

2016

An inverse eigenproblem for generalized reflexive matrices with normal $k+1$ -potencies

Wei-Ru Xu

East China Normal University, weiruxu@foxmail.com

Guo-Liang Chen

East China Normal University, glchen@math.ecnu.edu.cn

Follow this and additional works at: <http://repository.uwyo.edu/ela>



Part of the [Algebra Commons](#), and the [Dynamical Systems Commons](#)

Recommended Citation

Xu, Wei-Ru and Chen, Guo-Liang. (2016), "An inverse eigenproblem for generalized reflexive matrices with normal $k+1$ -potencies", *Electronic Journal of Linear Algebra*, Volume 31, pp. 100-119.

DOI: <https://doi.org/10.13001/1081-3810.3173>

This Article is brought to you for free and open access by Wyoming Scholars Repository. It has been accepted for inclusion in Electronic Journal of Linear Algebra by an authorized editor of Wyoming Scholars Repository. For more information, please contact scholcom@uwyo.edu.

AN INVERSE EIGENPROBLEM FOR GENERALIZED REFLEXIVE MATRICES WITH NORMAL $\{K + 1\}$ -POTENCIES*

WEI-RU XU[†] AND GUO-LIANG CHEN[†]

Abstract. Let $P, Q \in \mathbb{C}^{n \times n}$ be two normal $\{k+1\}$ -potent matrices, i.e., $PP^* = P^*P$, $P^{k+1} = P$, $QQ^* = Q^*Q$, $Q^{k+1} = Q$, $k \in \mathbb{N}$. A matrix $A \in \mathbb{C}^{n \times n}$ is referred to as generalized reflexive with two normal $\{k+1\}$ -potent matrices P and Q if and only if $A = PAQ$. The set of all $n \times n$ generalized reflexive matrices which rely on the matrices P and Q is denoted by $\mathcal{GR}^{n \times n}(P, Q)$. The left and right inverse eigenproblem of such matrices ask from us to find a matrix $A \in \mathcal{GR}^{n \times n}(P, Q)$ containing a given part of left and right eigenvalues and corresponding left and right eigenvectors. In this paper, first necessary and sufficient conditions such that the problem is solvable are obtained. A general representation of the solution is presented. Then an expression of the solution for the optimal Frobenius norm approximation problem is exploited. A stability analysis of the optimal approximate solution, which has scarcely been considered in existing literature, is also developed.

Key words. Left and right inverse eigenproblem, Optimal approximation problem, Generalized reflexive matrix, Moore-Penrose generalized inverse, $\{k+1\}$ -Potent matrix.

AMS subject classifications. 65F18, 15A51, 15A18, 15A12.

1. Introduction. Let $\text{rank}(A)$, A^* and A^\dagger be the rank, conjugate transpose and Moore-Penrose generalized inverse of a matrix $A \in \mathbb{C}^{n \times m}$, respectively. I_n , $\mathbf{0}$ and $i = \sqrt{-1}$ respectively signify the identity matrix of order n , zero matrix or vector with appropriate size and the imaginary unit. Moreover, for any matrix $A \in \mathbb{C}^{n \times m}$, $\mathcal{R}_A = I_n - AA^\dagger$ and $\mathcal{L}_A = I_m - A^\dagger A$ signify specified orthogonal projectors. Unless otherwise stated, $\|\cdot\|$ denotes the Frobenius norm.

Recently, in [10], the authors presented an n -by- n Hermitian reflexive matrix A with respect to a normal $\{k+1\}$ -potent matrix P , i.e., $A^* = A = PAP$ and $PP^* = P^*P$, $P^{k+1} = P$, $k \in \mathbb{N}$. Then they solved an inverse eigenvalue problem with an equality $AX = XD$ constraint, where $X \in \mathbb{C}^{n \times m}$ and $D = \text{diag}(d_1, \dots, d_m) \in \mathbb{R}^{m \times m}$ are given. We introduce the following definition.

DEFINITION 1.1. Let $P, Q \in \mathbb{C}^{n \times n}$ be two normal $\{k+1\}$ -potent matrices, i.e., $PP^* = P^*P$, $P^{k+1} = P$, $QQ^* = Q^*Q$, $Q^{k+1} = Q$, $k \in \mathbb{N}$. A matrix $A \in \mathbb{C}^{n \times n}$ is

*Received by the editors on November 24, 2015. Accepted for publication on January 3, 2016.
Handling Editor: Bryan L. Shader.

[†]Department of Mathematics, Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Shanghai, 200241, P.R. China (weiruxu@foxmail.com, glchen@math.ecnu.edu.cn). This work was supported by National Natural Science Foundation of China (No. 11471122).

referred to as *generalized reflexive* with respect to the matrices P and Q if and only if $A = PAQ$. The set of all $n \times n$ generalized reflexive matrices which rely on the matrices P and Q is denoted by $\mathcal{GR}^{n \times n}(P, Q)$.

In the above definition, the matrices in the set $\mathcal{GR}^{n \times n}(P, Q)$ are the generalized form of the generalized reflexive matrices in [5, 12]. Furthermore, the matrices P and Q could possibly be singular. From [15–17], we know that their spectra are contained in the set $\{0\} \cup \Omega_k$, where Ω_k is the set of all k -th roots of unity, i.e., $\Omega_k = \{e^{2\pi ij/k} : j = 0, 1, \dots, k - 1\}$. Similarly, Trench [24, 25] characterized the generalized matrices with k -involutory symmetries. But the left and right inverse eigenproblem for the matrices in the set $\mathcal{GR}^{n \times n}(P, Q)$ have not been discussed. We will consider two related problems. The first problem is the following.

PROBLEM 1. (Left and right inverse eigenproblem) Given the partial eigeninformation $X \in \mathbb{C}^{n \times m}$, $Y \in \mathbb{C}^{n \times \nu}$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{C}^{m \times m}$, $\Delta = \text{diag}(\mu_1, \mu_2, \dots, \mu_\nu) \in \mathbb{C}^{\nu \times \nu}$. Find $A \in \Omega$ to satisfy

$$AX = X\Lambda, \quad Y^*A = \Delta Y^*$$

such that A maintains the eigeninformation, where Ω is the set $\mathcal{GR}^{n \times n}(P, Q)$.

The prototype of this problem mainly arises in perturbation analysis of matrix eigenvalue [35], recursive matters [1] and has practical applications in the design and modification of mass-spring systems, dynamic structures, Hopfield neural networks, vibration in mechanic, civil engineering and aviation [7–9, 11, 14, 36, 39, 41]. When the matrices P and Q are nonsingular and $P^* = P, Q^* = Q, k = 2$, the above problem has been solved by Liang and Dai [20]. Furthermore, the inverse eigenvalue problems involving reflexive matrices with one equality constraint have drawn considerable interest in [13, 22, 23, 43]. In addition, assume $D = X\Lambda$ and $B = \Delta Y^*$. By taking transpose and renaming the variables, we could equally consider the system of matrix equations

$$\begin{cases} AX = B, \\ XC = D, \end{cases}$$

as in [18, 26–34], where the unique unknown $X \in \mathcal{GR}^{n \times n}(P, Q)$. The second problem is the optimal approximation problem.

PROBLEM 2. (Optimal approximation problem) Let \mathcal{S} be the solution set of the left and right inverse eigenproblem for the matrices in the set $\mathcal{GR}^{n \times n}(P, Q)$. Given a matrix $C \in \mathbb{C}^{n \times n}$, find $\hat{A} \in \mathcal{S}$ such that

$$\|\hat{A} - C\| = \min_{A \in \mathcal{S}} \|A - C\|.$$

The above problem is usually applied in the processes of test or recovery of linear systems due to incomplete data or revising given data. A preliminary estimate C of the unknown matrix A can be obtained by experimental observation values and the information of statistical distribution. The optimal of A is a matrix \hat{A} that satisfies the given matrix equations and prescribed structure for A and is the best approximation of C .

This paper is organized as follows. In Section 2, we first analyze the properties of the matrices in the set $\mathcal{GR}^{n \times n}(P, Q)$. Then the necessary and sufficient conditions for the left and right inverse eigenproblem of such matrices are obtained. In Section 3, when the solution set \mathcal{S} of the inverse eigenproblem is nonempty, we derive the unique solution of the optimal approximation problem. In Section 4, we perform a numerical algorithm of solving the optimal approximation problem. Its stability analysis is also presented, which appears rarely in some related literature (e.g., [19, 20, 40]). In Section 5, we present two illustrated examples. Finally, we draw some conclusions in Section 6.

2. Left and right inverse eigenproblem. In this section, we firstly discuss the properties of a matrix $A \in \mathcal{GR}^{n \times n}(P, Q)$. From Definition 1.1, we know that there are two unitary matrices $U, V \in \mathbb{C}^{n \times n}$ such that

$$(2.1) \quad P = U \text{diag}(\omega_1 I_{\mathbf{r}_1}, \dots, \omega_t I_{\mathbf{r}_t}, \mathbf{0}_{\mathbf{r}_{t+1}}) U^*,$$

$$(2.2) \quad Q = V \text{diag}(\nu_1 I_{\mathbf{s}_1}, \dots, \nu_g I_{\mathbf{s}_g}, \mathbf{0}_{\mathbf{s}_{g+1}}) V^*,$$

where $\omega_i, \nu_j \in \Omega_k$, $i = 1, \dots, t$, $j = 1, \dots, g$, $\sum_{i=1}^t \mathbf{r}_i = \text{rank}(P) = \mathbf{r}$, $\sum_{j=1}^g \mathbf{s}_j = \text{rank}(Q) = \mathbf{s}$ and $\mathbf{r}_{t+1} = n - \mathbf{r}$, $\mathbf{s}_{g+1} = n - \mathbf{s}$. Besides, ω_i are distinct for all $i = 1, \dots, t$, and ν_j are also distinct for all $j = 1, \dots, g$. Then for any matrix $A \in \mathcal{GR}^{n \times n}(P, Q)$, we partition the block matrix U^*AV as follows:

$$(2.3) \quad U^*AV = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,g} & A_{1,g+1} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,g} & A_{2,g+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{t,1} & A_{t,2} & \cdots & A_{t,g} & A_{t,g+1} \\ A_{t+1,1} & A_{t+1,2} & \cdots & A_{t+1,g} & A_{t+1,g+1} \end{pmatrix}.$$

Because $A = PAQ$, we can obtain

$$\begin{cases} A_{i,g+1} = \mathbf{0}, & i = 1, 2, \dots, t+1, \\ A_{t+1,j} = \mathbf{0}, & j = 1, 2, \dots, g, \\ A_{i,j} = \omega_i \nu_j A_{i,j}, & i = 1, 2, \dots, t, j = 1, 2, \dots, g \end{cases}$$

from (2.1), (2.2) and (2.3).

