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BIDERIVATIONS AND LINEAR COMMUTING MAPS ON SIMPLE GENERALIZED WITT ALGEBRAS OVER A FIELD*

ZHENGXIN CHEN†

Abstract. Let $\mathfrak{W}$ be a simple generalized Witt algebras over a field of characteristic zero. In this paper, it is proved that each anti-symmetric biderivation of $\mathfrak{W}$ is inner. As an application of biderivations, it is shown that a linear map ψ on $\mathfrak{W}$ is commuting if and only if ψ is a scalar multiplication map on $\mathfrak{W}$. The commuting automorphisms and derivations of $\mathfrak{W}$ are determined.

Key words. Biderivations, Commuting maps, Simple generalized Witt algebras.

AMS subject classifications. 17B05, 17B20, 17B40, 17B65.

1. Introduction.

The motivation to study commuting maps on the Kac-Moody algebras mainly comes from a survey paper [4] due to M. Brešar, where the author surveyed the development of the theory of commuting maps on associative algebras or rings and their applications by discussing the following topics:

(1) Various generalization of the notation of commuting maps;

(2) Commuting additive maps;

(3) Commuting traces of multiadditive maps;

(4) Commuting derivations;

(5) Applications of results of commuting maps to different areas, in particular to Lie theory.

Let $\mathcal{A}$ be an associative ring. A map $\varphi : \mathcal{A} \to \mathcal{A}$ is called *commuting* if

$$\varphi(x)x = x\varphi(x) \quad \text{for all } x \in \mathcal{A}. \tag{1.1}$$

Let us denote the commutator or the Lie product of the elements $x, y \in \mathcal{A}$ by $[x, y] = xy - yx$. Accordingly the equality (1.1) will be written as $[\varphi(x), x] = 0$. The identity mapping and zero mapping are two classical examples of commuting maps. The

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author of [4] showed that commuting maps on an associative algebra have significant applications to other important problems (e.g., Lie automorphisms, Lie derivations, biderivations, linear preserves, etc.) The principal task when treating a commuting map is to describe its form. Usually we consider commuting maps imposed with some restrictions, such as additive commuting maps, commuting traces, commuting automorphisms, commuting derivations, et al. (see [3, 6, 7, 9, 11, 12, 16, 17, 18, 19, 21]). We encourage the reader to read the well-written survey paper [4].

It is a natural question to define and determine the commuting maps on Lie algebras. Let $\mathfrak{g}$ be a Lie algebra with Lie product $[-,-]$ over a field F. A map $\varphi : \mathfrak{g} \to \mathfrak{g}$ is said to be *commuting* if

$$[\varphi(x), x] = 0 \quad \text{for all } x \in \mathfrak{g}. \tag{1.2}$$

In [23], the authors determined the biderivations of parabolic subalgebras of finite dimensional simple Lie algebras. So their linear commuting maps are scalar multiplication maps. In [10], the authors determined the commuting automorphisms and commuting derivations of certain nilpotent Lie algebras over commutative rings. In particular, the commuting automorphisms and commuting derivations of nilradicals of finite dimensional complex simple Lie algebras are completely determined. In [22], the authors proved that any biderivation of the infinite dimensional Schrödinger-Virasoro Lie algebra is inner, and so their linear commuting maps are completely determined. This paper is dedicated to determining the linear commuting maps on simple generalized Witt algebras over a field of characteristic zero, which are infinite dimensional simple Lie algebras.

Recall some basic notations and results about generalized Witt algebras. Let F be a field, I be a non-empty index set and G be an additive subgroup of $\Pi_{i\in I} F_i^+$, where F_i^+ $(i \in I)$ are copies of the additive group F. Let $\mathfrak{W} = W(G, I)$ be the Lie algebra over F with basis $\{\mathfrak{w}(\mathbf{a}, i) \mid \mathbf{a} \in G, i \in I\}$ and the multiplication

$$[\mathfrak{w}(\mathbf{a}, i), \mathfrak{w}(\mathbf{b}, j)] = a_j \mathfrak{w}(\mathbf{a}+\mathbf{b}, i) - b_i \mathfrak{w}(\mathbf{a}+\mathbf{b}, j), \tag{1.3}$$

where $i, j \in I$ and $\mathbf{a} = (a_i)_{i\in I}$, $\mathbf{b} = (b_i)_{i\in I} \in G$. The Lie algebra $\mathfrak{W}$ is infinite-dimensional if $G \neq 0$. Generalized Witt algebras have been considered by many authors over fields of positive characteristic (e.g., [13, 20, 24]) and over fields of characteristic zero (e.g., [1, 14, 15]). If $\mathrm{char}(F) = 0$, $G = \oplus_{i\in I} \mathbb{Z}_i$ and $|I| = n$ is finite, then by [15, Theorem 2], $\mathfrak{W}$ is simple and finitely generated by

$$\{\mathfrak{w}(\pm\mathbf{e_i}, j), \mathfrak{w}(\pm 2\mathbf{e_i}, j) \mid i, j \in I\},$$

where $\mathbf{e_i}$ is the elementary unit vector of $\mathbb{Z}^n$ with 1 in the ith position and 0 elsewhere, $G = \mathbb{Z}\mathbf{e_1} \oplus \mathbb{Z}\mathbf{e_2} \oplus \cdots \oplus \mathbb{Z}\mathbf{e_n}$.

