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SEMILINEAR PRESERVERS OF THE IMMANANTS IN THE SET OF DOUBLY STOCHASTIC MATRICES

M. ANTÓNIA DUFFNER† AND ROSÁRIO FERNANDES‡

Abstract. Let $S_n$ denote the symmetric group of degree $n$ and $M_n$ denote the set of all $n$-by-$n$ matrices over the complex field, $\mathbb{C}$. Let $\chi : S_n \rightarrow \mathbb{C}$ be an irreducible character of degree greater than 1 of $S_n$. The immanant $d_\chi : M_n \rightarrow \mathbb{C}$ associated with $\chi$ is defined by

$$d_\chi(X) = \sum_{\sigma \in S_n} \chi(\sigma) \prod_{j=1}^{n} X_{j\sigma(j)}, \quad X = [X_{jk}] \in M_n.$$  

Let $\Omega_n$ be the set of all $n$-by-$n$ doubly stochastic matrices, that is, matrices with nonnegative real entries and each row and column sum is one. We say that a map $T$ from $\Omega_n$ into $\Omega_n$ is semilinear if

- is semilinear if $T(\lambda S_1 + (1 - \lambda)S_2) = \lambda T(S_1) + (1 - \lambda)T(S_2)$ for all $S_1, S_2 \in \Omega_n$ and for all real number $\lambda$ such that $0 \leq \lambda \leq 1$;
- preserves $d_\chi$ if $d_\chi(T(S)) = d_\chi(S)$ for all $S \in \Omega_n$.

We characterize the semilinear surjective maps $T$ from $\Omega_n$ into $\Omega_n$ that preserve $d_\chi$, when the degree of $\chi$ is greater than one.

Key words. Immanants, Linear preserver problems, Doubly stochastic matrices.

AMS subject classifications. 15A69, 15A60, 15A42, 15A45, 15A04, 47B49.

1. Introduction. Let $M_n$ denote the set of all $n$-by-$n$ matrices over the complex field, $\mathbb{C}$. We denote by $I$ the identity in $M_n$. Let $S_n$ be the symmetric group of degree $n$. We denote by $id$ the identity in $S_n$. Let $\chi : S_n \rightarrow \mathbb{C}$ be an irreducible character of $S_n$ with degree greater than 1 (note that if the degree of $\chi$ is one then $\chi$ is the sign character or the principal character). The immanant $d_\chi$ is defined by

$$d_\chi(X) = \sum_{\sigma \in S_n} \chi(\sigma) \prod_{j=1}^{n} X_{j\sigma(j)}, \quad X = [X_{jk}] \in M_n.$$  

If the degree of the character $\chi$ is one, then $d_\chi$ is the determinant or the permanent. We denote the permanent by $per$,

$$per(X) = \sum_{\sigma \in S_n} \prod_{j=1}^{n} X_{j\sigma(j)}, \quad X = [X_{jk}] \in M_n.$$  

Let $\Omega_n$ denote the set of all $n$-by-$n$ doubly stochastic matrices, that is, matrices with nonnegative real entries and each row and column sum is one. $\Omega_n$ is a convex polyhedron in the euclidean $n^2$-space whose vertices are the $n$-by-$n$ permutation matrices, [2].

Definition 1.1. Let $T$ be a map from $\Omega_n$ into $\Omega_n$. We say that $T$
is a semilinear map if

\[ T(\lambda S_1 + (1 - \lambda)S_2) = \lambda T(S_1) + (1 - \lambda)T(S_2) \]

for all \( S_1, S_2 \in \Omega_n \) and for all real number \( \lambda \) such that \( 0 \leq \lambda \leq 1 \);

- preserves \( d_\chi \) if \( d_\chi(T(S)) = d_\chi(S) \) for all \( S \in \Omega_n \).

The behavior of the permanent on \( \Omega_n \) has been studied extensively. In \([9]\), the linear maps \( T \) from \( \Omega_n \) into \( \Omega_n \) which preserve the permanent are characterized, and in \([4]\), those that verify \( T(\Omega_n) = \Omega_n \). In this paper, we characterize the semilinear surjective maps \( T \) from \( \Omega_n \) into \( \Omega_n \) that preserve \( d_\chi \), where the character \( \chi \) has degree greater than one.

Let \( \alpha = (\alpha_1, \ldots, \alpha_r) \) be a partition of \( n \) of length \( r \), that is, a sequence of positive integers which are assumed to be nonincreasing and with sum equal to \( n \). \([2][3]\). Each partition \( \alpha = (\alpha_1, \ldots, \alpha_r) \) of \( n \) is related to a Young diagram, denoted by \([\alpha]\), which consists of \( r \) left justified rows of boxes, where the number of boxes in the \( i \)th row is \( \alpha_i \). The irreducible characters of \( S_n \) are in a bijective correspondence with the ordered partitions of \( n \). \([1]\). We identify the irreducible character \( \chi \) with the partition that corresponds to \( \chi \), or with the Young diagram \([\alpha]\) associated with \( \chi \).

Denote by \( P(\sigma) \) the permutation matrix associated with \( \sigma \in S_n \), that is,

\[ P(\sigma)_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j), \\ 0 & \text{otherwise.} \end{cases} \]

We denote by \( S^T \) the transpose of the matrix \( S \). Recall that \( (P(\sigma))^T = P(\sigma^{-1}) \).

The main result of this paper is the following theorem.

**Theorem 1.2.** Let \( \chi \) be an irreducible character of \( S_n \) of degree greater than one. Let \( T \) be a semilinear surjective map from \( \Omega_n \) into \( \Omega_n \). The map \( T \) preserves \( d_\chi \) if and only if there are \( \sigma, \alpha \in S_n \), with \( \chi(\sigma) = \chi(\alpha) \), such that one of the following conditions must hold:

1. \( T(S) = P(\sigma)P(\alpha)SP(\alpha^{-1}) \) for all \( S \in \Omega_n \).
2. \( T(S) = P(\sigma)P(\alpha)S^TP(\alpha^{-1}) \) for all \( S \in \Omega_n \).

Moreover, if \( \chi \neq [2, 2] \), then \( P(\sigma) = I \).

In Section 2, we shall present some preliminary definitions and propositions about the immanant of a matrix \( S \in \Omega_n \). To characterize the semilinear surjective maps \( T \) from \( \Omega_n \) into \( \Omega_n \) that preserve \( d_\chi \), we will consider several steps. So, in Section 3, we will prove that \( T \) must be injective. In Section 4, we will prove that the image by \( T \) of a permutation matrix is a permutation matrix. Finally, in Section 5, we will present the proof of the main result.

**2. Preliminares.** Let \( \chi \) be an irreducible character of \( S_n \). The boundary of the diagram \([\chi]\) is the set of boxes whose right edge, bottom edge, or bottom right vertex belong to the geometric boundary of the diagram. We will denote by \( p \) the number of boundary boxes of \([\chi]\). Note that if \( \chi \) is an irreducible character of \( S_n \) of degree greater than 1 then \( p \geq 3 \).

A set of successive boundary boxes whose deletion leads to another Young diagram is called a regular boundary part. The number of vertical steps of a regular boundary part is equal to the number of rows involved minus one.
The Murnaghan-Nakayama Rule is important to calculate the value of $\chi(\sigma)$, for $\sigma \in S_n$. For more details, see for example [1].

**Proposition 2.1.** (Murnaghan-Nakayama Rule) Let the disjoint cycles of $\sigma \in S_n$ have lengths $a_1, \ldots, a_q$ in any order. Determine all ways in which the diagram $[\chi]$ can be reduced to 0 by successively omitting regular boundary parts of lengths $a_1, \ldots, a_q$. Let the boundary parts occurring in the $s$th way contain $k_s$ vertical steps altogether. Then $\chi(\sigma) = \sum_s (-1)^{k_s}$.

