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A SINGULAR VALUE INEQUALITY RELATED TO A LINEAR MAP*

MINGHUA LIN†

Abstract. If $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is positive semidefinite with each block $n \times n$, it is proved that

$$2s_j\Big(\Phi(X)\Big) \le s_j\Big(\Phi(A+B)\Big), \quad j = 1, \ldots, n,$$

where $\Phi : X \mapsto X + (\operatorname{Tr} X)I$ and $s_j(\cdot)$ means the j-th largest singular value. This confirms a conjecture of the author in [M. Lin. A completely PPT map. *Linear Algebra Appl.*, 459:404–410, 2014.].

Key words. Singular value inequality, Block matrix, Linear map.

AMS subject classifications. 15A45, 15A60.

1. Introduction. In this article, capital letters are used for elements in $\mathbb{M}_n$, the set of $n \times n$ complex matrices. In [4], the main attention is paid to the linear map $\Phi : X \mapsto X + (\operatorname{Tr} X)I$ on $\mathbb{M}_n$. What remains unsolved in that paper is the following conjecture.

Conjecture 1.1. *If* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$, *where* $A, X, B \in \mathbb{M}_n$, *is positive semidefinite, then*

$$2s_j\Big(\Phi(X)\Big) \le s_j\Big(\Phi(A+B)\Big), \quad j = 1, \ldots, n. \tag{1.1}$$

The purpose of this note is to confirm the conjecture. It is noteworthy that without the appearance of Φ, inequality (1.1) may fail. Indeed, under the same condition, it is in general not true that

$$2s_1(X) \le s_1(A+B).$$

For an example, see [5, Example 3.4].

To proceed, let us fix some notation. The trace of X is denoted by $\operatorname{Tr} X$, the conjugate transpose of X is X^*. If the eigenvalues of X are real, then they are

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arranged nonincreasingly $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$; the singular values of X, denoted by $s_j(X)$, are similarly arranged. For two Hermitian matrices X and Y, we write $X \geq Y$ (or $Y \leq X$) to mean that $X - Y$ is positive semidefinite. If X is positive semidefinite, then it has a unique positive semidefinite square root, which is denoted by $X^{\frac{1}{2}}$. Finally, I stands for the $n \times n$ identity matrix.

2. Auxilliary results and proofs. We start with some lemmas. The first three are quite standard in matrix analysis.

Lemma 2.1. [2, p. 75] *For $M, N \in \mathbb{M}_n$,*

$$s_j(M+N) \leq s_j(M) + s_1(N), \quad j = 1, \ldots, n.$$

Lemma 2.2. [2, p. 262] *For $M, N \in \mathbb{M}_n$,*

$$2s_j(M^*N) \leq s_j(MM^* + NN^*), \quad j = 1, \ldots, n.$$

Recall that C is a contraction if $I \geq C^*C$.

Lemma 2.3. [3, p. 13] *If $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$, where $A, X, B \in \mathbb{M}_n$, is positive semidefinite, then there is a contraction C such that $X = A^{\frac{1}{2}}CB^{\frac{1}{2}}$.*

The next technical lemma is obviously an ad hoc one.

Lemma 2.4. *For $M, N \in \mathbb{M}_n$,*

$$\lambda_j(M^*M + N^*N) \leq \lambda_j(MM^* + NN^*) + \lambda_1(M^*M), \quad j = 1, \ldots, n.$$

Proof. Weyl's eigenvalue inequality ([2, p. 63]) states that if X, Y are Hermitian, then

$$\lambda_j(X+Y) \leq \lambda_1(X) + \lambda_j(Y), \quad j = 1, \ldots, n.$$

(It is interesting to note that Lemma 2.1 is an immediate consequence of this fact!) As $N^*N = U^*NN^*U$ for some unitary matrix U,

$$\begin{aligned} M^*M + N^*N &= M^*M + U^*NN^*U \\ &\leq M^*M + U^*(MM^* + NN^*)U. \end{aligned}$$

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Now applying Weyl's eigenvalue inequality gives the required result. □

COROLLARY 2.5. *For $M, N \in \mathbb{M}_n$,*

$$\lambda_j(M^*M + N^*N) \le \lambda_j(MM^* + NN^*) + \frac{1}{2}\operatorname{Tr}(M^*M + N^*N - M^*N - N^*M), \quad j = 1, \dots, n.$$

Proof. Replacing M, N with $M - N$, $M + N$, respectively, in Lemma 2.4 gives

$$\lambda_j(M^*M + N^*N) \le \lambda_j(MM^* + NN^*) + \frac{1}{2}\lambda_1((M-N)^*(M-N)).$$

But $(M-N)^*(M-N) \ge 0$, this implies

$$\begin{aligned}\lambda_1((M-N)^*(M-N)) &\le \operatorname{Tr}((M-N)^*(M-N)) \\ &= \operatorname{Tr}(M^*M + N^*N - M^*N - N^*M).\end{aligned}$$

Therefore, the corollary follows. □

Now we are ready to present:

Proof of Conjecture 1.1. First of all, note that for each j,

$$s_j\Big(A + B + (\operatorname{Tr}(A+B))I\Big) = \lambda_j(A+B) + \operatorname{Tr}(A+B),$$

so by Lemma 2.1 and Lemma 2.3, it suffices to show that for any contraction C

$$2\Big(s_j(A^{\frac{1}{2}}CB^{\frac{1}{2}}) + |\operatorname{Tr} A^{\frac{1}{2}}CB^{\frac{1}{2}}|\Big) \le \lambda_j(A+B) + \operatorname{Tr}(A+B). \tag{2.1}$$

By setting $M = C^*A^{\frac{1}{2}}$, $N = B^{\frac{1}{2}}$ in Lemma 2.2 gives

$$2s_j(A^{\frac{1}{2}}CB^{\frac{1}{2}}) \le \lambda_j(C^*AC + B).$$

Thus, inequality (2.1) would follow from

$$\lambda_j(C^*AC + B) - \lambda_j(A+B) \le \operatorname{Tr}(A+B) - 2|\operatorname{Tr} A^{\frac{1}{2}}CB^{\frac{1}{2}}|. \tag{2.2}$$

We may assume without loss of generality $\operatorname{Tr} A^{\frac{1}{2}}CB^{\frac{1}{2}} \ge 0$ in (2.2), as it is obvious that (2.2) is invariant by replacing C with $e^{i\theta}C$ for some θ.