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In the remainder of this article,

$$\mathfrak{W} = W(G, I)$$

will always denote the simple generalized Witt algebra over a field F, where

$$char(F) = 0, \;\; G = \mathbb{Z}^n, \;\; I = \{1, 2, \dots, n\}, \;\; n \geq 1.$$

Then $\mathfrak{W} = \oplus_{\mathbf{a} \in \mathbb{Z}^n} (\mathfrak{W})_{\mathbf{a}}$ is a $\mathbb{Z}^n$-graded Lie algebra respect to an infinite dimensional Cartan subalgebra $\mathcal{H} = \mathrm{span}_F\{\mathfrak{w}(\mathbf{0}, 1), \mathfrak{w}(\mathbf{0}, 2), \dots, \mathfrak{w}(\mathbf{0}, n)\}$, where $(\mathfrak{W})_{\mathbf{a}} = \mathrm{span}_F\{\mathfrak{w}(\mathbf{a},\, 1), \mathfrak{w}(\mathbf{a}, 2), \dots, \mathfrak{w}(\mathbf{a}, n)\}$. By [14, Proposition 4.2], $\mathfrak{W}$ is isomorphic to the derivation algebra of the Laurent polynomial ring $F[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ with commuting variables, and $\mathfrak{w}(\mathbf{a}, i) = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} x_i \frac{\partial}{\partial x_i}$ for $i \in I$, $\mathbf{a} = (\mathbf{a_j})_{\mathbf{j} \in \mathbf{I}}$.

In this article, we aim to determine the form of each linear commuting map on the simple generalized Witt algebra $\mathfrak{W}$ over a field of characteristic zero. To achieve this aim, we need firstly to describe anti-symmetric biderivations of $\mathfrak{W}$. For an associative ring $\mathfrak{R}$, a bilinear map $\varphi : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is called a *biderivation* of $\mathfrak{R}$ if it is a derivation with respect to both components, meaning that

$$\varphi(xy, z) = x\varphi(y, z) + \varphi(x, z)y \quad \text{and} \quad \varphi(x, yz) = \varphi(x, y)z + y\varphi(x, z)$$

for any $x, y, z \in \mathfrak{R}$. If $\mathfrak{R}$ is a noncommutative algebra then the map

$$\varphi(x, y) = \lambda[x, y], \quad \forall\, x, y \in \mathfrak{R},$$

where λ lies in the center of $\mathfrak{R}$, is a basic example of biderivation. Biderivations of this form are therefore called *inner biderivations*. Brešar et al. in [8] proved that all biderivations on noncommutative prime rings are inner. Zhang et al. in [25] showed that biderivations of nest algebras are usually inner, and they showed by examples that in some special cases non-inner biderivations do exist. D. Benkonvič in [2] extended the results of [25] and he proved that under certain conditions a biderivation of a triangular algebra is a sum of an extremal and an inner biderivation. In [23], the authors transfer the concept of biderivation from associative algebras to Lie algebras as follows. For an arbitrary Lie algebra $\mathfrak{g}$, we call a bilinear map $\varphi : \mathfrak{g} \to \mathfrak{g}$ a *biderivation* of $\mathfrak{g}$ if it is a derivation with respect to both components, meaning that

$$\varphi([x, y], z) = [x, \varphi(y, z)] + [\varphi(x, z), y], \quad \varphi(x, [y, z]) = [\varphi(x, y), z] + [y, \varphi(x, z)] \tag{1.4}$$

for all $x, y, z \in \mathfrak{g}$. A biderivation $\varphi : \mathfrak{g} \to \mathfrak{g}$ is called *anti-symmetric* if $\varphi(x, y) = -\varphi(y, x)$ for any $x, y \in \mathfrak{g}$. In this article we will firstly determine all anti-symmetric biderivations of the simple generalized Witt algebras $\mathfrak{W}$. As an application of biderivations, we describe the form of each linear commuting map of $\mathfrak{W}$. Finally, we prove that a commuting automorphism of $\mathfrak{W}$ must be the identity mapping, and a commuting derivation of $\mathfrak{W}$ must be the zero mapping.

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2. The biderivations of $\mathfrak{W}$. Recall the definition of an inner biderivation of a Lie algebra $\mathfrak{g}$ in [23].

DEFINITION 2.1. Let $\lambda \in F$, $\mathfrak{g}$ a Lie algebra over a filed F. The map $\varphi_\lambda : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, sending (x, y) to $\lambda[x, y]$, is a biderivation of $\mathfrak{g}$, called an *inner biderivation* of $\mathfrak{g}$.

REMARK 2.2. It is easy to see that any inner biderivation φ_λ is an anti-symmetric biderivation.

LEMMA 2.3. *Let $\mathfrak{g}$ be a Lie algebra over a field F with characteristic char$(F) \neq 2$, φ an anti-symmetric biderivation on $\mathfrak{g}$. Then*

$$[\varphi(x, y), [u, v]] = [[x, y], \varphi(u, v)] \tag{2.1}$$

for all $x, y, u, v \in \mathfrak{g}$. In particular,

$$[\varphi(x, y), [x, y]] = 0. \tag{2.2}$$

Proof. At first we compute $\varphi([x, u], [y, v])$ in two different ways. On one hand, since φ is a derivation in the first argument, we have that

$$\varphi([x, u], [y, v]) = [\varphi(x, [y, v]), u] + [x, \varphi(u, [y, v])].$$

Using the fact that φ is a derivation in the second argument, we further have that

$$\varphi([x, u], [y, v]) = [[\varphi(x, y), v], u]+[[y, \varphi(x, v)], u]+[x, [\varphi(u, y), v]]+[x, [y, \varphi(u, v)]]. \tag{2.3}$$

On the other hand, computing $\varphi([x, u], [y, v])$ in a different way we have that

$$\varphi([x, u], [y, v]) = [[x, \varphi(u, y)], v]+[[\varphi(x, y), u], v]+[y, [\varphi(x, v), u]]+[y, [x, \varphi(u, v)]]. \tag{2.4}$$

