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A COUNTEREXAMPLE TO A QUESTION OF BAPAT AND SUNDER*

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Abstract. A counterexample to a question of Bapat and Sunder is presented.

Key words. Permanent, Hadamard product, Oppenheim's inequality.

AMS subject classifications. 15A15.

1. Introduction. In [1], Bapat and Sunder raise the question of whether the inequality

$$\operatorname{per}(A \circ B) \leq \operatorname{per}(A) \prod_{j=1}^{n} b_{jj} \tag{1.1}$$

holds for positive semidefinite $n \times n$ matrices A and B. The quantity $\operatorname{per}(A)$ denotes the permanent of A and the notation $A \circ B$ is for the Hadamard (entrywise) product of A and B. This is the permanental version of Oppenheim's inequality. It is the objective of this article to provide a counterexample. The question is related to two other questions:

- The permanent on top conjecture, recently disproved by Shchesnovich [4] which would have implied (1.1) had it been true.
- The inequality $\operatorname{per}(A \circ B) \leq \operatorname{per}(A)\operatorname{per}(B)$ introduced by Chollet [2] and established in the case $n = 3$ by Gregorac and Hentzel [3]. This inequality would be a consequence of (1.1) had it been true. Chollet's conjecture remains open. For a relatively recent discussion of Chollet's conjecture, the reader may consult Zhang [5].

2. The counterexample. With $n = 7$, we take

$$A = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} e^{\frac{4}{5} i\pi} & \frac{1}{\sqrt{2}} e^{\frac{2}{5} i\pi} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} e^{-\frac{2}{5} i\pi} & \frac{1}{\sqrt{2}} e^{-\frac{4}{5} i\pi} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} e^{-\frac{4}{5} i\pi} & 1 & \cos\left(\frac{1}{5}\pi\right) e^{-\frac{1}{5} i\pi} & \cos\left(\frac{2}{5}\pi\right) e^{-\frac{2}{5} i\pi} & \cos\left(\frac{2}{5}\pi\right) e^{\frac{2}{5} i\pi} & \cos\left(\frac{1}{5}\pi\right) e^{\frac{1}{5} i\pi} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} e^{-\frac{2}{5} i\pi} & \cos\left(\frac{1}{5}\pi\right) e^{\frac{1}{5} i\pi} & 1 & \cos\left(\frac{1}{5}\pi\right) e^{-\frac{1}{5} i\pi} & \cos\left(\frac{2}{5}\pi\right) e^{-\frac{2}{5} i\pi} & \cos\left(\frac{2}{5}\pi\right) e^{\frac{2}{5} i\pi} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cos\left(\frac{2}{5}\pi\right) e^{\frac{2}{5} i\pi} & \cos\left(\frac{1}{5}\pi\right) e^{\frac{1}{5} i\pi} & 1 & \cos\left(\frac{1}{5}\pi\right) e^{-\frac{1}{5} i\pi} & \cos\left(\frac{2}{5}\pi\right) e^{-\frac{2}{5} i\pi} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} e^{\frac{2}{5} i\pi} & \cos\left(\frac{2}{5}\pi\right) e^{-\frac{2}{5} i\pi} & \cos\left(\frac{2}{5}\pi\right) e^{\frac{2}{5} i\pi} & \cos\left(\frac{1}{5}\pi\right) e^{\frac{1}{5} i\pi} & 1 & \cos\left(\frac{1}{5}\pi\right) e^{-\frac{1}{5} i\pi} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} e^{\frac{4}{5} i\pi} & \cos\left(\frac{1}{5}\pi\right) e^{-\frac{1}{5} i\pi} & \cos\left(\frac{2}{5}\pi\right) e^{-\frac{2}{5} i\pi} & \cos\left(\frac{2}{5}\pi\right) e^{\frac{2}{5} i\pi} & \cos\left(\frac{1}{5}\pi\right) e^{\frac{1}{5} i\pi} & 1 \end{pmatrix}$$

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and $B = A^T$. Then it is easy to check that A and B are hermitian positive semidefinite matrices of rank two (with eigenvalue $\frac{7}{2}$ of multiplicity two) and that

$$\operatorname{per}(A \circ B) = \frac{6185}{128},$$

that $\prod_{j=1}^{7} b_{jj} = 1$ and that $\operatorname{per}(A) = 45$. We find that

$$\frac{\operatorname{per}(A \circ B)}{\operatorname{per}(A) \prod_{j=1}^{7} b_{jj}} = \frac{1237}{1152} > 1.$$

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