In what follows, we will use this rule, namely, to state the following facts:

- If $\sigma$ is a cycle of length equal to $p$ then $\chi(\sigma) \neq 0$.
- If $\sigma$ is a cycle of length greater than $p$ then $\chi(\sigma) = 0$.
- If $\chi$ is a single hook, that is, an irreducible character $\chi = [\chi_1, \ldots, \chi_r]$ of $S_n$ such that $\chi_2 = \cdots = \chi_r = 1$, and $\sigma$ is the product of disjoint cycles of length greater than one, $\sigma_1, \ldots, \sigma_h$, with $h \geq 2$, and there is an integer $i$, such that $1 \leq i \leq h$ with the length of $\sigma_i$ greater than $\max\{\chi_1 - 1, r - 1\}$ then $\chi(\sigma) = 0$.
- If $\chi$ is a single hook and $\sigma$ is the product of two disjoint cycles of length greater than one, $\sigma_1, \sigma_2$, with the length of $\sigma_1$ equal to $\chi_1 - 1$ and the length of $\sigma_2$ equal to $r - 1$, or vice-versa, then $\chi(\sigma) \neq 0$.

In [9], M. Marcus and M. Newman proved the following result.

**Proposition 2.2.** If $S \in \Omega_n$, then

$$\text{per} S \leq 1.$$ 

Moreover, $\text{per} S = 1$ if and only if $S = P(\sigma)$, for some $\sigma \in S_n$.

If $\pi, \sigma \in S_n$, we denote by $\pi \circ \sigma$ the composition of these two permutations and we denote by $\sigma(k)$ the image of the value $k$ under the map $\sigma$. Furthermore, if $\pi \in S_n$ is a cycle, its length is denoted by $l(\pi)$.

**Remark 2.1.** Let $\chi$ be an irreducible character of $S_n$. We refer to [10, 11, 12, 13] for a general study in multilinear algebra.

1. $\chi(\sigma) \in \mathbb{Z}$ for all $\sigma \in S_n$, and

   $$\sum_{\sigma \in S_n} \chi(\sigma) = \begin{cases} 0 & \text{if } \chi \text{ is not the principal character}, \\ n! & \text{otherwise}. \end{cases}$$

2. $\chi(\sigma^{-1}) = \chi(\sigma)$ for all $\sigma \in S_n$, and $\chi(\pi \circ \sigma \circ \pi^{-1}) = \chi(\sigma)$ for all $\pi, \sigma \in S_n$.

3. $|\chi(\sigma)| \leq |\chi(id)|$ for all $\sigma \in S_n$.

4. If $n > 4$ and $\chi$ is a character of $S_n$ of degree greater than one, then $|\chi(\sigma)| < \chi(id)$ for all $\sigma \in S_n \setminus \{id\}$.

5. Using direct computation, if $\chi$ is a character of $S_n$ of degree greater than one and $\sigma \in S_n \setminus \{id\}$ verify $|\chi(\sigma)| = \chi(id)$ then $n = 4, \chi = [2, 2]$ and $\sigma \in \{(12)(34), (13)(24), (14)(23)\}$. Moreover, if $\chi = [2, 2]$ and $\sigma \in \{(12)(34), (13)(24), (14)(23)\}$, then $\chi(\pi \circ \sigma) = \chi(\pi), \forall \pi \in S_4$.

From the following proposition, we can conclude that whenever $\chi \neq [2, 2]$ and $S \in \Omega_n$, the maximum value of $d_{\chi}(S)$ is attained when $S = I$, and the minimum value is attained when $S = P(\tau)$, where $\chi(\tau) \leq \chi(\pi)$, for all $\pi \in S_n$. 


Proposition 2.3. Let \( \chi \) be an irreducible character of degree greater than 1 of \( S_n \). If \( S \in \Omega_n \) then \( d_\chi(S) \leq \chi(id) \), and the equality holds if and only if
\[
S = P(\sigma) \quad \text{and} \quad \chi(\sigma) = \chi(id).
\]

Moreover, \( d_\chi(S) \geq \chi(\tau) \), where \( \chi(\tau) \leq \chi(\pi) \) for all \( \pi \in S_n \), with equality if and only if
\[
S = P(\rho) \quad \text{and} \quad \chi(\rho) = \chi(\tau).
\]

Proof. Since
\[
|d_\chi(S)| = \left| \sum_{\sigma \in S_n} \chi(\sigma) \prod_{j=1}^n S_{j,\sigma(j)} \right| \leq \sum_{\sigma \in S_n} |\chi(\sigma)| \prod_{j=1}^n S_{j,\sigma(j)} \leq \sum_{\sigma \in S_n} \chi(id) \prod_{j=1}^n S_{j,\sigma(j)} = \chi(id) \text{per} S
\]
and since \( \text{per} S \leq 1 \), it follows that \( |d_\chi(S)| \leq \chi(id) \).

If \( \chi(id) = |d_\chi(S)| \leq \chi(id) \text{per} S \), then \( \text{per} S \geq 1 \). But as \( \text{per} S \leq 1 \), for all \( S \in \Omega_n \), then \( \text{per} S = 1 \). By Proposition 2.2 we have that \( S = P(\sigma) \) for some \( \sigma \in S_n \). By definition and hypothesis, \( \chi(id) = d_\chi(S) = d_\chi(P(\sigma)) = \chi(\sigma) \). Therefore, \( \chi(\sigma) = \chi(id) \).

Since \( \chi(\tau) < 0 \) if \( \chi(\tau) = \min \{ \chi(\sigma) : \sigma \in S_n \} \), we have that
\[
d_\chi(S) = \sum_{\sigma \in S_n} \chi(\sigma) \prod_{j=1}^n S_{j,\sigma(j)} \geq \sum_{\sigma \in S_n} \chi(\tau) \prod_{j=1}^n S_{j,\sigma(j)} = \chi(\tau) \sum_{\sigma \in S_n} \prod_{j=1}^n S_{j,\sigma(j)} = \chi(\tau) \text{per}(S) \geq \chi(\tau).
\]

Consequently, \( d_\chi(S) \geq \chi(\tau) \). If \( d_\chi(S) = \chi(\tau) \), then \( \text{per}(S) = 1 \). By Proposition 2.2, this implies that \( S = P(\sigma) \), for some \( \sigma \in S_n \). Because \( \chi(\tau) = d_\chi(S) = d_\chi(P(\sigma)) = \chi(\sigma) \) then \( \chi(\sigma) = \chi(\tau) \).

Corollary 2.4. Let \( \chi \) be an irreducible character of degree greater than 1 of \( S_n \). Let \( T \) be a map from \( \Omega_n \) into \( \Omega_n \) that preserves \( d_\chi \). If \( \chi \neq [2, 2] \), then \( T(I) = I \). Moreover, when \( \chi = [2, 2] \), there is \( \sigma \in S_n \) such that \( T(I) = P(\sigma) \) and \( \chi(\sigma) = \chi(id) \).

Proof. Since \( T(I) \in \Omega_n \) and \( d_\chi(T(I)) = d_\chi(I) = \chi(id) \), by last proposition, there is \( \sigma \in S_n \), such that \( T(I) = P(\sigma) \), with \( \chi(\sigma) = \chi(id) \). By Remark 2.1 we have that \( T(I) = I \) if \( \chi \neq [2, 2] \).

Remark 2.2.
1. Using last corollary we conclude that \( T(I) \) is invertible.
2. Using the main result of [11] (characterization of the subgroup of \( M_n \), \( S(S_n, \chi) = \{ A \in M_n ; d_\chi(AX) = d_\chi(X) \text{, for all } X \in M_n \} \)) we have that if \( \sigma \in S_n \) and \( \chi(\sigma) = \chi(id) \), then \( d_\chi(P(\sigma)S) = d_\chi(S) \) for all \( S \in \Omega_n \).

To prove the following lemmas, we will use the Murnaghan-Nakayama Rule (see the considerations at the beginning of this section and [11]).

Lemma 2.5. Let \( n \geq 4 \), and \( \chi \) be an irreducible character of \( S_n \) of degree greater than one. If \( i, j, k \in \{ 1, \ldots, n \} \), are distinct on pairs, then there are \( \sigma, \tau \in S_n \) such that
\[
\sigma(i) = j, \quad \sigma(k) = k, \quad \tau = \sigma \circ (ik), \quad \chi(\sigma) \neq 0, \quad \chi(\tau) = 0.
\]
Proof. Suppose that $\chi$ is not a single hook, and let $p$ be the number of boundary boxes of $[\chi]$. Then $p \leq n-1$. 