By comparing the two equalities (2.3), (2.4) we have that

$$[[x, y], \varphi(u, v)] - [\varphi(x, y), [u, v]] = [[x, v], \varphi(u, y)] - [\varphi(x, v), [u, y]]. \tag{2.5}$$

We set

$$\xi(x, y; u, v) = [[x, y], \varphi(u, v)] - [\varphi(x, y), [u, v]], \quad x, y, u, v \in \mathfrak{g}.$$

So the equality (2.5) implies that $\xi(x, y; u, v) = \xi(x, v; u, y)$ for any $x, y, u, v \in \mathfrak{g}$. Since φ is anti-symmetric, $\xi(x, y; u, v) = -\xi(x, y; v, u)$ for any $x, y, u, v \in \mathfrak{g}$. On one hand,

$$\xi(x, y; u, v) = -\xi(x, y; v, u) = -\xi(x, u; v, y) = \xi(x, u; y, v). \tag{2.6}$$

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On the other hand,

$$\xi(x,y;u,v) = \xi(x,v;u,y) = -\xi(x,v;y,u) = -\xi(x,u;y,v). \tag{2.7}$$

By the equalities (2.6), (2.7), we have $\xi(x,y;u,v) = -\xi(x,y;u,v)$. Since $\mathrm{char}(F) \neq 2$, then $\xi(x,y;u,v) = 0$. So $[\varphi(x,y),[u,v]] = [[x,y],\varphi(u,v)]$ for all $x,y,u,v \in \mathfrak{g}$. Considering the particular case when $u = x, v = y$ we have that

$$[\varphi(x,y),[x,y]] = [[x,y],\varphi(x,y)] = -[\varphi(x,y),[x,y]],$$

which shows that $[\varphi(x,y),[x,y]] = 0$. ◻

Remark 2.4. We should point out that two similar propositions, which are [22, Lemma 2.1] and [23, Lemma 2.2], hold only if the biderivations are anti-symmetric, and so the results about biderivations in [22] and [23] hold only for anti-symmetric biderivations. However, since the biderivations related to linear commuting maps in [22] and [23] are anti-symmetric, then the main results about linear commuting maps in [22] and [23] are absolutely true.

Lemma 2.5. *Assume that φ is an anti-symmetric biderivation on $\mathfrak{W}$. If $[x,y] = 0$ for $x,y \in \mathfrak{W}$, then $\varphi(x,y) = 0$.*

Proof. By Lemma 2.3,

$$[\varphi(x,y),[u,v]] = [[x,y],\varphi(u,v)] = 0 \tag{2.8}$$

for any $u,v \in \mathfrak{W}$. Since $\mathfrak{W}$ is a simple Lie algebra, then the derived subalgebra $[\mathfrak{W},\mathfrak{W}]$ coincides with $\mathfrak{W}$. Then the equality (2.8) shows that $\varphi(x,y)$ is in the center $Z(\mathfrak{W})$. Also since $\mathfrak{W}$ is simple, then $Z(\mathfrak{W}) = 0$. Thus, $\varphi(x,y) = 0$. ◻

Lemma 2.6. *Assume that φ is an anti-symmetric biderivation on $\mathfrak{W}$. For any $i \in I$, there is an element $\lambda_i \in F$ such that*

$$\varphi(\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i)) = \lambda_i[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i)] \tag{2.9}$$

for any nonzero $t \in F$, where λ_i is independent of the choice of t.

Proof. Without loss of generality, we choose a fixed $i \in I$ and a fixed nonzero $t \in F$. For any $k^{'} \neq i$, $[\mathfrak{w}(\mathbf{0},k^{'}),\mathfrak{w}(t\mathbf{e_i},i)] = 0$, then $\varphi(\mathfrak{w}(\mathbf{0},k^{'}),\ \mathfrak{w}(t\mathbf{e_i},i)) = 0$ by Lemma 2.5. So $[\mathfrak{w}(\mathbf{0},k^{'}),\varphi(\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i))] = \varphi([\mathfrak{w}(\mathbf{0},k^{'}),\mathfrak{w}(\mathbf{0},i)],\ \mathfrak{w}(t\mathbf{e_i},i)) - [\varphi(\mathfrak{w}(\mathbf{0},k^{'}),\mathfrak{w}(t\mathbf{e_i},i)),\ \mathfrak{w}(\mathbf{0},i)] = \varphi(0,\mathfrak{w}(t\mathbf{e_i},i)) - [0,\mathfrak{w}(\mathbf{0},i)] = 0$. Set

$$\varphi(\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i)) = \sum_{\mathbf{a}\in\mathbb{Z}^n,k\in I} c(\mathbf{a},k)\mathfrak{w}(\mathbf{a},k), \quad \text{where } c(\mathbf{a},k) \in F.$$

Then

$$[\mathfrak{w}(\mathbf{0},k^{'}), \sum_{\mathbf{a}\in\mathbb{Z}^n,k\in I} c(\mathbf{a},k)\mathfrak{w}(\mathbf{a},k)] = -\sum_{\mathbf{a}\in\mathbb{Z}^\mathbf{n},\mathbf{k}\in\mathbf{I}} a_{k'}c(\mathbf{a},k)\mathfrak{w}(\mathbf{a},k) = 0. \tag{2.10}$$

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Comparing the coefficients of $\mathfrak{w}(\mathbf{a},k)$ on the both sides of the equality (2.10), we have $a_{k'}c(\mathbf{a},k)=0$ for any $k\in I$ and $\mathbf{a}\in\mathbb{Z}^n$. For any $\mathbf{a}\notin F\mathbf{e}_i$, there is some $k^{'}\in I$ such that $k^{'}\neq i$ and $a_{k^{'}}\neq 0$, which implies that $c(\mathbf{a},k)=0$ for any $k\in I$. Thus, we can assume that

$$\varphi(\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i))=\sum_{m\in F,k\in I}c(m\mathbf{e_i},k)\mathfrak{w}(m\mathbf{e_i},k). \tag{2.11}$$