Thus, we have $\omega_i \nu_j = 1$ or $A_{i,j} = \mathbf{0}$ for any $i = 1, 2, \dots, t$ and $j = 1, 2, \dots, g$. Because $\omega_i, \nu_j \in \Omega_k$, it follows that $\omega_i = \overline{\nu_j}$ or $A_{i,j} = \mathbf{0}$ for each $i = 1, 2, \dots, t$ and $j = 1, 2, \dots, g$. Up to permutation similarity, the structure of the matrix U^*AV lies on the k -th roots of unity that appear in the Schur decompositions of the matrices P and Q . Assume $t \geq g$. Without loss of generality, the distinct eigenvalues of the matrices P and Q can be, respectively, arranged as

$$1, -1, \tau_3, \tau_4, \dots, \tau_p, \tau_{p+1}, \overline{\tau_{p+1}}, \tau_{p+2}, \overline{\tau_{p+2}}, \dots, \tau_{p+l}, \overline{\tau_{p+l}}, \tau_q, \tau_{q+1}, \dots, \tau_e, 0,$$

and

$$1, -1, \overline{\tau_3}, \overline{\tau_4}, \dots, \overline{\tau_p}, \tau_{p+1}, \overline{\tau_{p+1}}, \tau_{p+2}, \overline{\tau_{p+2}}, \dots, \tau_{p+l}, \overline{\tau_{p+l}}, \tau_h, \tau_{h+1}, \dots, \tau_f, 0,$$

where $p + 2l + e - q + 1 = t$, $p + 2l + f - h + 1 = g$, $e - q \geq f - h$ and $\tau_i \neq \overline{\tau_j}$ for any $i = q, q + 1, \dots, e$ and $j = h, h + 1, \dots, f$.

REMARK 2.1. The eigenvalues 1 and -1 may occur or not occur in the above arrangements. In order to consider a wider range of circumstances, we assume the eigenvalues 1 and -1 both appear in the above two arrangements. In addition, all the distinct eigenvalues of P and Q except zeros belong to the set Ω_k .

Based on the above arrangements, the matrix U^*AV is the following:

$$(2.4) \quad U^*AV = \text{diag}(A_{1,1}, A_{2,2}, A_{3,3}, \dots, A_{p,p}, \tilde{A}_{p+1,p+2}, \dots, \tilde{A}_{p+2l-1,p+2l}, \mathbf{0}),$$

where

$$(2.5) \quad \tilde{A}_{p+2s-1,p+2s} = \begin{pmatrix} \mathbf{0} & A_{p+2s-1,p+2s} \\ A_{p+2s,p+2s-1} & \mathbf{0} \end{pmatrix}$$

for all $s = 1, \dots, l$. Here,

$$A_{i,i} \in \mathbb{C}^{\mathbf{r}_i \times \mathbf{s}_i}, \quad i = 1, \dots, p,$$

$$A_{p+2s-1,p+2s} \in \mathbb{C}^{\mathbf{r}_{p+2s-1} \times \mathbf{s}_{p+2s}}, \quad A_{p+2s,p+2s-1} \in \mathbb{C}^{\mathbf{r}_{p+2s} \times \mathbf{s}_{p+2s-1}}, \quad s = 1, \dots, l,$$

$$\mathbf{r} = \sum_{i=1}^{p+2l} \mathbf{r}_i, \quad \mathbf{s} = \sum_{i=1}^{p+2l} \mathbf{s}_i.$$

REMARK 2.2. In (2.4), the block matrix $A_{1,1}$ is associated with the common eigenvalue 1 of P and Q . The block matrix $A_{2,2}$ is associated with the common eigenvalue -1 of P and Q . The block matrix $A_{j,j}$ is corresponding to the eigenvalue τ_j of P and the eigenvalue $\overline{\tau_j}$ of Q for any $j = 3, \dots, p$. For each $s = 1, \dots, l$,

the block matrix $A_{p+2s-1, p+2s}$ is corresponding to the eigenvalue τ_{p+s} of P and the eigenvalue $\overline{\tau_{p+s}}$ of Q . Meanwhile, the block matrix $A_{p+2s, p+2s-1}$ is corresponding to the eigenvalue $\overline{\tau_{p+s}}$ of P and the eigenvalue τ_{p+s} of Q .

Then we introduce the following result to solve the inverse eigenproblem later on.

LEMMA 2.3. ([4]) *Let $A_1 \in \mathbb{C}^{n \times m}$, $C_1 \in \mathbb{C}^{n \times p}$, $B_1 \in \mathbb{C}^{p \times q}$ and $D_1 \in \mathbb{C}^{m \times q}$ be given. The pair of matrix equations $A_1 Z = C_1$, $Z B_1 = D_1$ has a solution $Z \in \mathbb{C}^{m \times p}$ if and only if*

$$\mathcal{R}_{A_1} C_1 = \mathbf{0}, \quad D_1 \mathcal{L}_{B_1} = \mathbf{0}, \quad A_1 D_1 = C_1 B_1.$$

Moreover, the general solution can be expressed as

$$Z = A_1^\dagger C_1 + \mathcal{L}_{A_1} D_1 B_1^\dagger + \mathcal{L}_{A_1} R_1 \mathcal{R}_{B_1},$$

where $R_1 \in \mathbb{C}^{m \times p}$ is an arbitrary matrix.

Next, we can solve the left and right inverse eigenproblem for any matrix in the set $\mathcal{GR}^{n \times n}(P, Q)$ as follows.

THEOREM 2.4. *Given $X \in \mathbb{C}^{n \times m}$, $Y \in \mathbb{C}^{n \times \nu}$, $\Lambda \in \mathbb{C}^{m \times m}$ and $\Delta \in \mathbb{C}^{\nu \times \nu}$. Let $P, Q \in \mathbb{C}^{n \times n}$ be two normal $\{k+1\}$ -potent matrices and be given by (2.1) and (2.2), respectively. Partition*

$$(2.6) \quad U^* X = (\mathcal{X}_1^*, \mathcal{X}_2^*, \mathcal{X}_3^*, \dots, \mathcal{X}_p^*, \tilde{\mathcal{X}}_{p+1}^*, \dots, \tilde{\mathcal{X}}_{p+2l-1}^*, \mathcal{X}_{p+2l+1}^*)^*,$$

$$(2.7) \quad Y^* U = (\mathcal{Y}_1^*, \mathcal{Y}_2^*, \mathcal{Y}_3^*, \dots, \mathcal{Y}_p^*, \tilde{\mathcal{Y}}_{p+1}^*, \dots, \tilde{\mathcal{Y}}_{p+2l-1}^*, \mathcal{Y}_{p+2l+1}^*)$$

compatibly with the block row partitioning of the matrix $U^* A V$ in (2.4) and partition

$$(2.8) \quad V^* X = (X_1^*, X_2^*, X_3^*, \dots, X_p^*, \tilde{X}_{p+1}^*, \dots, \tilde{X}_{p+2l-1}^*, X_{p+2l+1}^*)^*,$$

$$(2.9) \quad Y^* V = (Y_1^*, Y_2^*, Y_3^*, \dots, Y_p^*, \tilde{Y}_{p+1}^*, \dots, \tilde{Y}_{p+2l-1}^*, Y_{p+2l+1}^*)$$

compatibly with the block column partitioning of the matrix $U^* A V$ in (2.4), where

$$\begin{aligned} \tilde{\mathcal{X}}_{p+2s-1}^* &= (\mathcal{X}_{p+2s-1}^* \quad \mathcal{X}_{p+2s}^*), & \tilde{\mathcal{Y}}_{p+2s-1}^* &= (\mathcal{Y}_{p+2s-1}^* \quad \mathcal{Y}_{p+2s}^*), \\ \tilde{X}_{p+2s-1}^* &= (X_{p+2s-1}^* \quad X_{p+2s}^*), & \tilde{Y}_{p+2s-1}^* &= (Y_{p+2s-1}^* \quad Y_{p+2s}^*) \end{aligned}$$

for any $s = 1, \dots, l$. Then there is a matrix $A \in \mathcal{GR}^{n \times n}(P, Q)$ such that $AX = X\Lambda$, $Y^* A = \Delta Y^*$ if and only if

$$(2.10) \quad \mathcal{L}_{\mathcal{Y}_i} \Delta Y_i^* = \mathbf{0}, \quad \mathcal{X}_i \Lambda \mathcal{L}_{X_i} = \mathbf{0}, \quad \mathcal{Y}_i^* \mathcal{X}_i \Lambda = \Delta Y_i^* X_i$$

hold for all $i = 1, \dots, p$,

(2.11)

$$\mathcal{L}_{\mathcal{Y}_{p+2s-1}} \Delta Y_{p+2s}^* = \mathbf{0}, \quad \mathcal{X}_{p+2s-1} \Lambda \mathcal{L}_{\mathcal{X}_{p+2s}} = \mathbf{0}, \quad \mathcal{Y}_{p+2s-1}^* \mathcal{X}_{p+2s-1} \Lambda = \Delta Y_{p+2s}^* \mathcal{X}_{p+2s},$$