If $\sigma \in S_n$ is a cycle of length $p$ such that $\sigma(i) = j, \sigma(k) = k$, since $\tau = \sigma \circ (ik)$ then $\tau$ is a cycle of length $p+1$. Using the Murnaghan-Nakayama Rule we have that $\chi(\sigma) \neq 0$ and $\chi(\tau) = 0$. 

Suppose that $\chi = [\chi_1,\ldots,\chi_{v+1}]$ is a single hook, with $\chi_1 = u > 1$ and $v \geq 1$. 

If $u-1 \geq v$, since $n = u + v \geq 4$ then $u - 1 + v \geq 3$. So, (note that $v \geq 1$ because $\chi$ has degree greater than one, $n \geq 4$) $u - 1 \geq 2$. Therefore, there exist $\sigma \in S_n$, and disjoint cycles $\sigma_1, \sigma_2$, where $\sigma = \sigma_1 \circ \sigma_2$, $l(\sigma_1) = u - 1$ and $l(\sigma_2) = v$, such that $\sigma_1(i) = j, \sigma(k) = k$. Consequently, $\tau = \sigma \circ (ik) = \sigma_1 \circ \sigma_2$ with $\tau_1, \tau_2 \in S_n$ and $l(\tau_1) = u, l(\tau_2) = v$. Using the Murnaghan-Nakayama Rule we have that $\chi(\sigma) \neq 0$ and $\chi(\tau) = 0$. 

If $u-1 < v$, then, there are $\sigma \in S_n$, disjoint cycles $\sigma_1, \sigma_2$, where $\sigma = \sigma_1 \circ \sigma_2$, $l(\sigma_1) = u - 1$ and $l(\sigma_2) = v$, such that $\sigma_2(i) = j$, and $\sigma(k) = k$. Therefore, $\tau = \sigma \circ (ik) = \sigma_1 \circ \sigma_2$ with $l(\tau_2) = v + 1$. Using the Murnaghan-Nakayama Rule we have that $\chi(\sigma) \neq 0$ and $\chi(\tau) = 0$. \[\Box\]

**Lemma 2.6.** Let $n \geq 3$, $i, j, k \in \{1, \ldots, n\}$, distinct on pairs and $\sigma, \tau \in S_n$ such that $\sigma(i) = j$, $\sigma(k) = k$, $\tau = \sigma \circ (ik)$.

Then for every $\pi \in S_n$, there are $s, l \in \{1, \ldots, n\}$ and $l \neq s$ that verify

$$\sigma^{-1}(s) \neq \pi(s), \quad \tau^{-1}(s) \neq \pi(s), \quad \sigma^{-1}(l) \neq \pi(l), \quad \tau^{-1}(l) \neq \pi(l).$$

**Proof.** Suppose that there is $\pi \in S_n$ with a unique $s \in \{1, \ldots, n\}$ such that $\pi(s) = t$, $\sigma(t) \neq s$, $\tau(t) \neq s$.

Consequently,

$$\text{if } l \neq s, \text{ then } \sigma^{-1}(l) = \pi(l) \text{ or } \tau^{-1}(l) = \pi(l).$$

Let $u$ and $v$ be elements such that $\sigma^{-1}(u) = t = \pi(s)$, $\tau^{-1}(v) = t = \pi(s)$, (note that $u \neq s$, $v \neq s$).

If $u = v$ then $\pi(u) = \sigma^{-1}(u) = t$ or $\pi(u) = \tau^{-1}(v) = t$. But $\pi(s) = t$, therefore we have a contradiction, $u = s$.

Consequently, $u \neq v$. Since $\tau = \sigma \circ (ik)$ then $(t = i, u = j, v = k)$ or $(t = k, u = k, v = j)$. We only prove the case $t = i, u = j, v = k$, because the proof of the other case is analogous. In the case that we will prove, $\sigma^{-1}(j) = i = \pi(s)$, $\tau^{-1}(k) = i = \pi(s)$.

Since $s \neq j, s \neq k$ then $\pi(j) = \sigma^{-1}(j)$ or $\pi(j) = \tau^{-1}(j)$. If $\pi(j) = \sigma^{-1}(j)$ then $\pi(j) = \sigma^{-1}(j) = \pi(s)$ and we can conclude that $s = j$ (impossible). So, $\pi(j) = \tau^{-1}(j) = k$. Since $s \neq k$ then $\pi(k) = \sigma^{-1}(k)$ or $\pi(k) = \tau^{-1}(k)$. If $\pi(k) = \tau^{-1}(k)$ then $\pi(k) = \tau^{-1}(k) = \pi(s)$ and we can conclude that $s = k$ (impossible). Therefore, $\pi(k) = \sigma^{-1}(k) = k$. But this implies that $\pi(j) = \pi(k) = k$ which is impossible. \[\Box\]
3. The injectivity of $T$. Let $\chi$ be an irreducible character of $S_n$ of degree greater than 1 and $T$ be a semilinear map from $\Omega_n$ into $\Omega_n$ that preserves $d_\chi$. In the main result of this section we will prove that $T$ must be injective.

**Theorem 3.1.** Let $\chi$ be an irreducible character of $S_n$ of degree greater than 1 and $T$ be a semilinear map from $\Omega_n$ into $\Omega_n$ that preserves $d_\chi$. Then $T$ is injective.

**Proof.** Let $S, S' \in \Omega_n$ such that $T(S) = T(S')$. Let $B \in \Omega_n$ and $x \in [0,1]$. Since

$$d_\chi(xS + (1-x)B) = d_\chi(T(xS + (1-x)B)) = d_\chi(T(xS + (1-x)B))$$

it follows that $d_\chi(xS + (1-x)B) = d_\chi(xS' + (1-x)B)$.

**Case (i)** Let $n \geq 4$. If $i, j, k \in \{1, \ldots, n\}$ are distinct on pairs, then by Lemma 2.5 there are $\sigma, \tau \in S_n$ such that $\sigma(i) = j, \sigma(k) = k, \tau = \sigma \circ (ik), \chi(\sigma) \neq 0, \chi(\tau) = 0$.

For each $b \in [0,1]$, let us consider the matrix

$$B_b = bP(\sigma) + (1-b)P(\tau).$$

So, for all $p \in \{1, \ldots, n\}$,

$$(B_b)_{p \sigma(p)} = \begin{cases} 1 & \text{if } p(p) = \sigma^{-1}(p) = \tau^{-1}(p), \\ b & \text{if } p(p) = \sigma^{-1}(p), \ p(p) \neq \tau^{-1}(p), \\ 1-b & \text{if } p(p) \neq \sigma^{-1}(p), \ p(p) = \tau^{-1}(p), \\ 0 & \text{otherwise}. \end{cases}$$

Now we will compute the coefficient of the term associated with $x$ of the polynomial

$$d_\chi(xS + (1-x)B_b) = \sum_{\pi \in S_n} \chi(\pi) \prod_{l=1}^{n} (xS + (1-x)B_b)_{\pi(l)}. $$

If there is $s \in \{1, \ldots, n\}$ such that for some $\pi \in S_n$, $\pi(s) \neq \sigma^{-1}(s)$ and $\pi(s) \neq \tau^{-1}(s)$ then

$$(xS + (1-x)B_b)_{\pi(s)} = xS_{\pi(s)}.$$ 

To obtain the coefficient of the term associated with $x$ of the polynomial $\chi(\pi) \prod_{l=1}^{n} (xS + (1-x)B_b)_{\pi(l)}$ the other terms of $\prod_{l=1, l \neq s}^{n} (xS + (1-x)B_b)_{\pi(l)}$ must verify $(B_b)_{\pi(l)} \neq 0$. Consequently, if $l \neq s$ then $\pi(l) = \sigma^{-1}(l)$ or $\pi(l) = \tau^{-1}(l)$. But this is impossible by Lemma 2.6. Therefore, if $s \in \{1, \ldots, n\}$ and $\pi \in S_n$, then $\pi(s) = \sigma^{-1}(s) \text{ or } \pi(s) = \tau^{-1}(s)$. Since $\tau = \sigma \circ (ik)$ then $\pi(s) = \sigma^{-1}(s) = \tau^{-1}(s)$, when $s \in \{1, \ldots, n\} \setminus \{j, k\}$. Because $\pi(j) = \sigma^{-1}(j)$ or $\pi(j) = \tau^{-1}(j)$, and $\pi(k) = \sigma^{-1}(k)$ or $\pi(k) = \tau^{-1}(k)$ then $\pi(j) = i$ or $\pi(j) = k$, and $\pi(k) = k$ or $\pi(k) = i$. But $\pi$ is a bijection, so we have two cases:

- If $\pi(j) = i$, then $\pi(k) = k$ and $\pi = \sigma^{-1}$.
- If $\pi(j) = k$, then $\pi(k) = i$ and $\pi = \tau^{-1}$.