By the equality (1.3), $[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i)]=-t\mathfrak{w}(t\mathbf{e_i},i)$. By Lemma 2.3, $[[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i)],\varphi(\mathfrak{w}(\mathbf{0},i),\,\mathfrak{w}(t\mathbf{e_i},i))]=0$, which implies that

$$[\mathfrak{w}(t\mathbf{e_i},i),\varphi(\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i))]=0.$$

By the equality (2.11), we have

$$[\mathfrak{w}(t\mathbf{e_i},i),\varphi(\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i))]= \tag{2.12}$$

$$-\sum_{m\in F,k\neq i,k\in I}mc(m\mathbf{e_i},k)\mathfrak{w}((t+m)\mathbf{e_i},k)+\sum_{m\in F}(t-m)c(m\mathbf{e_i},i)\mathfrak{w}((t+m)\mathbf{e_i},i).$$

By the above equality (2.12), for any $k\neq i$ and $m\neq 0$, $c(m\mathbf{e_i},k)=0$, and for any $m\neq t$, $c(m\mathbf{e_i},i)=0$. By the equality (2.11), we have

$$\varphi(\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i))=c(t\mathbf{e_i},i)\mathfrak{w}(t\mathbf{e_i},i)+\sum_{k\neq i,k\in I}c(\mathbf{0},k)\mathfrak{w}(\mathbf{0},k). \tag{2.13}$$

Fix a $k^{'}\in I$ such that $k^{'}\neq i$. We have $[\mathfrak{w}(\mathbf{e_{k'}},k^{'}),\mathfrak{w}(\mathbf{0},i)]=[\mathfrak{w}(\mathbf{e_{k'}},k^{'}),\mathfrak{w}(t\mathbf{e_i},i)]=0$, and so $[\mathfrak{w}(\mathbf{e_{k'}},k^{'}),\varphi(\mathfrak{w}(\mathbf{0},i),\,\mathfrak{w}(t\mathbf{e_i},i))]=\varphi([\mathfrak{w}(\mathbf{e_{k'}},k^{'}),\mathfrak{w}(\mathbf{0},i)],\mathfrak{w}(t\mathbf{e_i},i))-[\varphi(\mathfrak{w}(\mathbf{e_{k'}},k^{'}),\,\mathfrak{w}(t\mathbf{e_i},i)),\mathfrak{w}(\mathbf{0},i)]=\varphi(0,\mathfrak{w}(t\mathbf{e_i},i))-[0,\mathfrak{w}(\mathbf{0},i)]=0$. That is, $[\mathfrak{w}(\mathbf{e_{k'}},k^{'}),c(t\mathbf{e_i},i)\mathfrak{w}(t\mathbf{e_i},i)+\sum_{k\neq i,k\in I}c(\mathbf{0},k)\mathfrak{w}(\mathbf{0},k)]=c(\mathbf{0},k^{'})\mathfrak{w}(\mathbf{e_{k'}},k^{'})=0$, which implies that $c(\mathbf{0},k^{'})=0$. Hence, $c(\mathbf{0},k^{'})=0$ for any $k^{'}\neq i$, $k^{'}\in I$. Therefore, by the equality (2.13),

$$\varphi(\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i))=c(t\mathbf{e_i},i)\mathfrak{w}(t\mathbf{e_i},i)=-\frac{c(t\mathbf{e_i},i)}{t}[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i)].$$

For any nonzero $t^{'}\in F$, $t^{'}\neq t$, we can similarly obtain that

$$\varphi(\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t^{'}\mathbf{e_i},i))=-\frac{c^{'}(t^{'}\mathbf{e_i},i)}{t^{'}}[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t^{'}\mathbf{e_i},i)],\quad\text{where } c^{'}(t^{'}\mathbf{e_i},i)\in F.$$

The equality

$$[[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i)],\varphi(\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t^{'}\mathbf{e_i},i)]=[\varphi(\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i)),[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t^{'}\mathbf{e_i},i)]]$$

implies that

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$[[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i)],-\frac{c^{'}(t^{'}\mathbf{e_i},i)}{t^{'}}[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t^{'}\mathbf{e_i},i)]]=$

$$\Big[-\frac{c(t\mathbf{e_i},i)}{t}[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i)],[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t^{'}\mathbf{e_i},i)]\Big].$$

Note that $[[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i)],[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t^{'}\mathbf{e_i},i)]]=[-t\mathfrak{w}(t\mathbf{e_i},i),-t^{'}\mathfrak{w}(t^{'}\mathbf{e_i},i)]=tt^{'}(t-t^{'})\mathfrak{w}((t+t^{'})\mathbf{e_i},i)\neq 0$. Then $-\frac{c(t\mathbf{e_i},i)}{t}=-\frac{c^{'}(t^{'}\mathbf{e_i},i)}{t^{'}}$. So we can set

$$\lambda_i=-\frac{c(t\mathbf{e_i},i)}{t}.$$

Thus, λ_i is independent of the choice of t, $\varphi(\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i))=\lambda_i[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i)]$ for any nonzero $t\in F$. $\square$

LEMMA 2.7. *Assume that φ is an anti-symmetric biderivation on $\mathfrak{W}$. There is an element $\lambda\in F$ such that*

$$\varphi(\mathfrak{w}(\mathbf{0},j),\mathfrak{w}(t\mathbf{e_j},i))=\lambda[\mathfrak{w}(\mathbf{0},j),\mathfrak{w}(t\mathbf{e_j},i)]$$

for any $i,j\in I$ and any nonzero $t\in F$, where λ is independent of the choice of i,j and t.