(2.12)

$$\mathcal{L}_{\mathcal{Y}_{p+2s}} \Delta Y_{p+2s-1}^* = \mathbf{0}, \quad \mathcal{X}_{p+2s} \Lambda \mathcal{L}_{\mathcal{X}_{p+2s-1}} = \mathbf{0}, \quad \mathcal{Y}_{p+2s}^* \mathcal{X}_{p+2s} \Lambda = \Delta Y_{p+2s-1}^* \mathcal{X}_{p+2s-1}$$

hold for all $s = 1, \dots, l$ and $\mathcal{X}_{p+2l+1} \Lambda = \mathbf{0}$, $\Delta Y_{p+2l+1}^* = \mathbf{0}$. Furthermore, in this case the general solution is given by

$$(2.13) \quad A = U \text{diag}(A_{1,1}, A_{2,2}, A_{3,3}, \dots, A_{p,p}, \tilde{A}_{p+1,p+2}, \dots, \tilde{A}_{p+2l-1,p+2l}, \mathbf{0}) V^*,$$

with

$$A_{i,i} = (\mathcal{Y}_i^*)^\dagger \Delta Y_i^* + \mathcal{R}_{\mathcal{Y}_i} \mathcal{X}_i \Lambda \mathcal{X}_i^\dagger + \mathcal{R}_{\mathcal{Y}_i} E_i \mathcal{R}_{\mathcal{X}_i}, \quad i = 1, \dots, p,$$

and

$$\tilde{A}_{p+2s-1,p+2s} = \begin{pmatrix} \mathbf{0} & A_{p+2s-1,p+2s} \\ A_{p+2s,p+2s-1} & \mathbf{0} \end{pmatrix}, \quad s = 1, \dots, l,$$

where

$$A_{p+2s-1,p+2s} = (\mathcal{Y}_{p+2s-1}^*)^\dagger \Delta Y_{p+2s}^* + \mathcal{R}_{\mathcal{Y}_{p+2s-1}} \mathcal{X}_{p+2s-1} \Lambda \mathcal{X}_{p+2s}^\dagger + \mathcal{R}_{\mathcal{Y}_{p+2s-1}} F_s \mathcal{R}_{\mathcal{X}_{p+2s}},$$

$$A_{p+2s,p+2s-1} = (\mathcal{Y}_{p+2s}^*)^\dagger \Delta Y_{p+2s-1}^* + \mathcal{R}_{\mathcal{Y}_{p+2s}} \mathcal{X}_{p+2s} \Lambda \mathcal{X}_{p+2s-1}^\dagger + \mathcal{R}_{\mathcal{Y}_{p+2s}} G_s \mathcal{R}_{\mathcal{X}_{p+2s-1}}$$

and E_i, F_s, G_s are arbitrary matrices with appropriate sizes for any $i = 1, \dots, p$ and $s = 1, \dots, l$.

Proof. Assume the left and right inverse eigenproblem is solvable in $\mathcal{GR}^{n \times n}(P, Q)$, then the problem is equivalent to find a matrix $A \in \mathcal{GR}^{n \times n}(P, Q)$ such that $AX = X\Lambda$, $Y^*A = \Delta Y^*$. From (2.1) and (2.2), we have

$$(2.14) \quad U^*AVV^*X = U^*X\Lambda, \quad Y^*UU^*AV = \Delta Y^*V.$$

Substituting (2.4) and (2.6)-(2.9) into (2.14) yield

$$(2.15) \quad \begin{cases} A_{i,i}X_i = \mathcal{X}_i\Lambda, & i = 1, \dots, p, \\ \tilde{A}_{p+2s-1,p+2s}\tilde{X}_{p+2s-1} = \tilde{\mathcal{X}}_{p+2s-1}\Lambda, & s = 1, \dots, l, \\ \mathcal{X}_{p+2l+1}\Lambda = \mathbf{0}, \end{cases}$$

and

$$(2.16) \quad \begin{cases} \mathcal{Y}_i^*A_{i,i} = \Delta Y_i^*, & i = 1, \dots, p, \\ \tilde{\mathcal{Y}}_{p+2s-1}^*\tilde{A}_{p+2s-1,p+2s} = \Delta \tilde{Y}_{p+2s-1}^*, & s = 1, \dots, l, \\ \Delta Y_{p+2l+1}^* = \mathbf{0}. \end{cases}$$

Based on (2.5), Eqs. (2.15) and (2.16) are equivalent to the following

$$(2.17) \quad A_{i,i}X_i = \mathcal{X}_i\Lambda, \quad \mathcal{Y}_i^*A_{i,i} = \Delta Y_i^*, \quad i = 1, \dots, p,$$

$$(2.18) \quad A_{p+2s-1,p+2s}X_{p+2s} = \mathcal{X}_{p+2s-1}\Lambda, \quad \mathcal{Y}_{p+2s-1}^*A_{p+2s-1,p+2s} = \Delta Y_{p+2s}^*, \quad s = 1, \dots, l,$$

$$(2.19) \quad A_{p+2s,p+2s-1}X_{p+2s-1} = \mathcal{X}_{p+2s}\Lambda, \quad \mathcal{Y}_{p+2s}^*A_{p+2s,p+2s-1} = \Delta Y_{p+2s-1}^*, \quad s = 1, \dots, l,$$

and

$$(2.20) \quad \mathcal{X}_{p+2l+1}\Lambda = \mathbf{0}, \quad \Delta Y_{p+2l+1}^* = \mathbf{0}.$$

By Lemma 2.3, the matrix equations (2.17)-(2.19) are all consistent if and only if the equalities in (2.10)-(2.12) hold. In this case, the general solutions are respectively

$$(2.21) \quad A_{i,i} = (\mathcal{Y}_i^*)^\dagger \Delta Y_i^* + \mathcal{R}_{\mathcal{Y}_i} \mathcal{X}_i \Lambda X_i^\dagger + \mathcal{R}_{\mathcal{Y}_i} E_i \mathcal{R}_{X_i}, \quad i = 1, \dots, p,$$

$$(2.22) \quad A_{p+2s-1,p+2s} = (\mathcal{Y}_{p+2s-1}^*)^\dagger \Delta Y_{p+2s}^* + \mathcal{R}_{\mathcal{Y}_{p+2s-1}} \mathcal{X}_{p+2s-1} \Lambda X_{p+2s}^\dagger + \mathcal{R}_{\mathcal{Y}_{p+2s-1}} F_s \mathcal{R}_{X_{p+2s}}, \quad s = 1, \dots, l,$$

$$(2.23) \quad A_{p+2s,p+2s-1} = (\mathcal{Y}_{p+2s}^*)^\dagger \Delta Y_{p+2s-1}^* + \mathcal{R}_{\mathcal{Y}_{p+2s}} \mathcal{X}_{p+2s} \Lambda X_{p+2s-1}^\dagger + \mathcal{R}_{\mathcal{Y}_{p+2s}} G_s \mathcal{R}_{X_{p+2s-1}}, \quad s = 1, \dots, l.$$

By substituting the above equalities into (2.4) and (2.5), the general solution $A \in \mathcal{GR}^{n \times n}(P, Q)$ of the system of the matrix equations $AX = X\Lambda$, $Y^*A = \Delta Y^*$ is presented in (2.13).

Conversely, if the conditions (2.10)-(2.12) and (2.20) hold, then the system of the matrix equations $AX = X\Lambda$, $Y^*A = \Delta Y^*$ is consistent apparently. \square

For example, we analyze the following interesting Leontief model in economics.

EXAMPLE 2.5. The recursive inverse eigenvalue problem [1] which arises in Leontief economic model is to construct a matrix $A \in \mathbb{C}^{n \times n}$ such that

$$(2.24) \quad \begin{cases} A_i x_i = \lambda_i x_i, \\ y_i^* A_i = \mu_i y_i^*, \end{cases} \quad i = 1, 2, \dots, n,$$

where A_i is the i th leading principle submatrix of A and (μ_i, y_i) and (λ_i, x_i) are given left and right eigenpairs, respectively. Obviously, the recursive inverse eigenvalue problem is the left and right inverse eigenproblem with submatrix constraint.

If the required matrix A is in the set $\mathcal{GR}^{3 \times 3}(P, Q)$, where

$$P = \begin{pmatrix} -0.3333 & 0.6667 & 0.6667 \\ 0.6667 & -0.3333 & 0.6667 \\ 0.6667 & 0.6667 & -0.3333 \end{pmatrix}, \quad Q = \begin{pmatrix} 1.0000 & 0 & 0 \\ 0 & 0 & -1.0000 \\ 0 & -1.0000 & 0 \end{pmatrix},$$

i.e., $P^3 = P$, $Q^3 = Q$, then the orthogonal matrices U and V in (2.1) and (2.2) are, respectively,

$$U = \begin{pmatrix} 0.5774 & 0.4082 & 0.7071 \\ 0.5774 & 0.4082 & -0.7071 \\ 0.5774 & -0.8165 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1.0000 & 0 & 0 \\ 0 & 0.7071 & 0.7071 \\ 0 & -0.7071 & 0.7071 \end{pmatrix}.$$

Therefore, the matrix $U^*AV = \text{diag}(A_{11}, A_{22})$, where $A_{11} \in \mathbb{R}^{1 \times 2}$ and $A_{22} \in \mathbb{R}^{2 \times 1}$. In the following we input

$$X = (x_n) = (-0.2663, 0.2679, 0.9259)^*, \quad \Lambda = (\lambda_n) = -2.2351,$$

$$Y = (y_n) = (-0.3777, -0.9128, 0.1555)^*, \quad \Delta = (\mu_n) = 1.7351.$$

Substituting them into (2.24) yields the left and right inverse eigenproblem $A_3X = X\Lambda$, $Y^*A_3 = \Delta Y^*$. Here $A_3 = A$ is the required structured matrix. It is not difficult to verify that the conditions in (2.10) hold. We might as well take $E_i = \mathbf{0}$ in (2.21) for $i = 1, 2$. Then we have

$$A_{11} = \begin{pmatrix} 1.0001 & 2.0002 \end{pmatrix} \quad \text{and} \quad A_{22} = \begin{pmatrix} 1.9999 & 1.0002 \end{pmatrix}^*.$$

Finally, we can obtain the structured matrix

$$A = \begin{pmatrix} 0.5774 & 1.8940 & 0.2609 \\ 0.5774 & 0.8938 & -0.7393 \\ 0.5774 & -0.3381 & -1.9712 \end{pmatrix} \in \mathcal{GR}^{3 \times 3}(P, Q).$$

3. Optimal approximation problem. If the solution set \mathcal{S} of the left and right inverse eigenproblem for the matrices in the set $\mathcal{GR}^{n \times n}(P, Q)$ is nonempty, then we can obtain the optimal approximate solution corresponding to a given matrix $C \in \mathbb{C}^{n \times n}$. For simplicity, we give the following lemmas.