Therefore, the coefficient of the term associated with $x$ of the polynomial $d_\chi(xS + (1-x)B_b)$ appears when $\pi = \sigma^{-1}$ or $\pi = \tau^{-1}$.
As \(\chi(\tau^{-1}) = 0\), it is enough to compute \(\chi(\sigma^{-1}) \prod_{l=1}^{n}(xS + (1-x)B_{b})_{l\sigma^{-1}(l)}\). Since \(\sigma^{-1}(l) = l\), for all \(l \in \{1, \ldots, n\}\) and \(\tau^{-1}(l)) \neq l\) when \(l \in \{j, k\}\), then

\[
\chi(\sigma^{-1}) \prod_{l=1}^{n}(xS + (1-x)B_{b})_{l\sigma^{-1}(l)} = \chi(\sigma^{-1})(xS_{j\sigma^{-1}(j)} + (1-x)b)(xS_{k\sigma^{-1}(k)} + (1-x)b) \prod_{l \neq j, k}(xS_{l\sigma^{-1}(l)} + (1-x)).
\]

Consequently, the coefficient of the term associated with \(x\) in the polynomial \(d_{\chi}(xS + (1-x)B_{b})\) is

\[
\chi(\sigma^{-1})(xS_{j\sigma^{-1}(j)} - b) + (xS_{k\sigma^{-1}(k)} - b) + b^2 \sum_{l \neq j, k}(S_{l\sigma^{-1}(l)} - 1)).
\]

Since \(\sigma^{-1}(j) = i\) and \(\sigma^{-1}(k) = k\) then the coefficient of the term associated with \(x\) in the polynomial \(d_{\chi}(xS + (1-x)B_{b})\) is \(\chi(\sigma^{-1})(xS_{j\sigma^{-1}(j)} - b) + (xS_{k\sigma^{-1}(k)} - b) + b^2 \sum_{l \neq j, k}(S_{l\sigma^{-1}(l)} - 1)).
\]

Using the fact that

\[
d_{\chi}(xS + (1-x)B_{b}) = d_{\chi}(xS' + (1-x)B_{b})
\]

for all \(b \in [0, 1]\), we have that

\[
\chi(\sigma^{-1})(xS_{j\sigma^{-1}(j)} - b) + (xS_{k\sigma^{-1}(k)} - b) + b^2 \sum_{l \neq j, k}(S_{l\sigma^{-1}(l)} - 1)) = \chi(\sigma^{-1})(xS'_{j\sigma^{-1}(j)} - b) + (xS'_{k\sigma^{-1}(k)} - b) + b^2 \sum_{l \neq j, k}(S'_{l\sigma^{-1}(l)} - 1)),
\]

for all \(b \in [0, 1]\). Consequently,

\[
(S_{j\sigma^{-1}(j)} + S_{k\sigma^{-1}(k)})b + b^2 \left( \sum_{l \neq j, k}(S_{l\sigma^{-1}(l)} - 1)) - 2 \right) = (S'_{j\sigma^{-1}(j)} + S'_{k\sigma^{-1}(k)})b + b^2 \left( \sum_{l \neq j, k}(S'_{l\sigma^{-1}(l)} - 1)) - 2 \right)
\]

for all \(b \in [0, 1]\).

Then the coefficient of the term associated with \(b\) of the last polynomials are equal, i.e.,

\[
S_{j\sigma^{-1}(j)} + S_{k\sigma^{-1}(k)} = S'_{j\sigma^{-1}(j)} + S'_{k\sigma^{-1}(k)}
\]

(3.1) for all \(i, j, k \in \{1, \ldots, n\}\), distinct on pairs. Since \(n \geq 4\), there is \(p \notin \{i, j, k\}\) such that

\[
S_{j\sigma^{-1}(j)} + S'_{p\sigma^{-1}(p)} = S'_{j\sigma^{-1}(j)} + S'_{p\sigma^{-1}(p)},
\]

(3.2)

and subtracting the equalities (3.1) and (3.2), we obtain that

\[
S_{k\sigma^{-1}(k)} - S'_{k\sigma^{-1}(k)} = S_{p\sigma^{-1}(p)} - S'_{p\sigma^{-1}(p)}
\]

for all \(k, p \in \{1, \ldots, n\}\).

If \(c\) is the constant defined by \(c = S_{k\sigma^{-1}(k)} - S'_{k\sigma^{-1}(k)}\), then \(S_{k\sigma^{-1}(k)} = S'_{k\sigma^{-1}(k)} + c\), and by (3.1), we obtain \(S_{j\sigma^{-1}(j)} = S'_{j\sigma^{-1}(j)} - c\), for all \(i, j \in \{1, \ldots, n\}, i \neq j\).

As \(S, S' \in \Omega_n\), we have \(S_{jj} + \sum_{j=1,j \neq i}^{n}S_{ji} = 1\) and \(S'_{jj} + c + \sum_{j=1,j \neq i}^{n}(S'_{ji} - c) = 1\), which implies that \(\sum_{j=1}^{n}S'_{jj} + (2-n)c = 1\). Since \(n \neq 2\) then \(c = 0\), which means that \(S_{kk} = S'_{kk}\) and \(S_{ji} = S'_{ji}\), for all \(k, i, j \in \{1, \ldots, n\}\). Therefore, \(S = S'\), and \(T\) is injective.
Case (ii) Let \( n = 3 \) and \( \chi = [2,1] \). Let us consider \( \sigma = (ij) \) and \( \tau = (ijk) \) for \( \{i, j, k\} = \{1, 2, 3\} \). Then \( \chi(\sigma) = 0 \) and \( \chi(\tau) \neq 0 \). For each \( b \in [0, 1] \), consider the matrix \( B_b = bP(\sigma) + (1 - b)P(\tau) \). So, for all \( p \in \{1,2,3\} \) and \( \pi \in S_3 \),

\[
(B_b)_{\pi x(p)} = \begin{cases} 
1 & \text{if } \pi(p) = \sigma^{-1}(p) = \tau^{-1}(p), \\
\begin{array}{c}
b \quad \text{if } \pi(p) = \sigma^{-1}(p), \ \pi(p) \neq \tau^{-1}(p), \\
1-b & \text{if } \pi(p) \neq \sigma^{-1}(p), \ \pi(p) = \tau^{-1}(p), \\
0 & \text{otherwise.}
\end{array}
\end{cases}
\]

Then

\[
d_\chi(xS + (1-x)B_b) = \sum_{\pi \in S_3} \chi(\pi) \prod_{i=1}^{3} (xS + (1-x)B_b)_{\pi x(i)} \\
= \chi(\tau)(xS_{\tau(i)} + (1 - x)b)(xS_{\tau(j)})(xS_{\tau(k)}) + \chi(\tau^{-1})(xS_{\tau^{-1}(i)} + (1 - x)(1 - b)) \\
\cdot (xS_{\tau^{-1}(j)} + (1 - x))(xS_{\tau^{-1}(k)} + (1 - x)(1 - b)) + \chi(id)(xS_{ii})(xS_{jj})(xS_{kk} + (1 - x)b).
\]

Since \( \tau^{-1}(i) = k, \tau^{-1}(j) = i \) and \( \tau^{-1}(k) = j \), the coefficient of the term associated with \( x \) of the polynomial \( d_\chi(xS + (1-x)B_b) \) is \( \chi(\tau^{-1})(1 - b)((S_{ik} + S_{ji} + S_{kj} - 3) + b(-S_{ji} + 3)) \) for all \( b \in [0, 1] \).