Proof. If $|I|=n=1$, then the lemma holds by Lemma 2.6. Next assume that $|I|=n\geq 2$. By Lemma 2.6, for any $i\in I$, there is some $\lambda_i\in F$ such that $\varphi(\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i))=\lambda_i[\mathfrak{w}(\mathbf{0},i),\mathfrak{w}(t\mathbf{e_i},i)]$ for any nonzero $t\in F$. We will prove that $\lambda_i=\lambda_j$ for any $i\neq j,i,j\in I$, and $\lambda=\lambda_i$ satisfies the above condition.

Without loss of generality, we choose two fixed $i,j\in I$ and a fixed nonzero $t\in F$. By the equality (1.3), $[\mathfrak{w}(\mathbf{0},j),\mathfrak{w}(t\mathbf{e_j},i)]=-t\mathfrak{w}(t\mathbf{e_j},i)$. By Lemma 2.3, $[[\mathfrak{w}(\mathbf{0},j),\mathfrak{w}(t\mathbf{e_j},i)],\varphi(\mathfrak{w}(\mathbf{0},j),\mathfrak{w}(t\mathbf{e_j},i))]=0$. So we have an equality

$$[\mathfrak{w}(t\mathbf{e_j},i),\varphi(\mathfrak{w}(\mathbf{0},j),\mathfrak{w}(t\mathbf{e_j},i))]=0. \tag{2.14}$$

Set

$$\varphi(\mathfrak{w}(\mathbf{0},j),\mathfrak{w}(t\mathbf{e_j},i))=\sum_{\mathbf{a}\in\mathbb{Z}^n,k\in I}c(\mathbf{a},k)\mathfrak{w}(\mathbf{a},k),\quad\text{where } c(\mathbf{a},k)\in F.$$

By computation, $[\mathfrak{w}(t\mathbf{e_j},i),\varphi(\mathfrak{w}(\mathbf{0},j),\mathfrak{w}(t\mathbf{e_j},i))]=\sum\limits_{\mathbf{a}\in\mathbb{Z}^n,k\neq j,k\in I}-a_ic(\mathbf{a},k)\mathfrak{w}(\mathbf{a}+t\mathbf{e_j},k)$ $+\sum\limits_{\mathbf{a}\in\mathbb{Z}^n}c(\mathbf{a},j)(t\mathfrak{w}(\mathbf{a}+t\mathbf{e_j},i)-a_i\mathfrak{w}(\mathbf{a}+t\mathbf{e_j},j))=-\sum\limits_{\mathbf{a}\in\mathbb{Z}^n,k\neq i,k\in I}a_ic(\mathbf{a},k)\mathfrak{w}(\mathbf{a}+t\mathbf{e_j},k)+$ $\sum\limits_{\mathbf{a}\in\mathbb{Z}^n}(c(\mathbf{a},j)t-a_ic(\mathbf{a},i))\mathfrak{w}(\mathbf{a}+t\mathbf{e_j},i)$. By the above equality, if $a_i\neq 0$, then $c(\mathbf{a},k)=0$ for any $k\neq i$, $k\in I$. In particular, $c(\mathbf{a},j)=0$, which implies that $c(\mathbf{a},i)=\frac{t}{a_i}c(\mathbf{a},j)=0$ by the above equality. So if $a_i\neq 0$, we have that $c(\mathbf{a},k)=0$ for $k\in I$. Thus,

$$\varphi(\mathfrak{w}(\mathbf{0},j),\mathfrak{w}(t\mathbf{e_j},i))=\sum_{\mathbf{a}\in\mathbb{Z}^n,a_i=0,k\in I}c(\mathbf{a},k)\mathfrak{w}(\mathbf{a},k). \tag{2.15}$$

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For any $k^{'} \in I$ with $k^{'} \neq j$, $[\mathfrak{w}(\mathbf{0}, k^{'}), \mathfrak{w}(t\mathbf{e_j}, i)] = 0$, and so $\varphi(\mathfrak{w}(\mathbf{0}, k^{'}), \mathfrak{w}(t\mathbf{e_j}, i)) = 0$ by Lemma 2.5. Thus, $[\mathfrak{w}(\mathbf{0}, k^{'}), \varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i))] = \varphi([\mathfrak{w}(\mathbf{0}, k^{'}), \mathfrak{w}(\mathbf{0}, j)], \mathfrak{w}(t\mathbf{e_j}, i)) - [\varphi(\mathfrak{w}(\mathbf{0}, k^{'}), \mathfrak{w}(t\mathbf{e_j}, i)), \mathfrak{w}(\mathbf{0}, j)] = \varphi(0, \mathfrak{w}(t\mathbf{e_j}, i)) - [0, \mathfrak{w}(\mathbf{0}, j)] = 0$. By the equality (2.15), we have

$$\Big[\mathfrak{w}(\mathbf{0}, k^{'}), \sum_{\mathbf{a} \in \mathbb{Z}^n, a_i = 0, k \in I} c(\mathbf{a}, k)\mathfrak{w}(\mathbf{a}, k)\Big] = -\sum_{\mathbf{a} \in \mathbb{Z}^n, a_i = 0, k \in I} a_{k'} c(\mathbf{a}, k)\mathfrak{w}(\mathbf{a}, k) = 0,$$

which implies that $a_{k'} c(\mathbf{a}, k) = 0$ for any $k \in I$ and $\mathbf{a} \in \mathbb{Z}^n$ with $a_i = 0$. If $\mathbf{a} \notin F\mathbf{e_j}$ and $a_i = 0$, then there is some $k^{'} \in I$ such that $k^{'} \neq i$ or j, $a_{k'} \neq 0$, which implies that $c(\mathbf{a}, k) = 0$ for any $k \in I$. Thus, by the equality (2.15), we have that

$$\varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i)) = \sum_{m \in F, k \in I} c(m\mathbf{e_j}, k)\mathfrak{w}(m\mathbf{e_j}, k). \tag{2.16}$$