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Now setting $M = A^{\frac{1}{2}}C$, $N = B^{\frac{1}{2}}$ in Corollary 2.5 yields

$$\begin{aligned}
&\lambda_j(C^*AC + B) - \lambda_j(A^{\frac{1}{2}}CC^*A^{\frac{1}{2}} + B)\\
&\leq \frac{1}{2}\operatorname{Tr}(C^*AC + B - C^*A^{\frac{1}{2}}B^{\frac{1}{2}} - B^{\frac{1}{2}}A^{\frac{1}{2}}C)\\
&\leq \operatorname{Tr}(C^*AC + B - C^*A^{\frac{1}{2}}B^{\frac{1}{2}} - B^{\frac{1}{2}}A^{\frac{1}{2}}C)\\
&= \operatorname{Tr}(C^*AC + B) - 2\operatorname{Tr} A^{\frac{1}{2}}CB^{\frac{1}{2}}\\
&\leq \operatorname{Tr}(A + B) - 2\operatorname{Tr} A^{\frac{1}{2}}CB^{\frac{1}{2}}.
\end{aligned}$$

As C is a contraction, $A^{\frac{1}{2}}CC^*A^{\frac{1}{2}} \leq A$. The desired inequality (2.2) follows by noting that

$$\lambda_j(A^{\frac{1}{2}}CC^*A^{\frac{1}{2}} + B) \leq \lambda_j(A + B).$$

Hence, Conjecture 1.1 is confirmed.

3. Concluding remarks. The geometric mean of two positive definite matrices $A, B \in \mathbb{M}_n$ is defined by $A\sharp B = B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}}$. The notion of geometric mean can be uniquely extended to all $A, B \geq 0$ by a limit from above:

$$A\sharp B := \lim_{\epsilon \to 0}(A + \epsilon I_n)\sharp(B + \epsilon I_n).$$

For more information about the matrix geometric mean, we refer to [3, Chapter 4].

The noncommutative AM-GM inequality says that

$$\frac{A + B}{2} \geq A\sharp B.$$

It is natrual to consider some possible improvements of (1.1). We remark that the following strenghening of (1.1) is not valid in general:

$$s_j\Big(\Phi(X)\Big) \leq s_j\Big(\Phi(A\sharp B)\Big), \quad j = 1, \ldots, n. \tag{3.1}$$

For example, consider $X = A^{\frac{1}{2}}B^{\frac{1}{2}}$, which guarantees that $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is positive semidefinite. It is known that

$$\prod_{j=1}^{k} s_j(A^{\frac{1}{2}}B^{\frac{1}{2}}) \geq \prod_{j=1}^{k} \lambda_j(A^{\frac{1}{2}}B^{\frac{1}{2}}) \geq \prod_{j=1}^{k} \lambda_j(A\sharp B), \quad k = 1, \ldots, n.$$

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In particular, we can find some positive definite matrices A, B such that $\mathrm{Tr}\, A^{\frac{1}{2}} B^{\frac{1}{2}} > \mathrm{Tr}\, A \sharp B$ and $\mathrm{Tr}\, AB > \mathrm{Tr}(A\sharp B)^2$. Now for these fixed A and B,

$$\begin{aligned}
\sum_{j=1}^{n} s_j^2\Big(\Phi(X)\Big) &= \mathrm{Tr}\, \Big(A^{\frac{1}{2}} B^{\frac{1}{2}} + (\mathrm{Tr}\, A^{\frac{1}{2}} B^{\frac{1}{2}})I\Big)^* \Big(A^{\frac{1}{2}} B^{\frac{1}{2}} + (\mathrm{Tr}\, A^{\frac{1}{2}} B^{\frac{1}{2}})I\Big) \\
&= \mathrm{Tr}\, AB + (n+2)\Big(\mathrm{Tr}\, A^{\frac{1}{2}} B^{\frac{1}{2}}\Big)^2 \\
&> \mathrm{Tr}(A\sharp B)^2 + (n+2)\Big(\mathrm{Tr}\, A\sharp B\Big)^2 \\
&= \mathrm{Tr}\, \Big(A\sharp B + (\mathrm{Tr}\, A\sharp B)I\Big)^2 = \sum_{j=1}^{n} s_j^2\Big(\Phi(A\sharp B)\Big)
\end{aligned}$$

would contradict (3.1).

As pointed out by a referee, if $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is PPT (that is, both $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ and $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ are positive semidefinite), then (3.1) is valid. Indeed, this follows from [6, Lemma 4.2] (see also [1, Lemma 3.1]) and (1.1).

In contrast to (3.1), there is some numerical evidence for the the following assertion:

CONJECTURE 3.1. *If* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$, *where* $A, X, B \in \mathbb{M}_n$, *is positive semidefinite, then*

$$s_j\Big(\Phi(X)\Big) \le s_j\Big(\Phi(A)\sharp\Phi(B)\Big), \quad j = 1, \ldots, n.$$

It seems the proof in Section 2 does not lead to this improvement.

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