Using the fact that

\[
d_\chi(xS + (1-x)B_b) = d_\chi(xS' + (1-x)B_b),
\]

we have that

\[
\chi(\tau^{-1})(1 - b)((S_{ij} + S_{ji} + S_{kj} - 3) + b(-S_{ji} + 3)) = \chi(\tau^{-1})(1 - b)((S'_{ij} + S'_{ji} + S'_{kj} - 3) + b(-S'_{ji} + 3))
\]

for all \( b \in [0, 1] \). So, the coefficient of the term associated with \( b^2 \) of last polynomials are equal and this implies that

\[
S_{ji} = S'_{ji}
\]

for all \( i \neq j \). Since \( S_{ii} + S_{ji} + S_{ki} = 1 = S'_{ii} + S'_{ji} + S'_{ki} \), then \( S_{ii} = S'_{ii} \), for all \( i \in \{1,2,3\} \). Consequently, \( S = S' \). So, \( T \) is injective. \( \square \)

4. The image of a permutation matrix by \( T \). Let \( C \subseteq \Omega_n \) be a convex polyhedron. An element \( S \in C \) is a vertex of \( C \), if \( S \) satisfies:

\[
\forall S_1, S_2 \in C : \ S = \alpha S_1 + (1 - \alpha) S_2, \ \text{with } \alpha \in [0, 1[ , \ \text{it follows } S_1 = S_2 = S.
\]

Let \( T \) be a semilinear map from \( \Omega_n \) into \( \Omega_n \) that preserves \( d_\chi \). Since \( \Omega_n \) and \( T(\Omega_n) \) are convex polyhedrons, and the permutation matrices are the vertices of \( \Omega_n \) (see [2]), in the next step we will see that if \( \sigma \in S_n \), then \( T(P(\sigma)) \) is a vertex of \( T(\Omega_n) \).

Proposition 4.1. Let \( \chi \) be an irreducible character of degree greater than 1 of \( S_n \). Let \( T \) be a semilinear map from \( \Omega_n \) into \( \Omega_n \) that preserves \( d_\chi \). If \( \sigma \in S_n \), then \( T(P(\sigma)) \) is a vertex of the convex polyhedron \( T(\Omega_n) \).

Proof. Let \( S_1, S_2 \in \Omega_n \) and \( \sigma \in S_n \) such that \( T(P(\sigma)) = \alpha T(S_1) + (1 - \alpha) T(S_2) \), for some \( \alpha \in [0, 1[. \) Then by semilinearity of \( T \) we have \( T(P(\sigma)) = T(\alpha S_1 + (1 - \alpha) S_2) \). Using Theorem 3.1, \( P(\sigma) = \alpha S_1 + (1 - \alpha) S_2, \)
with $\alpha \in [0, 1]$. As $P(\sigma)$ is a vertex of $\Omega_n$, then $S_1 = S_2 = P(\sigma)$, which means that $T(S_1) = T(S_2) = T(P(\sigma))$, and $T(P(\sigma))$ is a vertex of $T(\Omega_n)$. $\square$

In what follows, we consider that the semilinear map $T$ from $\Omega_n$ into $\Omega_n$ is surjective. Since $T$ preserves $d_\chi$, we have that $T$ is bijective and $T(\Omega_n) = \Omega_n$.

**Corollary 4.2.** Let $\chi$ be an irreducible character of degree greater than 1 of $S_n$. Let $T$ be a semilinear surjective map from $\Omega_n$ into $\Omega_n$ that preserves $d_\chi$. Then for each $\sigma \in S_n$ there is a $\pi \in S_n$ such that $T(P(\sigma)) = P(\pi)$, where $\chi(\sigma) = \chi(\pi)$.

**Definition 4.3.** We say that two matrices $S_1$ and $S_2$ are equal to one in the position $(i, j)$, if $(S_1)_{ij} = (S_2)_{ij} = 1$.

We denote by $c[S_1, S_2]$ the number of positions where $S_1$ and $S_2$ are equal to one. Consequently, if $P$ is a permutation matrix and $S \in \Omega_n$, then $c[P, S]$ is equal to the number of ones of the matrix $xP + (1 - x)S$, for all $x \in [0, 1]$. In particular $c[I, S]$ is equal to the number of ones in the main diagonal of $S$.

**Proposition 4.4.** Let $\chi$ be an irreducible character of degree greater than 1 of $S_n$. Let $T$ be a semilinear surjective map from $\Omega_n$ into $\Omega_n$ that preserves $d_\chi$. Let $\sigma \in S_n$ such that $\chi(\sigma) \neq 0$ and $S \in \Omega_n$. If $T(P(\sigma)) = P(\pi)$ and $T(S) = S'$, then

$$\sum_{j=1}^{n} S_{j\sigma^{-1}(j)} = \sum_{j=1}^{n} S'_{j\pi^{-1}(j)}.$$

**Proof.** Let $x \in [0, 1]$. First we will compute the coefficient of the term associated with $x$ of the polynomial $d_\chi(xS + (1 - x)P(\sigma)) = \sum_{\tau \in S_n} \chi(\tau) \prod_{j=1}^{n} (xS + (1 - x)P(\sigma))_{\tau(j)}$. If $\tau \neq \sigma^{-1}$, then there is $s \in \{1, \ldots, n\}$ such that $(xS + (1 - x)P(\sigma))_{\tau(s)} = xS_{\sigma(\tau(s))}$. Since $\tau$ and $\sigma$ are bijections, there are, at least two integers $s, h \in \{1, \ldots, n\}$ with $s \neq h$ and $(xS + (1 - x)P(\sigma))_{\tau(s)} = xS_{\sigma(\tau(s))}$, $(xS + (1 - x)P(\sigma))_{\tau(h)} = xS_{\sigma(\tau(h))}$. Consequently, $\prod_{j=1}^{n} (xS + (1 - x)P(\sigma))_{\tau(j)}$ is a polynomial with the coefficient associated with $x$ equal to zero. So, the coefficient of the term associated with $x$ of the polynomial $d_\chi(xS + (1 - x)P(\sigma))$ is obtained when $\tau = \sigma^{-1}$ and is equal to

$$\chi(\sigma^{-1}) \sum_{j=1}^{n} (S_{j\sigma^{-1}(j)} - 1).$$

As $d_\chi(xS + (1 - x)P(\sigma)) = d_\chi(xS' + (1 - x)P(\pi))$ we have that

$$\chi(\sigma^{-1}) \sum_{j=1}^{n} (S'_{j\sigma^{-1}(j)} - 1) = \chi(\pi^{-1}) \sum_{j=1}^{n} (S_{j\sigma^{-1}(j)} - 1).$$

Consequently, we get the desired conclusion using Corollary 4.2 and the fact that $\chi(\sigma) \neq 0$. $\square$

**Corollary 4.5.** Let $\chi$ be an irreducible character of degree greater than 1 of $S_n$. Let $T$ be a semilinear surjective map from $\Omega_n$ into $\Omega_n$ that preserves $d_\chi$. Let $\sigma \in S_n$ such that $\chi(\sigma) \neq 0$ and $\rho \in S_n$. If $T(P(\sigma)) = P(\pi)$, then

$$c[P(\sigma), P(\rho)] = c[P(\pi), T(P(\rho))].$$
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Proof. By Proposition 4.4

\[ \sum_{j=1}^{n} P(\rho)_{j\sigma^{-1}(j)} = \sum_{j=1}^{n} T(P(\rho))_{j\sigma^{-1}(j)}. \]

So we get the desired conclusion. \( \square \)

Lemma 4.6. Let \( \chi \) be an irreducible character of degree greater than 1 of \( S_n \). Let \( T \) be a semilinear surjective map from \( \Omega_n \) into \( \Omega_n \) that preserves \( d_\chi \). Let \( \rho, \theta \in S_n \) such that \( T(P(\rho)) = P(\theta) \). If \( \rho \) is a transposition then \( \theta \) is a transposition, and if \( \rho \) is a cycle of length three then \( \theta \) is a cycle of length three.