For any $k^{'} \neq i$ or j, $[\mathfrak{w}(\mathbf{e_{k'}}, k^{'}), \mathfrak{w}(m\mathbf{e_j}, i)] = [\mathfrak{w}(\mathbf{e_{k'}}, k^{'}), \mathfrak{w}(\mathbf{0}, j)] = 0$, which implies that $[\mathfrak{w}(\mathbf{e_{k'}}, k^{'}), \varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(m\mathbf{e_j}, i))] = \varphi([\mathfrak{w}(\mathbf{e_{k'}}, k^{'}), \mathfrak{w}(\mathbf{0}, j)], \mathfrak{w}(m\mathbf{e_j}, i)) - [\varphi(\mathfrak{w}(\mathbf{e_{k'}}, k^{'}), \mathfrak{w}(m\mathbf{e_j}, i)), \mathfrak{w}(\mathbf{0}, j)] = \varphi(0, \mathfrak{w}(m\mathbf{e_j}, i)) - [0, \mathfrak{w}(\mathbf{0}, j)] = 0$. By the equality (2.16), $[\mathfrak{w}(\mathbf{e_{k'}}, k^{'}), \varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i))] = \sum\limits_{m \in F, k \in I} c(m\mathbf{e_j}, k)[\mathfrak{w}(\mathbf{e_{k'}}, k^{'}), \mathfrak{w}(m\mathbf{e_j}, k)]$ $= \sum\limits_{m \in F} c(m\mathbf{e_j}, k^{'})\ \mathfrak{w}(\mathbf{e_{k'}} + m\mathbf{e_j}, k^{'}) = 0$, which implies that $c(m\mathbf{e_j}, k^{'}) = 0$ for any $m \in F$. Thus, $c(m\mathbf{e_j}, k) = 0$ for any $m \in F$ and $k \neq i$ or j in the equality (2.16). Then we have

$$\varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i)) = \sum_{m \in F} (c(m\mathbf{e_j}, i)\mathfrak{w}(m\mathbf{e_j}, i) + c(m\mathbf{e_j}, j)\mathfrak{w}(m\mathbf{e_j}, j)). \tag{2.17}$$

By Lemma 2.3,

$$[[\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(\mathbf{e_j}, j)], \varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i))] = [\varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(\mathbf{e_j}, j)), [\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i)]].$$

By Lemma 2.6, $\varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, j)) = \lambda_j[\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, j)]$. So $[[\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(\mathbf{e_j}, j)], \varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i))] = [-\mathfrak{w}(\mathbf{e_j}, j), \sum\limits_{m \in F} (c(m\mathbf{e_j}, i)\mathfrak{w}(m\mathbf{e_j}, i) + c(m\mathbf{e_j}, j)\ \mathfrak{w}(m\ \mathbf{e_j}, j))] = \sum\limits_{m \in F} (mc(m\mathbf{e_j}, i)\mathfrak{w}((m+1)\mathbf{e_j}, i) + (m-1)c(m\mathbf{e_j}, j)\mathfrak{w}((m+1)\mathbf{e_j}, j))$, and $[\varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(\mathbf{e_j}, j)), [\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i)]] = [\lambda_j[\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(\mathbf{e_j}, j)], [\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i)]] = [-\lambda_j \mathfrak{w}(\mathbf{e_j}, j), -t\,\mathfrak{w}(t\mathbf{e_j}, i)] = -\lambda_j t^2 \mathfrak{w}((t+1)\mathbf{e_j}, i)$. So we have an equality

$$\sum_{m \in F} (mc(m\mathbf{e_j}, i)\mathfrak{w}((m+1)\mathbf{e_j}, i) + (m-1)c(m\mathbf{e_j}, j)\mathfrak{w}((m+1)\mathbf{e_j}, j)) = -\lambda_j t^2 \mathfrak{w}((t+1)\mathbf{e_j}, i). \tag{2.18}$$

Comparing the coefficients of $\mathfrak{w}((t+1)\mathbf{e_j}, i)$ (resp., $\mathfrak{w}((t+1)\mathbf{e_j}, j)$) on the both sides of the equality (2.18), we have $tc(t\mathbf{e_j}, i) = -\lambda_j t^2$ (resp., $(t-1)c(t\mathbf{e_j}, j) = 0$). Then

$$c(t\mathbf{e_j}, i) = -\lambda_j t, \quad c(t\mathbf{e_j}, j) = 0 \quad \text{for any } t \neq 1.$$

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Similarly, for $m \neq t$, comparing the coefficients of $\mathfrak{w}((m+1)\mathbf{e_j}, i)$ (resp., $\mathfrak{w}((m+1)\mathbf{e_j}, j)$) on the both sides of the equality (2.18), we have $mc(m\mathbf{e_j}, i) = 0$ (resp., $(m-1)c(m\mathbf{e_j}, j) = 0$). Then, in the above equality (2.18),

$$c(m\mathbf{e_j}, i) = 0 \quad \text{for } m \neq t \text{ or } 0, c(m\mathbf{e_j}, j) = 0 \quad \text{for } m \neq t \text{ or } 1.$$

So

$$\varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i)) = -\lambda_j t\mathfrak{w}(t\mathbf{e_j}, i) + c(\mathbf{0}, i)\mathfrak{w}(\mathbf{0}, i) + c(\mathbf{e_j}, j)\mathfrak{w}(\mathbf{e_j}, j). \tag{2.19}$$

By a similar process, we can obtain that

$$\varphi(\mathfrak{w}(\mathbf{0}, i), \mathfrak{w}(t'\mathbf{e_i}, j)) = -\lambda_i t'\mathfrak{w}(t'\mathbf{e_i}, j) + c'(\mathbf{0}, j)\mathfrak{w}(\mathbf{0}, j) + c'(\mathbf{e_i}, i)\mathfrak{w}(\mathbf{e_i}, i)$$

for any nonzero $t' \in F$, where $c'(\mathbf{0}, j), c'(\mathbf{e_i}, i) \in F$. Next we will prove the following claim.