LEMMA 3.1. ([3]) *Let $A_2 \in \mathbb{C}^{p \times n}$, $B_2 \in \mathbb{C}^{m \times q}$ and $C_2 \in \mathbb{C}^{p \times q}$. The linear matrix equation $A_2 Z B_2 = C_2$ has a solution $Z \in \mathbb{C}^{n \times m}$ if and only if*

$$A_2 A_2^\dagger C_2 B_2^\dagger B_2 = C_2.$$

Furthermore, in this case the general solution is given by

$$Z = A_2^\dagger C_2 B_2^\dagger + R_2 - A_2^\dagger A_2 R_2 B_2 B_2^\dagger,$$

where $R_2 \in \mathbb{C}^{n \times m}$ is an arbitrary matrix.

LEMMA 3.2. Let $\Phi \in \mathbb{C}^{p \times p}$ and $\Psi \in \mathbb{C}^{q \times q}$ be orthogonal projection matrices and $T \in \mathbb{C}^{p \times q}$. Then the general solution of the least-squares problem

$$\min_{L \in \mathbb{C}^{p \times q}} \|\Phi L \Psi + T\|$$

is expressed as

$$L = -\Phi T \Psi + W - \Phi W \Psi,$$

where $W \in \mathbb{C}^{p \times q}$ is an arbitrary matrix.

Proof. Let

$$f(L) = \|\Phi L \Psi + T\|^2.$$

Then $f(L)$ is a convex, continuous and differentiable function with respect to L . Therefore, if $f(L) = \min$ then $\frac{\partial f(L)}{\partial L} = 0$. Because of $\Phi^* = \Phi = \Phi^2$ and $\Psi^* = \Psi = \Psi^2$, then

$$\frac{\partial f(L)}{\partial L} = 2\Phi(L + T)\Psi.$$

When $\frac{\partial f(L)}{\partial L} = 0$, we have $\Phi L \Psi = -\Phi T \Psi$. By Lemma 3.1 and $\Phi^\dagger = \Phi, \Psi^\dagger = \Psi$, it follows that $L = -\Phi T \Psi + W - \Phi W \Psi$, where $W \in \mathbb{C}^{p \times q}$. \square

Now, the explicit solution of the optimal approximation problem can be derived as follows.

THEOREM 3.3. Let $X \in \mathbb{C}^{n \times m}, Y \in \mathbb{C}^{n \times \nu}, \Lambda \in \mathbb{C}^{m \times m}$ and $\Delta \in \mathbb{C}^{\nu \times \nu}$ be given by Theorem 2.4. Assume the solution set $\mathcal{S} \subseteq \mathcal{GR}^{n \times n}(P, Q)$ of the left and right inverse eigenproblem is nonempty. Given $C \in \mathbb{C}^{n \times n}$ and partition

$$(3.1) \quad U^* C V = \begin{pmatrix} C_{1,1} & \cdots & C_{1,p+2l} & C_{1,p+2l+1} \\ \vdots & \ddots & \vdots & \vdots \\ C_{p+2l,1} & \cdots & C_{p+2l,p+2l} & C_{p+2l,p+2l+1} \\ C_{p+2l+1,1} & \cdots & C_{p+2l+1,p+2l} & C_{p+2l+1,p+2l+1} \end{pmatrix}$$

conformally with the same block partitioning of (2.4). Then the optimal approximation problem has a unique solution \hat{A} and \hat{A} can be expressed as

$$(3.2) \quad \hat{A} = U \text{diag}(\hat{A}_{1,1}, \hat{A}_{2,2}, \hat{A}_{3,3}, \dots, \hat{A}_{p,p}, \tilde{A}_{p+1,p+2}, \dots, \tilde{A}_{p+2l-1,p+2l}, \mathbf{0}) V^*,$$

with

$$\widehat{A}_{i,i} = (\mathcal{Y}_i^*)^\dagger \Delta Y_i^* + \mathcal{R}_{\mathcal{Y}_i} \mathcal{X}_i \Lambda X_i^\dagger + \mathcal{R}_{\mathcal{Y}_i} C_{i,i} \mathcal{R}_{X_i}, \quad i = 1, \dots, p,$$

and

$$\widetilde{A}_{p+2s-1,p+2s} = \begin{pmatrix} \mathbf{0} & \widehat{A}_{p+2s-1,p+2s} \\ \widehat{A}_{p+2s,p+2s-1} & \mathbf{0} \end{pmatrix}, \quad s = 1, \dots, l,$$

where

$$\begin{aligned} \widehat{A}_{p+2s-1,p+2s} = & (\mathcal{Y}_{p+2s-1}^*)^\dagger \Delta Y_{p+2s}^* + \mathcal{R}_{\mathcal{Y}_{p+2s-1}} \mathcal{X}_{p+2s-1} \Lambda X_{p+2s}^\dagger \\ & + \mathcal{R}_{\mathcal{Y}_{p+2s-1}} C_{p+2s-1,p+2s} \mathcal{R}_{X_{p+2s}}, \end{aligned}$$

$$\begin{aligned} \widehat{A}_{p+2s,p+2s-1} = & (\mathcal{Y}_{p+2s}^*)^\dagger \Delta Y_{p+2s-1}^* + \mathcal{R}_{\mathcal{Y}_{p+2s}} \mathcal{X}_{p+2s} \Lambda X_{p+2s-1}^\dagger \\ & + \mathcal{R}_{\mathcal{Y}_{p+2s}} C_{p+2s,p+2s-1} \mathcal{R}_{X_{p+2s-1}} \end{aligned}$$

for all $s = 1, \dots, l$.

Proof. Assume the solution set $\mathcal{S} \subseteq \mathcal{GR}^{n \times n}(P, Q)$ of the left and right inverse eigenproblem is nonempty. From (2.13), it is apparent to verify that \mathcal{S} is a closed convex set. Since $\mathbb{C}^{n \times n}$ is a uniformly convex Banach space under the Frobenius norm [6, p. 22], there exists a unique solution for the inverse eigenproblem. By the unitary invariance of the Frobenius norm and (2.13), the optimal approximation problem is equivalent to

$$(3.3) \quad \min_{A \in \mathcal{S}} \|U^* AV - U^* CV\|^2.$$

Because the matrix $U^* CV$ has the same partitioning with any matrix $U^* AV$, (3.3) is equivalent to

$$\begin{aligned} & \sum_{i=1}^p \|A_{i,i} - C_{i,i}\|^2 + \sum_{s=1}^l \|A_{p+2s-1,p+2s} - C_{p+2s-1,p+2s}\|^2 \\ & + \sum_{s=1}^l \|A_{p+2s,p+2s-1} - C_{p+2s,p+2s-1}\|^2 = \min. \end{aligned}$$

Obviously, the above equality is equivalent to

$$(3.4) \quad \|A_{i,i} - C_{i,i}\| = \min, \quad i = 1, \dots, p,$$

$$(3.5) \quad \|A_{p+2s-1,p+2s} - C_{p+2s-1,p+2s}\| = \min, \quad s = 1, \dots, l,$$

and

$$(3.6) \quad \|A_{p+2s,p+2s-1} - C_{p+2s,p+2s-1}\| = \min, \quad s = 1, \dots, l.$$

By Lemma 3.2 and (2.21), the solutions of (3.4) in E_i are respectively

$$E_i = \mathcal{R}_{\mathcal{Y}_i} C_{i,i} \mathcal{R}_{X_i} + \overline{E}_i - \mathcal{R}_{\mathcal{Y}_i} \overline{E}_i \mathcal{R}_{X_i},$$

where \overline{E}_i are arbitrary matrices with appropriate sizes for all $i = 1, \dots, p$. Then by substituting the formula of E_i into (2.21), it follows that the best approximate solutions of (3.4) are

$$(3.7) \quad \widehat{A}_{i,i} = (\mathcal{Y}_i^*)^\dagger \Delta Y_i^* + \mathcal{R}_{\mathcal{Y}_i} \mathcal{X}_i \Lambda X_i^\dagger + \mathcal{R}_{\mathcal{Y}_i} C_{i,i} \mathcal{R}_{X_i}, \quad i = 1, \dots, p.$$