Proof. Let \( \rho \) be a cycle of length \( 2 \leq l \leq 3 \), such that \( T(P(\rho)) = P(\theta) \), then by Corollary 4.5

\[ c[I, P(\rho)] = n - l = c[I, P(\theta)]. \]

If \( l = 2 \), then there are \( i, j \in \{1, \ldots, n\} \) such that \( i \neq j \), \( P(\theta)_{ii} = P(\theta)_{jj} = 0 \), \( P(\theta)_{kk} = 1 \), for all \( k \neq i, j \), and consequently, \( P(\theta)_{ij} = P(\theta)_{ji} = 1 \).

The case \( l = 3 \) can be proved using the same arguments. \( \square \)

A semilinear map \( T \) is called unital if \( T(I) = I \). When \( T \) is a semilinear map from \( \Omega_n \) into \( \Omega_n \) the case of a nonunital map can be reduced to the unital case by considering the semilinear map \( \Phi \) defined by \( \Phi(S) = T(I)^{-1}T(S) \), since \( T(I) \) is invertible. Recall that by Corollary 2.4 if the irreducible character of degree greater than one, \( \chi \), verifies \( \chi \neq [2, 2] \) and \( T \) preserves \( d_\chi \) then \( T(I) = I \).

Proposition 4.7. Let \( \chi \) be an irreducible character of degree greater than 1 of \( S_n \). Let \( T \) be a semilinear unital surjective map from \( \Omega_n \) into \( \Omega_n \) that preserves \( d_\chi \). Then there is \( \alpha \in S_n \) such that for all \( i, j \in \{1, \ldots, n\}, i \neq j \),

\[ T(P(ij)) = P(\alpha(i)\alpha(j)). \]

Proof. First we will prove two claims.

If \( X \) is a set, we denote by \(|X|\) the cardinality of \( X \).

Claim 1. Let \( i, j, l, a, e, c, d \in \{1, \ldots, n\} \) with \( i, j, l \) distinct on pairs. If \( T(P(ij)) = P(\alpha e) \) and \( T(P(il)) = P(\alpha d) \) then \(|\{a, e, c, d\}| = 3\).

Proof of Claim 1. Using Lemma 4.6 since \( T \) is injective, \(|\{a, e, c, d\}| \neq 2\).

Suppose that \(|\{a, e, c, d\}| = 4\), which does not happen if \( n = 3 \). Let \( S = bP(ij) + (1 - b)P(il) \), with \( b \in [0, 1] \). Since, \( T(S) = bP(\alpha e) + (1 - b)P(\alpha d) \), where \( b \in [0, 1] \), and \( d_\chi(xS + (1 - x)I) = d_\chi(xT(S) + (1 - x)I) \), then the coefficient of the term associated with \( x^2b^2 \) of both polynomials must be equal.

First we will compute the term associated with \( x^2b^2 \) of the polynomial

\[ d_\chi(xS + (1 - x)I) = \sum_{\pi \in S_n} \chi(\pi) \prod_{s=1}^{n} (xS + (1 - x)I)_{s\pi(s)}. \]
Therefore, the coefficient of the term associated with 

When \( \pi \in S_n \),

\[
(S)_{s\pi(s)} = (bP(ij) + (1 - b)P(il))_{s\pi(s)} = \begin{cases} 
1 & \text{if } s \in \{1, \ldots, n\} \setminus \{i, j, l\}, \quad \pi(s) = s, \\
b & \text{if } (s, \pi(s)) \in \{(l, l), (i, j), (j, i)\}, \\
1 - b & \text{if } (s, \pi(s)) \in \{(l, i), (i, l), (j, j)\}, \\
0 & \text{otherwise}.
\end{cases}
\]

Consequently, when \( \pi \in S_n \),

\[
(xS + (1 - x)I)_{s\pi(s)} = \begin{cases} 
1 & \text{if } s \in \{1, \ldots, n\} \setminus \{i, j, l\}, \quad \pi(s) = s, \\
1 - x & \text{if } (s, \pi(s)) = (i, i), \\
1 - xb & \text{if } (s, \pi(s)) = (j, j), \\
1 - x(1 - b) & \text{if } (s, \pi(s)) = (l, l), \\
xb & \text{if } (s, \pi(s)) \in \{(i, j), (j, i)\}, \\
x(1 - b) & \text{if } (s, \pi(s)) \in \{(i, i), (i, l)\}, \\
0 & \text{otherwise}.
\end{cases}
\]

So, if \( \pi \notin \{id, (ij), (il)\} \) and there is \( h \in \{1, \ldots, n\} \setminus \{i, j, l\} \) with \( \pi(h) \neq h \) then \( (xS + (1 - x)I)_{h\pi(h)} = 0 \) and \( \chi(\pi) \prod_{s=1}^n (xS + (1 - x)I)_{s\pi(s)} = 0 \). Consequently, if \( \chi(\pi) \prod_{s=1}^n (xS + (1 - x)I)_{s\pi(s)} \neq 0 \) then \( \pi(h) = h \), for all \( h \in \{1, \ldots, n\} \setminus \{i, j, l\} \) and \( \pi \in \{id, (ij), (il), (jl), (ijl)\} \).

If \( \pi = (jl) \) or \( \pi = (ijl) \) then \( (xS + (1 - x)I)_{j\pi(j)} = 0 \) and \( \chi(\pi) \prod_{s=1}^n (xS + (1 - x)I)_{s\pi(s)} = 0 \).

If \( \pi = (ilj) \) then \( (xS + (1 - x)I)_{j\pi(l)} = 0 \) and \( \chi(\pi) \prod_{s=1}^n (xS + (1 - x)I)_{s\pi(s)} = 0 \).

So, \( d(xS + (1 - x)I) = \chi(id)(1 - x(1 - b))(xb)(xb) + \chi(id)(1 - x(1 - b))(1 - xb)(1 - x) + \chi(id)(1 - xb)(x(1 - b))^2 \).

Therefore, the coefficient of the term associated with \( x^2b^2 \) of the polynomial \( d(xS + (1 - x)I) \) is

\[-\chi(id) + \chi(ij) + \chi(id).\]

Now we will compute the term associated with \( x^2b^2 \) of the polynomial

\[d(\chi(xT(S) + (1 - x)I)) = \sum_{\pi \in S_n} \chi(\pi) \prod_{s=1}^n (xT(S) + (1 - x)I)_{s\pi(s)}.\]

When \( \pi \in S_n \),

\[
(T(S))_{s\pi(s)} = (bP(ae) + (1 - b)P(cd))_{s\pi(s)} = \begin{cases} 
1 & \text{if } s \in \{1, \ldots, n\} \setminus \{a, e, c, d\}, \quad \pi(s) = s, \\
b & \text{if } (s, \pi(s)) \in \{(a, c), (e, a), (c, c), (d, d)\}, \\
1 - b & \text{if } (s, \pi(s)) \in \{(c, d), (d, c), (a, a), (e, e)\}, \\
0 & \text{otherwise}.
\end{cases}
\]

Consequently, when \( \pi \in S_n \),

\[
(xT(S) + (1 - x)I)_{s\pi(s)} = \begin{cases} 
1 & \text{if } s \in \{1, \ldots, n\} \setminus \{a, e, c, d\}, \quad \pi(s) = s, \\
1 - xb & \text{if } (s, \pi(s)) \in \{(a, a), (a, e)\}, \\
1 - x(1 - b) & \text{if } (s, \pi(s)) \in \{(d, d), (c, c)\}, \\
xb & \text{if } (s, \pi(s)) \in \{(a, e), (e, a)\}, \\
x(1 - b) & \text{if } (s, \pi(s)) \in \{(c, d), (d, c)\}, \\
0 & \text{otherwise}.
\end{cases}
\]
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So, if \( \pi \not\in \{id, (ae), (cd)\} \) and there is \( h \in \{1, \ldots, n\} \setminus \{a, e, c, d\} \) with \( \pi(h) \neq h \) then \( (xT(S) + (1-x)I)_{h\pi(h)} = 0 \) and \( \chi(\pi) \prod_{s=1}^{n} (xT(S) + (1-x)I)_{s\pi(s)} = 0 \). Consequently, if \( \chi(\pi) \prod_{s=1}^{n} (xT(S) + (1-x)I)_{s\pi(s)} \neq 0 \) then \( \pi(h) = h \), for all \( h \in \{1, \ldots, n\} \setminus \{a, e, c, d\} \).