Claim. $\lambda_i = \lambda_j$ for any $i \neq j$, and $c(\mathbf{0}, i) = c(\mathbf{e_j}, j) = c'(\mathbf{0}, j) = c'(\mathbf{e_i}, i) = 0$ for any $i, j \in I$.

Choose a $t' \in F$ with $t' \neq t$. By Lemma 2.3, we have the equality $[[\mathfrak{w}(\mathbf{0}, i), \mathfrak{w}(t'\mathbf{e_i}, j)], \varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i))] - [\varphi(\mathfrak{w}(\mathbf{0}, i), \mathfrak{w}(t'\mathbf{e_i}, j)), [\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i)]] = 0$. By computation, $[[\mathfrak{w}(\mathbf{0}, i), \mathfrak{w}(t'\mathbf{e_i}, j)], \varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i))] - [\varphi(\mathfrak{w}(\mathbf{0}, i), \mathfrak{w}(t'\mathbf{e_i}, j)), [\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i)]] = [-t'\mathfrak{w}(t'\mathbf{e_i}, j), -\lambda_j t\mathfrak{w}(t\mathbf{e_j}, i) + c(\mathbf{0}, i)\mathfrak{w}(\mathbf{0}, i) + c(\mathbf{e_j}, j)\mathfrak{w}(\mathbf{e_j}, j))] - [-\lambda_i t'\mathfrak{w}(t'\mathbf{e_i}, j) + c'(\mathbf{0}, j)\mathfrak{w}(\mathbf{0}, j) + c'(\mathbf{e_i}, i)\mathfrak{w}(\mathbf{e_i}, i), -t\mathfrak{w}(t\mathbf{e_j}, i)] = \lambda_j tt'(t'\mathfrak{w}(t'\mathbf{e_i} + t\mathbf{e_j}, j) - t\mathfrak{w}(t'\mathbf{e_i} + t\mathbf{e_j}, i)) - t'^2 c(\mathbf{0}, i)\mathfrak{w}(t'\mathbf{e_i}, j) + t'c(\mathbf{e_j}, j)\mathfrak{w}(t'\mathbf{e_i} + \mathbf{e_j}, j) - \lambda_i tt'(t'\mathfrak{w}(t'\mathbf{e_i} + t\mathbf{e_j}, j) - t\mathfrak{w}(t'\mathbf{e_i} + t\mathbf{e_j}, i)) - t^2 c'(\mathbf{0}, j)\mathfrak{w}(t\mathbf{e_j}, i) + c'(\mathbf{e_i}, i)t\mathfrak{w}(\mathbf{e_i} + t\mathbf{e_j}, i) = (\lambda_j - \lambda_i)tt'^2\mathfrak{w}(t'\mathbf{e_i} + t\mathbf{e_j}, j) + (\lambda_i - \lambda_j)t^2 t'\mathfrak{w}(t'\mathbf{e_i} + t\mathbf{e_j}, i) + t'c(\mathbf{e_j}, j)\mathfrak{w}(t'\mathbf{e_i} + \mathbf{e_j}, j) + tc'(\mathbf{e_i}, i)\mathfrak{w}(\mathbf{e_i} + t\mathbf{e_j}, i) - t'^2 c(\mathbf{0}, i)\mathfrak{w}(t'\mathbf{e_i}, j) - t^2 c'(\mathbf{0}, j)\mathfrak{w}(t\mathbf{e_j}, i)$. If $t \neq 1, t' \neq 1$, then the coefficients on the right-hand side of the above equality are zero, which implies that the claim holds. If $t = 1$, then $t' \neq 1$, and $[[\mathfrak{w}(\mathbf{0}, i), \mathfrak{w}(t'\mathbf{e_i}, j)], \varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i))] - [\varphi(\mathfrak{w}(\mathbf{0}, i), \mathfrak{w}(t'\mathbf{e_i}, j)), [\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i)]] = (t'\lambda_j - t'\lambda_i + c(\mathbf{e_j}, j))t'\ \mathfrak{w}(t'\mathbf{e_i} + \mathbf{e_j}, j) + (\lambda_i - \lambda_j)t'\mathfrak{w}(t'\mathbf{e_i} + \mathbf{e_j}, i) - t'^2 c(\mathbf{0}, i)\mathfrak{w}(t'\mathbf{e_i}, j) - c'(\mathbf{0}, j)\mathfrak{w}(\mathbf{e_j}, i) + c'(\mathbf{e_i}, i)\mathfrak{w}(\mathbf{e_i} + \mathbf{e_j}, i)$. Thus, the coefficients on the right-hand side of this equality are zero, and so the claim holds. Similarly, for the case $t' = 1$, the claim also holds.

