–194, 2013.
- [2] I. Brnetić, K.A. Khan, and J. Pečarić. Refinement of Jensen's inequality with applications to cyclic mixed symmetric means and Cauchy means. *J. Math. Inequal.*, 9:1309–1321, 2015.
- [3] T. Furuta, J. Mičić, J. Pečarić, and Y. Seo. *Mond-Pečarić Method in Operator Inequalities, Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*. Element, Zagreb, 2005.
- [4] L. Horváth, K.A. Khan, and J. Pečarić. *Combinatorial Improvements of Jensen's Inequality, Classical and New Refinements of Jensen's Inequality with Applications*. Element, Zagreb, 2014.
- [5] L. Horváth, K.A. Khan, and J. Pečarić. Cyclic refinements of the discrete and integral form of Jensen's inequality with applications. *Analysis*, to appear.
- [6] C.A. McCarthy. *cp. Israel J. Math.*, 5:249–271, 1967.
- [7] M.K. Khan and M. Hanif. On some refinements of Jensen's inequality. *J. Approx. Theory*, 93:313–330, 1998.
- [8] M. Kian and M.S. Moslehian. Refinements of operator Jensen-Mercer inequality. *Electron. J. Linear Algebra*, 26:742–753, 2013.
- [9] M. Khosravi, J.S. Aujla, S.S. Dragomir, and M.S. Moslehian. Refinements of Choi-Davis-Jensen's inequality. *Bull. Math. Anal. Appl.*, 3:127–133, 2011.
- [10] J. Mičić and J. Pečarić. Order among power means of positive operators. *Sci. Math. Jpn.*, 677–693, e-2009.
- [11] B. Mond and J. Pečarić. Convex inequalities in Hilbert space. *Houston J. Math.*, 19:405–420, 1993.
- [12] B. Mond and J. Pečarić. Converses of Jensen's inequality for several operators. *Rev. Anal. Numér. Théor. Approx.*, 23:179–183, 1994.
- [13] M. Niezgodá. Choi-Davis-Jensen's inequality and generalized inverses of linear operators. *Electron. J. Linear Algebra*, 26:406–416, 2013.
- [14] W. Rudin. *Functional Analysis*. McGraw-Hill, Inc., 1991.