If \( \pi(r) \in \{\pi(a), \pi(c)\} \subseteq \{c, d\} \) or \( \pi(r) \in \{\pi(e), \pi(d)\} \subseteq \{a, e\} \), then \( (xT(S) + (1-x)I)_{r\pi(r)} = 0 \) and \( \chi(\pi) \prod_{s=1}^{n} (xT(S) + (1-x)I)_{s\pi(s)} = 0 \).

So, \( d_{\chi}(xT(S) + (1-x)I) = \chi(ae)(1-x(1-b))^{2}(xb)^{2} + \chi(id)(1-x(1-b))^{2}(1-xb)^{2} + \chi(cd)(1-xb)^{2}(1-b)^{2} \). Therefore, the coefficient of the term associated with \( x^{2}b^{2} \) of the polynomial \( d_{\chi}(xT(S) + (1-x)I) \) is

\[
-2\chi(id) + \chi(ae) + \chi(cd).
\]

Since the polynomials \( d_{\chi}(xS + (1-x)I) \) and \( d_{\chi}(xT(S) + (1-x)I) \) are equal then the coefficients of the term associated with \( x^{2}b^{2} \) of each polynomial are equal, i.e.,

\[
-\chi(id) + \chi(ij) + \chi(il) = -2\chi(id) + \chi(ae) + \chi(cd).
\]

Because \( \chi(id) \neq 0 \), we obtain a contradiction. Consequently, \(|\{a, e, c, d\}| = 3\). \( \blacksquare \)

Claim 2. Let \( i, j, l, a, e, d \in \{1, \ldots, n\} \) with \( i, j, l \) distinct on pairs and \( a, e, d \) distinct on pairs. If \( T(P(ij)) = P(\{ae\}) \) and \( T(P(il)) = P(ad) \), then

\[
T(P(jl)) = P(ed).
\]

Proof of Claim 2. If \( T(P(jl)) = P(gf) \), using Claim 1, we conclude that \(|\{a, e, g, f\}| = 3 \) and \(|\{a, d, g, f\}| = 3\).

Let us assume that \( g = a \). Then \( f \neq a, f \neq e \) and \( f \neq d \), and consequently \(|\{a, e, d, f\}| = 4\).

Let \( S = b_{1}P(ij) + b_{2}P(il) + (1 - (b_{1} + b_{2}))P(jl) \), with \( b_{1}, b_{2} \in [0,1] \) and \( b_{1} + b_{2} \leq 1 \). Since, \( d_{\chi}(xS + (1-x)I) = d_{\chi}(xT(S) + (1-x)I) \), then the coefficient of the term associated with \( x^{2}b_{1}b_{2} \) of both polynomials must be equal.

First we will compute the term associated with \( x^{4}b_{1}b_{2} \) of the polynomial

\[
d_{\chi}(xS + (1-x)I) = \sum_{\pi \in S_{n}} \chi(\pi) \prod_{s=1}^{n} (xS + (1-x)I)_{s\pi(s)}.
\]

When \( \pi \in S_{n} \),

\[
(S)_{s\pi(s)} = \begin{cases} 
1 & \text{if } s \in \{1, \ldots, n\} \setminus \{i, j, l\}, \quad \pi(s) = s, \\
b_{1} & \text{if } (s, \pi(s)) \in \{(l, i), (i, j), (j, i)\}, \\
b_{2} & \text{if } (s, \pi(s)) \in \{(j, l), (i, l), (l, i)\}, \\
1 - (b_{1} + b_{2}) & \text{if } (s, \pi(s)) \in \{(i, i), (j, l), (l, j)\}, \\
0 & \text{otherwise}.
\end{cases}
\]
Consequently, when $\pi \in S_n$,

\[
(xS + (1 - x)I)_{x\pi(s)} = \begin{cases} 
1 & \text{if } s \in \{1, \ldots, n\} \setminus \{i, j, l\}, \quad \pi(s) = s, \\
1 - x(b_1 + b_2) & \text{if } (s, \pi(s)) = (i, i), \\
1 - x(1 - b_2) & \text{if } (s, \pi(s)) = (j, i), \\
1 - x(1 - b_1) & \text{if } (s, \pi(s)) = (l, i), \\
x b_1 & \text{if } (s, \pi(s)) \in \{(i, j), (j, i)\}, \\
x b_2 & \text{if } (s, \pi(s)) \in \{(i, l), (l, i)\}, \\
x(1 - (b_1 + b_2)) & \text{if } (s, \pi(s)) \in \{(l, j), (j, l)\}, \\
0 & \text{otherwise.} 
\end{cases}
\]

So, if $\pi \in S_n$ and there is $h \in \{1, \ldots, n\} \setminus \{i, j, l\}$ with $\pi(h) \neq h$ then $(xS + (1 - x)I)_{h\pi(h)} = 0$ and $\chi(\pi) \prod_{s=1}^{n} (xS + (1 - x)I)_{h\pi(s)} = 0$. If $\pi \in S_n$ and for all $h \in \{1, \ldots, n\} \setminus \{i, j, l\}$, $\pi(h) = h$ then $(xS + (1 - x)I)_{h\pi(h)} = 1$. Consequently, the degree of the polynomial $d_\chi(xS + (1 - x)I)$ is less than or equal to three. Therefore, the coefficient of the term associated with $x^4b_1b_2$ of the polynomial $d_\chi(xS + (1 - x)I)$ is zero.

Now we will compute the term associated with $x^4b_1b_2$ of the polynomial $d_\chi(xT(S) + (1 - x)I) = \sum_{\pi \in S_n} \chi(\pi) \prod_{s=1}^{n} (xT(S) + (1 - x)I)_{s\pi(s)}$. When $\pi \in S_n$,

\[
(T(S))_{s\pi(s)} = (b_1P(ae) + b_2P(ad) + (1 - (b_1 + b_2))P(af))_{s\pi(s)} = \begin{cases} 
1 & \text{if } s \in \{1, \ldots, n\} \setminus \{a, c, d\}, \quad \pi(s) = s, \\
b_1 & \text{if } (s, \pi(s)) \in \{(a, e), (e, a)\}, \\
b_2 & \text{if } (s, \pi(s)) \in \{(a, d), (d, a)\}, \\
1 - (b_1 + b_2) & \text{if } (s, \pi(s)) \in \{(a, f), (f, a)\}, \\
1 - b_1 & \text{if } (s, \pi(s)) = (e, c), \\
1 - b_2 & \text{if } (s, \pi(s)) = (d, a), \\
b_1 + b_2 & \text{if } (s, \pi(s)) = (f, f), \\
0 & \text{otherwise.} 
\end{cases}
\]

Consequently, when $\pi \in S_n$,

\[
(xT(S) + (1 - x)I)_{s\pi(s)} = \begin{cases} 
1 & \text{if } s \in \{1, \ldots, n\} \setminus \{a, c, d\}, \quad \pi(s) = s, \\
x b_1 & \text{if } (s, \pi(s)) \in \{(a, e), (e, a)\}, \\
x b_2 & \text{if } (s, \pi(s)) \in \{(a, d), (d, a)\}, \\
x(1 - (b_1 + b_2)) & \text{if } (s, \pi(s)) \in \{(a, f), (f, a)\}, \\
1 - x b_1 & \text{if } (s, \pi(s)) = (e, c), \\
1 - x b_2 & \text{if } (s, \pi(s)) = (d, a), \\
1 - x(1 - (b_1 + b_2)) & \text{if } (s, \pi(s)) = (f, f), \\
1 - x & \text{if } (s, \pi(s)) = (a, a), \\
0 & \text{otherwise.} 
\end{cases}
\]

