Electronic Journal of Linear Algebra

Volume 31 *Volume 31: (2016)* Article 6

2016

A counterexample to a question of Bapat and Sunder

Stephen W. Drury
Department of Mathematics & Statistics, McGill University, Montreal, drury@math.mcgill.ca

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 69-70, February 2016

ELA
http:/repository.uwyo.edu/ela

A COUNTEREXAMPLE TO A QUESTION OF BAPAT AND SUNDER*

STEPHEN W. DRURY†

Abstract. A counterexample to a question of Bapat and Sunder is presented.

Key words. Permanent, Hadamard product, Oppenheim's inequality.

AMS subject classifications. 15A15.

1. Introduction. In [1], Bapat and Sunder raise the question of whether the inequality

$$\operatorname{per}(A \circ B) \leq \operatorname{per}(A) \prod_{j=1}^{n} b_{jj} \tag{1.1}$$

holds for positive semidefinite $n \times n$ matrices A and B. The quantity $\operatorname{per}(A)$ denotes the permanent of A and the notation $A \circ B$ is for the Hadamard (entrywise) product of A and B. This is the permanental version of Oppenheim's inequality. It is the objective of this article to provide a counterexample. The question is related to two other questions:

- The permanent on top conjecture, recently disproved by Shchesnovich [4] which would have implied (1.1) had it been true.
- The inequality $\operatorname{per}(A \circ B) \leq \operatorname{per}(A)\operatorname{per}(B)$ introduced by Chollet [2] and established in the case $n = 3$ by Gregorac and Hentzel [3]. This inequality would be a consequence of (1.1) had it been true. Chollet's conjecture remains open. For a relatively recent discussion of Chollet's conjecture, the reader may consult Zhang [5].

2. The counterexample. With $n = 7$, we take

$$A=\begin{pmatrix}
1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & 1 & \frac{1}{\sqrt{2}}e^{\frac{4}{5}i\pi} & \frac{1}{\sqrt{2}}e^{\frac{2}{5}i\pi} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}e^{-\frac{2}{5}i\pi} & \frac{1}{\sqrt{2}}e^{-\frac{4}{5}i\pi} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}e^{-\frac{4}{5}i\pi} & 1 & \cos\left(\frac{1}{5}\pi\right)e^{-\frac{1}{5}i\pi} & \cos\left(\frac{2}{5}\pi\right)e^{-\frac{2}{5}i\pi} & \cos\left(\frac{2}{5}\pi\right)e^{\frac{2}{5}i\pi} & \cos\left(\frac{1}{5}\pi\right)e^{\frac{1}{5}i\pi} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}e^{-\frac{2}{5}i\pi} & \cos\left(\frac{1}{5}\pi\right)e^{\frac{1}{5}i\pi} & 1 & \cos\left(\frac{1}{5}\pi\right)e^{-\frac{1}{5}i\pi} & \cos\left(\frac{2}{5}\pi\right)e^{-\frac{2}{5}i\pi} & \cos\left(\frac{2}{5}\pi\right)e^{\frac{2}{5}i\pi} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cos\left(\frac{2}{5}\pi\right)e^{\frac{2}{5}i\pi} & \cos\left(\frac{1}{5}\pi\right)e^{\frac{1}{5}i\pi} & 1 & \cos\left(\frac{1}{5}\pi\right)e^{-\frac{1}{5}i\pi} & \cos\left(\frac{2}{5}\pi\right)e^{-\frac{2}{5}i\pi} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}e^{\frac{2}{5}i\pi} & \cos\left(\frac{2}{5}\pi\right)e^{-\frac{2}{5}i\pi} & \cos\left(\frac{2}{5}\pi\right)e^{\frac{2}{5}i\pi} & \cos\left(\frac{1}{5}\pi\right)e^{\frac{1}{5}i\pi} & 1 & \cos\left(\frac{1}{5}\pi\right)e^{-\frac{1}{5}i\pi} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}e^{\frac{4}{5}i\pi} & \cos\left(\frac{1}{5}\pi\right)e^{-\frac{1}{5}i\pi} & \cos\left(\frac{2}{5}\pi\right)e^{-\frac{2}{5}i\pi} & \cos\left(\frac{2}{5}\pi\right)e^{\frac{2}{5}i\pi} & \cos\left(\frac{1}{5}\pi\right)e^{\frac{1}{5}i\pi} & 1
\end{pmatrix}$$

*Received by the editors on November 30, 2015. Accepted for publication on January 26, 2016. Handling Editor: Ravi Bapat.

†Department of Mathematics and Statistics, McGill University, Montreal, Canada H3A 0B9 (drury@math.mcgill.ca).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 69-70, February 2016

ELA
http:/repository.uwyo.edu/ela

and $B = A^T$. Then it is easy to check that A and B are hermitian positive semidefinite matrices of rank two (with eigenvalue $\frac{7}{2}$ of multiplicity two) and that

$$\mathrm{per}(A \circ B) = \frac{6185}{128},$$

that $\prod_{j=1}^{7} b_{jj} = 1$ and that $\mathrm{per}(A) = 45$. We find that

$$\frac{\mathrm{per}(A \circ B)}{\mathrm{per}(A) \prod_{j=1}^{7} b_{jj}} = \frac{1237}{1152} > 1.$$

REFERENCES

[1] R.B. Bapat and V.S. Sunder. On majorization and Schur products. *Linear Algebra Appl.*, 72:107–117, 1985.
[2] J. Chollet. Is there a permanental analogue to Oppenheim's inequality? *Amer. Math. Monthly*, 84(1):57–59, 1982.
[3] R.J. Gregorac and I.R. Hentzel. A note on the analogue of Oppenheim's inequality for permanents. *Linear Algebra Appl.*, 94:109–112, 1987.
[4] V.S. Shchesnovich. The permanent-on-top conjecture is false. *Linear Algebra Appl.*, 490:196–201, 2016.
[5] F. Zhang. An analytic approach to a permanent conjecture. *Linear Algebra Appl.*, 438:1570–1579, 2013.