Similarly, by Lemma 3.2, (2.22) and (2.23), the solutions of (3.5) in F_s and the solutions of (3.6) in G_s are respectively

$$F_s = \mathcal{R}_{\mathcal{Y}_{p+2s-1}} C_{p+2s-1,p+2s} \mathcal{R}_{X_{p+2s}} + \overline{F}_s - \mathcal{R}_{\mathcal{Y}_{p+2s-1}} \overline{F}_s \mathcal{R}_{X_{p+2s}},$$

$$G_s = \mathcal{R}_{\mathcal{Y}_{p+2s}} C_{p+2s,p+2s-1} \mathcal{R}_{X_{p+2s-1}} + \overline{G}_s - \mathcal{R}_{\mathcal{Y}_{p+2s}} \overline{G}_s \mathcal{R}_{X_{p+2s-1}},$$

where \overline{F}_s and \overline{G}_s are arbitrary matrices with appropriate sizes for all $s = 1, \dots, l$. Then by substituting F_s and G_s into (2.22) and (2.23), we can obtain that the best approximation solutions of (3.5) and (3.6) are respectively

$$(3.8) \quad \widehat{A}_{p+2s-1,p+2s} = (\mathcal{Y}_{p+2s-1}^*)^\dagger \Delta Y_{p+2s}^* + \mathcal{R}_{\mathcal{Y}_{p+2s-1}} \mathcal{X}_{p+2s-1} \Lambda X_{p+2s}^\dagger + \mathcal{R}_{\mathcal{Y}_{p+2s-1}} C_{p+2s-1,p+2s} \mathcal{R}_{X_{p+2s}},$$

$$(3.9) \quad \widehat{A}_{p+2s,p+2s-1} = (\mathcal{Y}_{p+2s}^*)^\dagger \Delta Y_{p+2s-1}^* + \mathcal{R}_{\mathcal{Y}_{p+2s}} \mathcal{X}_{p+2s} \Lambda X_{p+2s-1}^\dagger + \mathcal{R}_{\mathcal{Y}_{p+2s}} C_{p+2s,p+2s-1} \mathcal{R}_{X_{p+2s-1}}$$

for all $s = 1, \dots, l$. Therefore, we can get the unique solution \widehat{A} as described in (3.2) by substituting (3.7), (3.8) and (3.9) into (2.13). \square

In Theorem 3.3, if $C = \mathbf{0}$, we can immediately obtain the following result.

COROLLARY 3.4. *In Theorem 2.4, if $AX = X\Lambda$, $Y^*A = \Delta Y^*$ is consistent in the set $\mathcal{GR}^{n \times n}(P, Q)$, then its unique solution with the least norm is*

$$A_0 = U \text{diag}(A_{1,1}^0, A_{2,2}^0, A_{3,3}^0, \dots, A_{p,p}^0, A_{p+1,p+2}^0, \dots, A_{p+2l-1,p+2l}^0, \mathbf{0}) V^*$$

in which

$$A_{i,i}^0 = (\mathcal{Y}_i^*)^\dagger \Delta Y_i^* + \mathcal{R}_{\mathcal{Y}_i} \mathcal{X}_i \Lambda X_i^\dagger, \quad i = 1, \dots, p,$$

and

$$A_{p+2s-1,p+2s}^0 = \begin{pmatrix} \mathbf{0} & A_{p+2s-1,p+2s}^0 \\ A_{p+2s,p+2s-1}^0 & \mathbf{0} \end{pmatrix}, \quad s = 1, \dots, l,$$

where

$$A_{p+2s-1,p+2s}^0 = (\mathcal{Y}_{p+2s-1}^*)^\dagger \Delta Y_{p+2s}^* + \mathcal{R}_{\mathcal{Y}_{p+2s-1}} \mathcal{X}_{p+2s-1} \Lambda X_{p+2s}^\dagger,$$

$$A_{p+2s,p+2s-1}^0 = (\mathcal{Y}_{p+2s}^*)^\dagger \Delta Y_{p+2s-1}^* + \mathcal{R}_{\mathcal{Y}_{p+2s}} \mathcal{X}_{p+2s} \Lambda X_{p+2s-1}^\dagger$$

for all $s = 1, \dots, l$.

4. Numerical algorithm and its stability analysis. Based on Theorem 3.3, we propose the following algorithm for computing the optimal approximate solution over the matrices in $\mathcal{GR}^{n \times n}(P, Q)$.

ALGORITHM 4.1.

Input: $X \in \mathbb{C}^{n \times m}$, $Y \in \mathbb{C}^{n \times \nu}$, $\Lambda \in \mathbb{C}^{m \times m}$, $\Delta \in \mathbb{C}^{\nu \times \nu}$, $C \in \mathbb{C}^{n \times n}$ and two normal $\{k + 1\}$ -potent matrices $P, Q \in \mathbb{C}^{n \times n}$.

Output: $\hat{A} \in \mathbb{C}^{n \times n}$.

Begin

1. Compute $\mathbf{r}_1, \dots, \mathbf{r}_{t+1}$ and U by (2.1) and calculate $\mathbf{s}_1, \dots, \mathbf{s}_{g+1}$ and V by (2.2).
2. Partition U^*X , Y^*U , V^*X , Y^*V according to (2.6)-(2.9) respectively.
3. Compute X_i^\dagger , \mathcal{Y}_i^\dagger , $i = 1, \dots, p + 2l$.
4. If the conditions in (2.10)-(2.12) and (2.20) are all satisfied, then continue. Otherwise, we stop.
5. Partition the matrix U^*CV by (3.1).
6. Compute $\hat{A}_{i,i}$ according to (3.7) for all $i = 1, \dots, p$.
7. Compute $\hat{A}_{p+2s-1,p+2s}$ and $\hat{A}_{p+2s,p+2s-1}$ by (3.8) and (3.9) respectively for all $s = 1, \dots, l$.
8. Compute $\hat{A}_{p+2s-1,p+2s}$ by (2.5) for all $s = 1, \dots, l$.
9. Compute \hat{A} by (3.2).

End

Based on the above algorithm, we begin to discuss its computational complexity. We mainly consider the case when P and Q have dense eigenvector matrices U and V , respectively.

For Step 1, using spectral decomposition to compute U and V requires $\mathcal{O}(n^3)$ operations. For Step 2, since U and V are dense, it requires $\mathcal{O}(n^2(m + \nu))$ operations. For Step 3, it requires $\mathcal{O}(\mathbf{s}_i^2 m + m^3 + \mathbf{r}_i^2 \nu + \nu^3)$ operations to compute X_i^\dagger and \mathcal{Y}_i^\dagger by using singular value decomposition. Then the cost of this step is $\mathcal{O}((\mathbf{s}_1^2 + \dots + \mathbf{s}_{p+2l}^2)m + m^3 + (\mathbf{r}_1^2 + \dots + \mathbf{r}_{p+2l}^2)\nu + \nu^3)$. For Step 4, if we compute $\mathcal{L}_{\mathcal{Y}_i}$ and \mathcal{L}_{X_i} for all $i = 1, \dots, p + 2l$ firstly, it needs $\mathcal{O}(\nu^2 \mathbf{r} + m^2 \mathbf{s})$ operations. Secondly, it requires $\mathcal{O}(n(\nu + m))$ operations to compute ΔY_i^* and $\mathcal{X}_i \Lambda$ for all $i = 1, \dots, p + 2l + 1$. Therefore,

the cost of Step 4 is $\mathcal{O}((m^2 + \nu^2 + m\nu)(\mathbf{r} + \mathbf{s}) + n(\nu + m))$. For Step 5, because of the density of U and V , it requires $\mathcal{O}(n^3)$ operations. For Step 6, if we compute $(\mathcal{Y}_i^*)^\dagger \Delta Y_i^*$ as $(\mathcal{Y}_i^*)^\dagger (\Delta Y_i^*)$, $\mathcal{Y}_i \mathcal{Y}_i^\dagger \mathcal{X}_i \Lambda X_i^\dagger$ as $(\mathcal{Y}_i \mathcal{Y}_i^\dagger)(\mathcal{X}_i \Lambda) X_i^\dagger$, $\mathcal{Y}_i \mathcal{Y}_i^\dagger C_{i,i}$ as $(\mathcal{Y}_i \mathcal{Y}_i^\dagger) C_{i,i}$, $C_{i,i} X_i X_i^\dagger$ as $C_{i,i}(X_i X_i^\dagger)$, and $\mathcal{Y}_i \mathcal{Y}_i^\dagger C_{i,i} X_i X_i^\dagger$ as $(\mathcal{Y}_i \mathcal{Y}_i^\dagger C_{i,i})(X_i X_i^\dagger)$, then the operations required is $\mathcal{O}(\sum_{i=1}^p (\nu \mathbf{r}_i + m \mathbf{s}_i + \mathbf{r}_i \mathbf{s}_i)(\mathbf{r}_i + \mathbf{s}_i))$. For Step 7, the cost is $\mathcal{O}(\sum_{i=p+1}^{p+2l} (\nu \mathbf{r}_i + m \mathbf{s}_i + \mathbf{r}_i \mathbf{s}_i)(\mathbf{r}_i + \mathbf{s}_i))$ by using the same strategy with Step 6. Step 8 obviously requires no operations. Finally, since U and V are dense, the matrix \hat{A} can be computed as $U(1 : n, 1 : \mathbf{r}) \text{diag}(\hat{A}_{1,1}, \hat{A}_{2,2}, \hat{A}_{3,3}, \dots, \hat{A}_{p,p}, \tilde{A}_{p+1,p+2}, \dots, \tilde{A}_{p+2l-1,p+2l}) V(1 : n, 1 : \mathbf{s})^*$. Then it requires $\mathcal{O}(n^2 \mathbf{s} + n \mathbf{r} \mathbf{s})$ operations. Therefore, the total complexity of the algorithm is $\mathcal{O}(n^3 + n^2(m + \nu + \mathbf{s}) + n(\nu + m + \mathbf{r} \mathbf{s}) + m^3 + \nu^3 + (m^2 + \nu^2 + m\nu)(\mathbf{r} + \mathbf{s}) + \sum_{i=1}^{p+2l} (\nu \mathbf{r}_i + m \mathbf{s}_i + \mathbf{r}_i \mathbf{s}_i)(\mathbf{r}_i + \mathbf{s}_i))$. In practice, $m \ll n$ and $\nu \ll n$.