Finally, we set

$$\lambda = \lambda_1 = \lambda_2 = \cdots = \lambda_n,$$

then λ is independent of the choice of i, j and t. By the equality (2.19), $\varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i)) = -\lambda t\mathfrak{w}(t\mathbf{e_j}, i) = \lambda[\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i)]$. □

THEOREM 2.8. *Every anti-symmetric biderivation φ of $\mathfrak{W}$ is inner.*

Proof. By Lemma 2.7, there is an element $\lambda \in F$ such that $\varphi(\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i)) = \lambda\ [\mathfrak{w}(\mathbf{0}, j), \mathfrak{w}(t\mathbf{e_j}, i)]$ for any nonzero $t \in F$ and $i, j \in I$. For any $x, y \in \mathfrak{W}$,

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$[\varphi(x,y),\mathfrak{w}(t\mathbf{e_j},i)] = [\varphi(x,\,y),-\frac{1}{t}[\mathfrak{w}(\mathbf{0},j),\,\mathfrak{w}(t\mathbf{e_j},\,i)]] = -\frac{1}{t}[\varphi(x,y),[\mathfrak{w}(\mathbf{0},j),\,\mathfrak{w}(t\mathbf{e_j},i)]] = -\frac{1}{t}[[x,y],\varphi(\mathfrak{w}(\mathbf{0},j),\mathfrak{w}(t\mathbf{e_j},i))] = -\frac{1}{t}[[x,\,y],\,\lambda[\mathfrak{w}(\mathbf{0},j),\,\mathfrak{w}(t\mathbf{e_j},i)]] = -\frac{1}{t}[[x,y],-\lambda t\,\mathfrak{w}(t\mathbf{e_j},i)]] = [\lambda[x,y],\mathfrak{w}(t\mathbf{e_j},i)]$. So $[\varphi(x,y)-\lambda[x,y],\,\mathfrak{w}(t\mathbf{e_j},i)] = 0$ for any nonzero $t\in F$ and $i,j\in I$. Since the set

$$\{\mathfrak{w}(\pm\mathbf{e_j},i),\mathfrak{w}(\pm 2\mathbf{e_j},i)\mid i,j\in I\}$$

generates the Lie algebra $\mathfrak{W}$, then $\varphi(x,y)-\lambda[x,y]\in Z(\mathfrak{W})=0$. So $\varphi(x,y)=\lambda[x,y]$ for any $x,y\in\mathfrak{W}$. In other words, $\varphi=\varphi_\lambda$ is an inner biderivation. □

3. Linear commuting maps on $\mathfrak{W}$. We now apply the Theorem 2.8 to describe the linear commuting maps on the generalized Witt algebra $\mathfrak{W}$. Recall that a linear commuting map ψ on $\mathfrak{W}$ subject to $[\psi(x),x]=0$ for any $x\in\mathfrak{W}$. Obviously, if ψ on $\mathfrak{W}$ is such a map, then $[\psi(x),y]=[x,\psi(y)]$ for any $x,y\in\mathfrak{W}$.

THEOREM 3.1. *A linear map ψ on $\mathfrak{W}$ is commuting if and only if ψ is a scalar multiplication map on $\mathfrak{W}$.*

Proof. The sufficient direction is obvious. We now prove the necessary direction. Let ψ be a linear commuting map of $\mathfrak{W}$. We construct a map φ from $\mathfrak{W}\times\mathfrak{W}\to\mathfrak{W}$ by setting

$$\varphi(x,y)=[\psi(x),y]. \tag{3.1}$$

Obviously, φ is bilinear. By computation, for $x,y,z\in\mathfrak{W}$, $\varphi(x,[y,z])=[\psi(x),[y,z]]=[[\psi(x),y],z]+[y,[\psi(x),z]]=[\varphi(x,y),z]+[y,\varphi(x,z)]$. So φ is a derivation with respect to the second component. Since $[\psi(x),y]=[x,\psi(y)]$, it is easy to see that φ is also a derivation with respect to the first component. Hence, φ is a biderivation on $\mathfrak{W}$. Furthermore, $\varphi(x,y)=[\psi(x),y]=[x,\psi(y)]=-[\psi(y),x]=-\varphi(y,x)$, and so φ is anti-symmetric. By Theorem 2.8, φ is an inner biderivation, so we can find $\lambda\in F$ such that $\varphi(x,y)=\lambda[x,y]$ for any $x,y\in\mathfrak{W}$. Thus, $[\psi(x)-\lambda x,y]=[\psi(x),y]-\lambda[x,y]=\varphi(x,y)-\lambda[x,y]=0$, which implies that $\psi(x)-\lambda x\in Z(\mathfrak{W})=0$. Therefore $\psi(x)=\lambda x$ for any $x\in\mathfrak{W}$. Thus, the theorem holds. □

COROLLARY 3.2. *(1) Let ψ be a commuting automorphism of $\mathfrak{W}$. Then ψ is the identity mapping.*

(2) Let ψ be a commuting derivation of $\mathfrak{W}$. Then ψ is the zero mapping.

Proof. Let ψ is a linear commuting map of $\mathfrak{W}$. By Theorem 3.1, we may assume that $\psi=\lambda\cdot 1_{\mathfrak{W}}$ for some $\lambda\in F$. Obviously, $\psi(0)=0$.

For any $0\neq x\in\mathfrak{W}$, there is some $y\in\mathfrak{W}$ such that $[x,y]\neq 0$.

(1) If ψ is a commuting automorphism of $\mathfrak{W}$, then

$$\psi([x,y])=[\psi(x),\psi(y)], \tag{3.2}$$

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That is, $\lambda[x,y] = [\lambda x, \lambda y] = \lambda^2[x,y]$. So $\lambda^2 = \lambda$. Thus, $\lambda = 0$ or 1. By the invertibility of ψ, $\lambda = 1$. Thus, $\psi(x) = x = 1_{\mathfrak{W}}(x)$, and so (1) follows.

(2) If ψ is a commuting derivation of $\mathfrak{W}$, then

$$\psi([x,y]) = [\psi(x), y] + [x, \psi(y)], \tag{3.3}$$

we have $\lambda[x,y] = [\lambda x, y] + [x, \lambda y] = 2\lambda[x,y]$. So $\lambda = 0$. Thus, $\psi(x) = 0$, and so (2) follows. ∎

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