Finally, we give a stability analysis for Algorithm 4.1, that is, we study how the solution of Theorem 3.3 is affected by a small perturbation of C . Then the corresponding result is given as follows.

THEOREM 4.2. *Given $C^{(i)} \in \mathbb{C}^{n \times n}$, $i = 1, 2$. Let*

$$\hat{A}^{(i)} = \arg \min_{A \in \mathcal{S}} \|A - C^{(i)}\|$$

for $i = 1, 2$. Then there exists a constant α independent of $C^{(i)}$, $i = 1, 2$ such that

$$\|\hat{A}^{(2)} - \hat{A}^{(1)}\| \leq \alpha \|C^{(2)} - C^{(1)}\|.$$

Proof. It obvious to know that $\hat{A}^{(i)}$ are, respectively, the solutions of Theorem 3.3 corresponding to $C^{(i)}$ for $i = 1, 2$. Then

$$\hat{A}^{(i)} = A_0 + U \text{diag}(\mathcal{R}_{\mathcal{Y}_1} C_{1,1}^{(i)} \mathcal{R}_{X_1}, \dots, \mathcal{R}_{\mathcal{Y}_p} C_{p,p}^{(i)} \mathcal{R}_{X_p}, \Phi_1^{(i)}, \dots, \Phi_l^{(i)}, \mathbf{0}) V^*, \quad i = 1, 2,$$

in which

$$\Phi_s^{(i)} = \begin{pmatrix} \mathbf{0} & \mathcal{R}_{\mathcal{Y}_{p+2s-1}} C_{p+2s-1,p+2s}^{(i)} \mathcal{R}_{X_{p+2s}} \\ \mathcal{R}_{\mathcal{Y}_{p+2s}} C_{p+2s,p+2s-1}^{(i)} \mathcal{R}_{X_{p+2s-1}} & \mathbf{0} \end{pmatrix}$$

for all $s = 1, \dots, l$, where A_0 is given by Corollary 3.4 and $C_{j,j}^{(i)}$, $C_{p+2s-1,p+2s}^{(i)}$, $C_{p+2s,p+2s-1}^{(i)}$, $j = 1, \dots, p$, $s = 1, \dots, l$ are sub-blocks of $U^* C^{(i)} V$ as described in (3.1). Note that $C_{p+2l+1,p+2l+1}^{(i)}$ are also sub-blocks of $U^* C^{(i)} V$ for all $i = 1, 2$. Let

$$\alpha = \|\text{diag}(\mathcal{R}_{\mathcal{Y}_1}, \dots, \mathcal{R}_{\mathcal{Y}_{p+2l-1}}, \mathcal{R}_{\mathcal{Y}_{p+2l}})\|_2 \cdot \|\text{diag}(\mathcal{R}_{X_1}, \dots, \mathcal{R}_{X_{p+2l-1}}, \mathcal{R}_{X_{p+2l}})\|_2.$$

Then it follows that

$$\begin{aligned}
 & \|\widehat{A}^{(2)} - \widehat{A}^{(1)}\| \\
 = & \left\| U \operatorname{diag} \left(\mathcal{R}_{Y_1} (C_{1,1}^{(2)} - C_{1,1}^{(1)}) \mathcal{R}_{X_1}, \dots, \mathcal{R}_{Y_p} (C_{p,p}^{(2)} - C_{p,p}^{(1)}) \mathcal{R}_{X_p}, \right. \right. \\
 & \left. \left. \Phi_1^{(2)} - \Phi_1^{(1)}, \dots, \Phi_l^{(2)} - \Phi_l^{(1)}, \mathbf{0} \right) V^* \right\| \\
 = & \left\| \operatorname{diag} \left(\mathcal{R}_{Y_1}, \dots, \mathcal{R}_{Y_p}, \mathcal{R}_{Y_{p+1}}, \mathcal{R}_{Y_{p+2}}, \dots, \mathcal{R}_{Y_{p+2l-1}}, \mathcal{R}_{Y_{p+2l}}, \mathbf{0} \right) \cdot \operatorname{diag} \left(C_{1,1}^{(2)} \right. \right. \\
 & \left. \left. - C_{1,1}^{(1)}, \dots, C_{p,p}^{(2)} - C_{p,p}^{(1)}, \Sigma_1^{(2)} - \Sigma_1^{(1)}, \dots, \Sigma_l^{(2)} - \Sigma_l^{(1)}, C_{p+2l+1, p+2l+1}^{(2)} - \right. \right. \\
 & \left. \left. C_{p+2l+1, p+2l+1}^{(1)}, \dots, \mathcal{R}_{X_1}, \dots, \mathcal{R}_{X_p}, \mathcal{R}_{X_{p+1}}, \mathcal{R}_{X_{p+2}}, \dots, \mathcal{R}_{X_{p+2l-1}}, \right. \right. \\
 & \left. \left. \mathcal{R}_{X_{p+2l}}, \mathbf{0} \right) \right\| \\
 \leq & \alpha \|U^*(C^{(2)} - C^{(1)})V\| = \alpha \|C^{(2)} - C^{(1)}\|,
 \end{aligned}$$

where

$$\Sigma_s^{(i)} = \begin{pmatrix} \mathbf{0} & C_{p+2s-1, p+2s}^{(i)} \\ C_{p+2s, p+2s-1}^{(i)} & \mathbf{0} \end{pmatrix}$$

for all $s = 1, \dots, l$. \square

5. Numerical examples. In this section, we will use MATLAB R2013a which has a machine precision of around 10^{-15} to verify the effectiveness of Algorithm 4.1.

EXAMPLE 5.1. Let X, Y, Λ, Δ, C and two 8×8 normal $\{9\}$ -potent matrices P and Q be given as follows:

$$X = \begin{pmatrix} 0.5133 & 0.5133 & 0.1988 & 0.3586 \\ 0.1703 + 0.1724i & 0.1703 - 0.1724i & -0.1948 & -0.2946 \\ 0.1932 + 0.1608i & 0.1932 - 0.1608i & -0.1684 & -0.2509 \\ -0.1463 - 0.1454i & -0.1463 + 0.1454i & -0.7620 & -0.5407 \\ -0.0265 - 0.0033i & -0.0265 + 0.0033i & 0.4989 & 0.6531 \\ -0.3695 + 0.1691i & -0.3695 - 0.1691i & 0.1054 & -0.0001 \\ -0.3466 + 0.1575i & -0.3466 - 0.1575i & 0.1317 & 0.0436 \\ 0.0121 - 0.5111i & 0.0121 + 0.5111i & 0.1904 & 0.0309 \end{pmatrix},$$

$$Y = \begin{pmatrix} -0.0391 & 0.0100 & -0.0669 \\ -0.0391 & 0.0100 & 0.0176 \\ -0.0391 & 0.0100 & 0.0176 \\ 0.2970 & -0.0342 & -0.0106 \\ 0.2076 & -0.0225 & -0.0031 \\ -0.7350 & 0.7040 & -0.7194 \\ 0.5368 & -0.7082 & 0.6908 \\ -0.1890 & 0.0310 & -0.0106 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} 1.9474 + 0.4464i & 0 & 0 & 0 \\ 0 & 1.9474 - 0.4464i & 0 & 0 \\ 0 & 0 & -0.8724 & 0 \\ 0 & 0 & 0 & -0.1589 \end{pmatrix},$$

$$\Delta = \begin{pmatrix} -0.8724 & 0 & 0 \\ 0 & -0.1589 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} 0.2012 + 0.3780i & -0.1524 + 0.0244i & -0.1524 - 0.2256i & 0.2500 + 0.1768i \\ -0.1524 + 0.0244i & 0.2012 + 0.3780i & -0.1524 - 0.2256i & 0.2500 + 0.1768i \\ -0.1524 - 0.2256i & -0.1524 - 0.2256i & 0.2012 + 0.6280i & 0.2500 + 0.1768i \\ 0.2500 + 0.1768i & 0.2500 + 0.1768i & 0.2500 + 0.1768i & -0.6036 - 0.1768i \\ 0.4369 + 0.0934i & 0.0833 - 0.2601i & 0.0833 - 0.0101i & 0.2500 - 0.1768i \\ 0.0833 - 0.2601i & 0.4369 + 0.0934i & 0.0833 - 0.0101i & 0.2500 - 0.1768i \\ 0.0833 - 0.0101i & 0.0833 - 0.0101i & 0.4369 - 0.1566i & 0.2500 - 0.1768i \\ 0.2500 - 0.1768i & 0.2500 - 0.1768i & 0.2500 - 0.1768i & 0.1036 + 0.1768i \\ 0.4369 + 0.0934i & 0.0833 - 0.2601i & 0.0833 - 0.0101i & 0.2500 - 0.1768i \\ 0.0833 - 0.2601i & 0.4369 + 0.0934i & 0.0833 - 0.0101i & 0.2500 - 0.1768i \\ 0.0833 - 0.0101i & 0.0833 - 0.0101i & 0.4369 - 0.1566i & 0.2500 - 0.1768i \\ 0.2500 - 0.1768i & 0.2500 - 0.1768i & 0.2500 - 0.1768i & 0.1036 + 0.1768i \\ 0.2012 + 0.3780i & -0.1524 + 0.0244i & -0.1524 - 0.2256i & 0.2500 + 0.1768i \\ -0.1524 + 0.0244i & 0.2012 + 0.3780i & -0.1524 - 0.2256i & 0.2500 + 0.1768i \\ -0.1524 - 0.2256i & -0.1524 - 0.2256i & 0.2012 + 0.6280i & 0.2500 + 0.1768i \\ 0.2500 + 0.1768i & 0.2500 + 0.1768i & 0.2500 + 0.1768i & -0.6036 - 0.1768i \end{pmatrix},$$

$$Q = \begin{pmatrix} 0.0061 - 0.0875i & 0.0061 - 0.0875i & 0.0061 - 0.0875i & 0.2418 + 0.1482i \\ 0.0061 - 0.0875i & 0.0061 - 0.0875i & 0.0061 - 0.0875i & 0.2418 + 0.1482i \\ 0.0061 - 0.0875i & 0.0061 - 0.0875i & 0.0061 - 0.0875i & 0.2418 + 0.1482i \\ 0.2418 + 0.1482i & 0.2418 + 0.1482i & 0.2418 + 0.1482i & -0.4653 - 0.5590i \\ 0.2418 + 0.1482i & 0.2418 + 0.1482i & 0.2418 + 0.1482i & 0.2418 + 0.1482i \\ 0.2418 - 0.1347i & 0.2418 - 0.1347i & 0.2418 - 0.1347i & 0.2418 - 0.1347i \\ 0.0061 + 0.1010i & 0.0061 + 0.1010i & 0.0061 + 0.1010i & 0.0061 + 0.1010i \\ 0.2500 & 0.2500 & 0.2500 & 0.2500 \\ 0.2418 + 0.1482i & 0.2418 - 0.1347i & 0.0061 + 0.1010i & 0.2500 \\ 0.2418 + 0.1482i & 0.2418 - 0.1347i & 0.0061 + 0.1010i & 0.2500 \\ 0.2418 + 0.1482i & 0.2418 - 0.1347i & 0.0061 + 0.1010i & 0.2500 \\ 0.2418 + 0.1482i & 0.2418 - 0.1347i & 0.0061 + 0.1010i & 0.2500 \\ -0.4653 - 0.5590i & 0.2418 - 0.1347i & 0.0061 + 0.1010i & 0.2500 \\ 0.2418 - 0.1347i & -0.4653 + 0.5724i & 0.0061 + 0.1010i & 0.2500 \\ 0.0061 + 0.1010i & 0.0061 + 0.1010i & 0.7132 - 0.6061i & 0.2500 \\ 0.2500 & 0.2500 & 0.2500 & -0.7500 \end{pmatrix}$$

and $C = \text{magic}(8)$, where $\text{magic}(8)$ is the magic matrix of order 8.

Through computing by MATLAB, the spectra of the matrices P and Q are

$$\sigma(P) = (1, -1, 0.7071 + 0.7071i, 0.7071 + 0.7071i, -0.7071 + 0.7071i, \\ -0.7071 - 0.7071i, i, 0)$$

and

$$\sigma(Q) = (1, -1, 0.7071 - 0.7071i, -0.7071 + 0.7071i, -0.7071 - 0.7071i, \\ -0.7071 - 0.7071i, 0, 0),$$

respectively. Then we have

$$r_1 = r_2 = r_4 = r_5 = r_6 = r_7 = 1, r_3 = 2,$$

and

$$s_1 = s_2 = s_3 = s_4 = 1, s_5 = s_6 = 2.$$

By Algorithm 4.1, we can easily calculate the unitary matrices U and V . Then partition U^*X , Y^*U , V^*X , Y^*V according to (2.6)-(2.9) respectively. We can find that the conditions in (2.10)-(2.12) and (2.20) hold. Therefore, by (3.1) we can partition the block matrix $U^*CV = (C_{i,j})_{7 \times 6}$. Then it produces that $\hat{A}_{1,1} = (0)$, $\hat{A}_{2,2} = (2)$, $\hat{A}_{3,3} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\hat{A}_{4,5} = (1, 2)$ and $\hat{A}_{5,4} = (2)$. Finally, we obtain that \hat{A} is the following

$$\begin{pmatrix} 0.6111 & 0.6111 & 0.6111 & -0.2054 & 0.0117 & -0.1193 & -1.1385 & -0.3819 \\ 0.3025 & 0.3025 & 0.3025 & -0.5140 & -0.2969 & -0.4279 & 0.7131 & -0.3819 \\ 0.3232 & 0.3232 & 0.3232 & -0.4933 & -0.2762 & -0.4072 & 0.5891 & -0.3819 \\ -0.1041 & -0.1041 & -0.1041 & -0.9206 & -0.7035 & 0.9544 & -0.1637 & 1.1456 \\ -0.1043 & -0.1043 & -0.1043 & 0.7122 & 0.4951 & 0.6261 & -1.1385 & -0.3819 \\ -0.4129 & -0.4129 & -0.4129 & 0.4036 & 0.1865 & 0.3175 & 0.7131 & -0.3819 \\ -0.3922 & -0.3922 & -0.3922 & 0.4242 & 0.2072 & 0.3381 & 0.5891 & -0.3819 \\ -0.2232 & -0.2232 & -0.2232 & 0.5933 & 0.3762 & -1.2817 & -0.1637 & 1.1456 \end{pmatrix}.$$

Thus, we have $\|\hat{A} - P\hat{A}Q\| = 1.1815 \times 10^{-15}$, it indicates that $\hat{A} \in \mathcal{GR}^{8 \times 8}(P, Q)$. Besides, $\|\hat{A}X - X\Lambda\| = 5.5032 \times 10^{-15}$ and $\|Y^*\hat{A} - \Delta Y^*\| = 3.7114 \times 10^{-15}$. It follows that \hat{A} is the solution of the system of matrix equations $AX = X\Lambda$, $Y^*A = \Delta Y^*$.

EXAMPLE 5.2. In Example 5.1, let the given matrix $C = \mathbf{0}$. By Corollary 3.4, we have $A_{1,1}^0 = (0)$, $A_{2,2}^0 = (2)$, $A_{3,3}^0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $A_{4,5}^0 = (1, 2)$ and $A_{5,4}^0 = (2)$. Then the unique solution A_0 of the matrix equations $AX = X\Lambda$, $Y^*A = \Delta Y^*$ with the least norm is exactly equal to \hat{A} . Now, we perturb A_0 to obtain a matrix

$C(\epsilon) = A_0 + \epsilon \cdot \text{hilb}(8) \notin \mathcal{GR}^{8 \times 8}(P, Q)$, where $\text{hilb}(8)$ is the Hilbert matrix of order 8. In Example 5.1, we have verified that the necessary and sufficient conditions in (2.10)-(2.12) and (2.20) are satisfied. By using Algorithm 4.1, we obtain the solution $\hat{A}(\epsilon)$ of the optimal approximation problem corresponding to $C(\epsilon)$. In Figure 5.1, we plot two quantities $\lg \|\hat{A}(\epsilon) - A_0\|$ and $\lg \|\hat{A}(\epsilon) - C(\epsilon)\|$ for ϵ from 10^{-10} to 10^{10} . We can see from Figure 5.1 that $\hat{A}(\epsilon)$ approximates to $C(\epsilon)$ as ϵ tends to zero. However, for any $\epsilon \leq 1$, $\hat{A}(\epsilon) = A_0$ almost up to the machine precision.

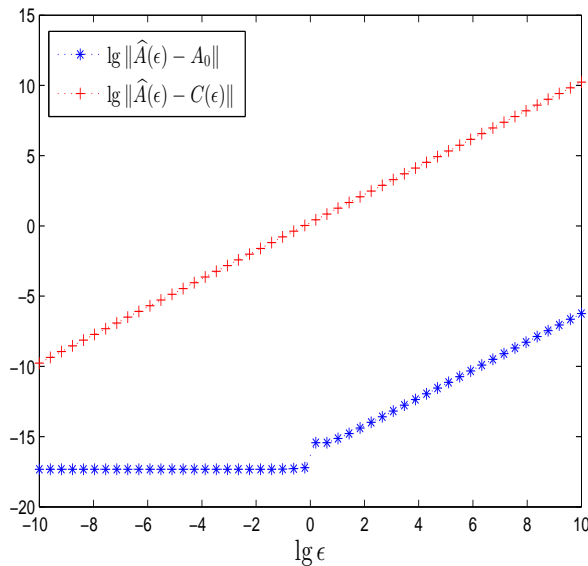


FIG. 5.1. $\lg \|\hat{A}(\epsilon) - A_0\|$ and $\lg \|\hat{A}(\epsilon) - C(\epsilon)\|$ versus $\lg \epsilon$.

Meanwhile, we also plot three quantities $\lg \|\hat{A}(\epsilon)X - X\Lambda\|$, $\lg \|Y^* \hat{A}(\epsilon) - \Delta Y^*\|$ and $\lg \|P \hat{A}(\epsilon)Q - \hat{A}(\epsilon)\|$ for ϵ from 10^{-10} to 10^{10} in Figure 5.2. Seen from Figure 5.2, we know that $\hat{A}(\epsilon)X = X\Lambda$, $Y^* \hat{A}(\epsilon) = \Delta Y^*$ and $P \hat{A}(\epsilon)Q = \hat{A}(\epsilon)$ for any $\epsilon \leq 10$. It implies that the system of the matrix equations $AX = X\Lambda$, $Y^*A = \Delta Y^*$ is consistent in the set $\mathcal{GR}^{8 \times 8}(P, Q)$.

Therefore, the above examples clearly verifies that Algorithm 4.1 is effective and feasible to solve the optimal approximation problem. We also note that as the give matrix C approaches to a solution A of the left and right inverse eigenproblem, C becomes closer to the unique solution \hat{A} of the optimal approximation problem.

6. Conclusions. In view of [10], we have solved the left and right inverse eigenproblem for the generalized reflexive matrices with normal $\{k + 1\}$ -potent matrices P and Q . It allows the singularities of the matrices P and Q . When the matrices

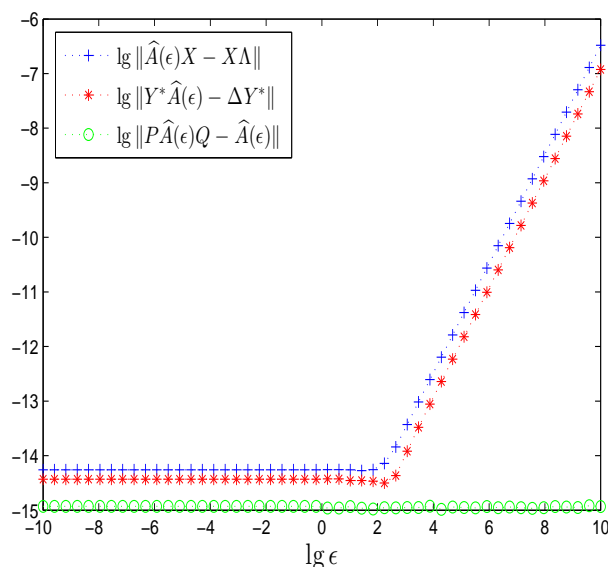


FIG. 5.2. $\lg \|\hat{A}(\epsilon)X - X\Lambda\|$, $\lg \|Y^*\hat{A}(\epsilon) - \Delta Y^*\|$ and $\lg \|P\hat{A}(\epsilon)Q - \hat{A}(\epsilon)\|$ versus $\lg \epsilon$.

P and Q are nonsingular and $P^* = P, Q^* = Q, k = 2$, our results are the same as that in [20] and can extend the previous results in [19]. Similarly, we can also solve the left and right inverse eigenproblem for the generalized anti-reflexive matrices A with two normal $\{k + 1\}$ -potencies such that $PAQ = -A$, where P and Q are the normal $\{k + 1\}$ -potent matrices. The left and right inverse eigenproblem in our paper is the inverse eigenproblem with two equalities constraint. However, in [2, 10, 13, 21–23, 37, 38, 42, 43] the authors considered the inverse eigenproblems with one equality constraint. Therefore, our results can generalize their results to some extent. Additionally, a numerical algorithm and two illustrated examples have both been presented to verify the associated optimal approximation problem effectively.

REFERENCES

- [1] M. Arav, D. Hershkowitz, V. Mehrmann, and H. Schneider. The recursive inverse eigenvalue problem. *SIAM J. Matrix Anal. Appl.*, 22:392–412, 2000.
- [2] Z.-J. Bai and R.H. Chan. Inverse eigenproblem for centrosymmetric and centroskew matrices and their approximation. *Theoret. Comput. Sci.*, 315:309–318, 2004.
- [3] J.K. Baksalary and R. Kala. The matrix equation $AXB + CYD = E$. *Linear Algebra Appl.*, 30:141–147, 1980.
- [4] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses: Theory and Applications*. Springer, New York, 2003.

- [5] H.-C. Chen. Generalized reflexive matrices: Special properties and applications. *SIAM J. Matrix Anal. Appl.*, 19:140–153, 1998.
- [6] E.W. Cheney. *Introduction to Approximation Theory*. McGraw-Hill, New York, 1966.
- [7] M.T. Chu. Inverse eigenvalue problem. *SIAM Rev.*, 40:1–39, 1998.
- [8] M.T. Chu and G.H. Golub. *Inverse Eigenvalue Problems: Theory, Algorithms, and Application*. Oxford Science Publications, Oxford University Press, 2005.
- [9] S.J. Cox, M. Embree, and J.M. Hokanson. One can hear the composition of a string: Experiments with an inverse eigenvalue problem. *SIAM Rev.*, 54:157–178, 2012.
- [10] S. Gigola, L. Lebtahi, and N. Thome. The inverse eigenvalue problem for a Hermitian reflexive matrix and the optimization problem. *J. Comput. Appl. Math.*, 291:449–457, 2015.
- [11] S. Gigola, L. Lebtahi, and N. Thome. Inverse eigenvalue problem for normal J -hamiltonian matrices. *Appl. Math. Lett.*, 48:36–40, 2015.
- [12] A. Herrero and N. Thome. Using the GSVD and the lifting technique to find $\{P, k + 1\}$ reflexive and anti-reflexive solutions of $AXB = C$. *Appl. Math. Lett.*, 24:1130–1141, 2011.
- [13] G.-X. Huang and F. Yin. An inverse eigenproblem and an associated approximation problem for generalized reflexive and anti-reflexive matrices. *J. Comput. Appl. Math.*, 235:2888–2895, 2011.
- [14] S.-L. James Hu and H. Li. A systematic linear space approach to solving partially described inverse eigenvalue problems. *Inverse Probl.*, 24:035014–035026, 2008.
- [15] L. Lebtahi, O. Romero, and N. Thome. Characterizations of $\{K, s + 1\}$ -potent matrices and applications. *Linear Algebra Appl.*, 436:293–306, 2012.
- [16] L. Lebtahi, O. Romero, and N. Thome. Relations between $\{K, s + 1\}$ -potent matrices and different classes of complex matrices. *Linear Algebra Appl.*, 438:1517–1531, 2013.
- [17] L. Lebtahi, O. Romero, and N. Thome. Algorithms for $\{K, s + 1\}$ -potent matrix constructions. *J. Comput. Appl. Math.*, 249:157–162, 2013.
- [18] F. Li, X. Hu, and L. Zhang. The generalized reflexive solution for a class of matrix equations ($AX = B, XC = D$). *Acta Math. Sci. (Engl. Ser.)*, 28:185–193, 2008.
- [19] F. Li, X. Hu, and L. Zhang. Left and right inverse eigenpairs problem of generalized centrosymmetric matrices and its optimal approximation problem. *Appl. Math. Comput.*, 212:481–487, 2009.
- [20] M.-L. Liang and L.-F. Dai. The left and right inverse eigenvalue problems of generalized reflexive and anti-reflexive matrices. *J. Comput. Appl. Math.*, 234:743–749, 2010.
- [21] Z. Liu and H. Faßbender. An inverse eigenvalue problem and associated approximation problem for generalized K -centrohermitian matrices. *J. Comput. Appl. Math.*, 206:578–585, 2007.
- [22] Z. Liu, L. Lu, and K. Wang. The inverse eigenvalue problem of generalized reflexive matrices and its approximation. *Appl. Math. Comput.*, 218:1611–1616, 2011.
- [23] Z.-Y. Peng. The inverse eigenvalue problem for Hermitian anti-reflexive matrices and its approximation. *Appl. Math. Comput.*, 162:1377–1389, 2005.
- [24] W.F. Trench. Characterization and properties of matrices with k -involutory symmetries. *Linear Algebra Appl.*, 429:2278–2290, 2008.
- [25] W.F. Trench. Characterization and properties of matrices with k -involutory symmetries II. *Linear Algebra Appl.*, 432:2782–2797, 2010.
- [26] Q.W. Wang. A system of four matrix equations over von Neumann regular rings and its applications. *Acta Math. Sin. (Engl. Ser.)*, 21:323–334, 2005.
- [27] Q.W. Wang. Bisymmetric and centrosymmetric solutions to systems of real quaternion matrix equations. *Comput. Math. Appl.*, 49:641–650, 2005.
- [28] Q.W. Wang. The general solution to a system of real quaternion matrix equations. *Comput. Math. Appl.*, 49:665–675, 2005.
- [29] Q.W. Wang, H.X. Chang, and C.Y. Lin. P -(skew) symmetric common solutions to a pair of quaternions matrix equations. *Appl. Math. Comput.*, 195:721–732, 2008.

- [30] Q.W. Wang and C.K. Li. Ranks and the least-norm of the general solution to a system of quaternion matrix equations. *Linear Algebra Appl.*, 430:1626–1640, 2009.
- [31] Q.W. Wang, G.J. Song, and X. Liu. Maximal and minimal ranks of the common solution of some linear matrix equations over an arbitrary division ring with applications. *Algebra Colloq.*, 16:293–308, 2009.
- [32] Q.W. Wang, J.W. van der Woude, and H.X. Chang. A system of real quaternion matrix equations with applications. *Linear Algebra Appl.*, 431:2291–2303, 2009.
- [33] Q.W. Wang and Z.C. Wu. Common Hermitian solutions to some operator equations on Hilbert C^* -modules. *Linear Algebra Appl.*, 432:3159–3171, 2010.
- [34] Q.W. Wang, Y. Zhou, and Q. Zhang. Ranks of the common solution to six quaternion matrix equations. *Acta Math. Appl. Sin. (Engl. Ser.)*, 27:443–462, 2011.
- [35] J.H. Wilkinson. *The Algebraic Eigenvalue Problem*. Oxford University Press, Oxford, 1965.
- [36] D. Xie. Least-squares solution for inverse eigenpair problem of nonnegative definite matrices. *Comput. Math. Appl.*, 40:1241–1251, 2000.
- [37] D. Xie, X. Hu, and L. Zhang. The solvability conditions for inverse eigenproblem of symmetric and anti-persymmetric matrices and its approximation. *Numer. Linear Algebra Appl.*, 10:223–234, 2003.
- [38] D. Xie and Y. Sheng. Inverse eigenproblem of anti-symmetric and persymmetric matrices and its approximation. *Inverse Probl.*, 19:217–225, 2003.
- [39] W.-R. Xu and G.-L. Chen. Submatrix constrained inverse eigenvalue problem involving generalised centrohermitian matrices in vibrating structural model correction. *East Asian J. Appl. Math.*, 6:42–59, 2016.
- [40] F. Yin and G.-X. Huang. Left and right inverse eigenvalue problem of (R, S) -symmetric matrices and its optimal approximation problem. *Appl. Math. Comput.*, 219:9261–9269, 2013.
- [41] L. Zhang. A class of inverse eigenvalue problems for symmetric matrices. *Numer. Math. J. Chin. Univ.*, 12:65–71, 1990.
- [42] F.-Z. Zhou, X.-Y. Hu, and L. Zhang. The solvability conditions for the inverse eigenvalue problems of centro-symmetric matrices. *Linear Algebra Appl.*, 364:147–160, 2003.
- [43] F.-Z. Zhou. The solvability conditions for the inverse eigenvalue problems of reflexive matrices. *J. Comput. Appl. Math.*, 188:180–189, 2006.