If $\pi \notin \{(a, e), (a, f), (a, d), (d, a)\}$ then there is $h \in \{1, \ldots, n\}$ with $\pi(h) \notin \{h, (a, e), (a, f), (a, d), (d, a)\}$. Consequently, $(xT(S) + (1 - x)I)_{h\pi(h)} = 0$. Then $\chi(\pi) \prod_{h=1}^{n} (xT(S) + (1 - x)I)_{h\pi(h)} = 0$. So, $d_\chi(xT(S) + (1 - x)I) = \chi(id)(1 - x b_1)(1 - x b_2)(1 - x(1 - (b_1 + b_2)) + \chi(af)(x b_1)^2(1 - x b_2)(1 - x(1 - (b_1 + b_2))) + \chi(af)(x b_2)^2(1 - x b_1)(1 - x(1 - (b_1 + b_2))) + \chi(af)x(1 - (b_1 + b_2))^2(1 - x b_1)(1 - x b_2) = \chi(id)(1 - x(b_1 + b_2) + 1 + x^2(b_1 + b_2 + b_1 b_2) - x^3 b_1 b_2)(1 - x(1 - (b_1 + b_2))) + \cdots + \chi(af)x(1 - 2 b_1 - 2 b_2 + b_1^2 + b_2^2 + 2 b_1 b_2)(1 - x(b_1 + b_2) + x^2 b_1 b_2).$
Then the coefficient of the polynomial \( d_\chi(xT(S) + (1 - x)I) \) associated with \( x^4b_1b_2 \) is \( \chi(id) + \chi(af) \).

Since \( d_\chi(xS + (1 - x)I) = d_\chi(xT(S) + (1 - x)I) \), then the coefficient of the term associated with \( x^4b_1b_2 \) of both polynomials must be equal, i.e.,

\[
0 = \chi(id) + \chi(af).
\]

But this is impossible by Remark 2.1 when \( \chi \neq [2, 2] \), and because \( \chi(id) = 2 \), \( \chi(af) = 0 \), when \( \chi = [2, 2] \). Thus, \( g \neq a \), and using the same argument, we have that \( f \neq a \). Therefore, \( g = e \), so \( f = d \) since \( \{|a, d, g, f\}| = 3 \), and we conclude that \( T(P(jl)) = P(ed) \).

For all \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \) let us consider \( k \in \{1, \ldots, n\} \), such that \( k \neq i, k \neq j \). Let us assume that \( \{i, j, k\} = \{1, 2, 3\} \).

Using Lemma 4.6 there are \( j_1, j_2, j_3, j_4 \in \{1, \ldots, n\} \) such that \( T(P(12)) = P(j_1j_2), T(P(13)) = P(j_3j_4) \). By Claim 1 \( |\{j_1, j_2, j_3, j_4\}| = 3 \). Let \( \alpha(1) = i_1 \) where \( i_1 \in \{j_1, j_2\} \cap \{j_3, j_4\} \). Let \( \alpha(2) = i_2 \), where \( i_2 \in \{j_1, j_2\} \setminus \{i_1\} \), and \( \alpha(3) = i_3 \), where \( i_3 \in \{j_3, j_4\} \setminus \{i_1\} \).

Using this construction, we can define a function

\[
\alpha: \{1, \ldots, n\} \rightarrow \{1, \ldots, n\},
\]

where \( \alpha(r) = i_r \).

Using Claim 2 and the injectivity of \( T \), we conclude that \( \alpha \in S_n \). □

5. Proof of the main result. Let \( \chi \) be an irreducible character of degree greater than 1 of \( S_n \). In this section, we characterize the semilinear surjective maps \( T \) from \( \Omega_n \) into \( \Omega \) that preserve \( d_\chi \) (Theorem 1.2).

By the Murnaghan-Nakayama Rule (mentioned in Section 2), if \( \chi \) is an irreducible character of \( S_n \) and \( p \) is the number of boundary boxes of the Young Diagram associated with \( \chi \), then \( \chi(\xi) \neq 0 \) whenever \( \xi \) is a cycle of length \( p \). On what follows we consider \( \alpha \in S_n \) obtained using Proposition 4.7.

**Proposition 5.1.** Let \( \chi \) be an irreducible character of \( S_n \) of degree greater than one and \( p \) be the number of boundary boxes of the Young Diagram associated with \( \chi \). Let \( T \) be a semilinear unital surjective map from \( \Omega_n \) into \( \Omega_n \) that preserves \( d_\chi \). Let \( \xi \in S_n \) be a cycle of length \( p \) and \( T(P(\xi)) = P(\rho) \). Then

\[
\rho = \alpha \circ \xi \circ \alpha^{-1}
\]

or

\[
\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}.
\]

**Proof.** Let \( \xi = (i_1i_2 \cdots i_p) \). Then, by Corollary 4.5 \( c[I, P(\xi)] = c[I, P(\rho)] = n - p \). Let \( S = P(i_1i_2) \). Then

\[
S' = T(S) = T(P(i_1i_2)) = P(\alpha(i_1)\alpha(i_2))
\]

and, by Corollary 4.5

\[
c[P(\xi), P(i_1i_2)] = n - p + 1 = c[P(\rho), P(\alpha(i_1)\alpha(i_2))],
\]
i.e., $\rho^{-1}(\alpha(i_1)) = \alpha(i_2)$, or $\rho^{-1}(\alpha(i_2)) = \alpha(i_1)$, and both cases cannot happen at the same time because $\rho$ is not a transposition.

Repeating the same argument with $S = P(i_t i_{t+1})$, where $t \in \{2, \ldots, p-1\}$, and using the bijectivity of $\rho$ and $\alpha$, we must have

1) $\rho^{-1}(\alpha(i_1)) = \alpha(i_2)$, $\rho^{-1}(\alpha(i_2)) = \alpha(i_3)$, $\ldots$, $\rho^{-1}(\alpha(i_p)) = \alpha(i_1)$, or

2) $\rho^{-1}(\alpha(i_1)) = \alpha(i_p)$, $\rho^{-1}(\alpha(i_p)) = \alpha(i_{p-1})$, $\ldots$, $\rho^{-1}(\alpha(i_2)) = \alpha(i_1)$.

So by definition of $\xi$ we have that

1) $\alpha^{-1} \circ \rho^{-1} \circ \alpha = \xi^{-1}$, or

2) $\alpha^{-1} \circ \rho^{-1} \circ \alpha = \xi$.

Then $\rho = \alpha \circ \xi \circ \alpha^{-1}$ or $\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}$. $\Box$

In Proposition 5.2 we will prove that if $\beta, \gamma \in S_n$ are cycles of length $p$, then we cannot have $T(P(\beta)) = P(\alpha \circ \beta \circ \alpha^{-1})$, and $T(P(\gamma)) = P(\alpha \circ \gamma^{-1} \circ \alpha^{-1})$.

**Proposition 5.2.** Let $\chi$ be an irreducible character of $S_n$ of degree greater than one and $p$ be the number of boundary boxes of the Young Diagram associated with $\chi$. Let $T$ be a semilinear unital surjective map from $\Omega_n$ into $\Omega_n$ that preserves $d_{\chi}$. Suppose that if $p = 4$ then $n \neq p$. Let $\xi \in S_n$ be a cycle of length $p$ and $T(P(\xi)) = P(\rho)$.

1) If $\rho = \alpha \circ \xi \circ \alpha^{-1}$ then $T(P(\theta)) = P(\alpha \circ \theta \circ \alpha^{-1})$ whenever $\theta \in S_n$ is a cycle of length $p$.

2) If $\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}$ then $T(P(\theta)) = P(\alpha \circ \theta^{-1} \circ \alpha^{-1})$ whenever $\theta \in S_n$ is a cycle of length $p$.

**Proof.** We will prove part 1). The proof will be divided into two cases:

**Case 1.** Let $p \neq n$.

**Claim 1.** If $i, j \in \{1, \ldots, n\}$ with $i \neq j$, verify $\xi(i) = i$ and $\xi(j) \neq j$ then $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi \circ (ij) \circ \alpha^{-1})$.

**Proof of Claim 1.** Since $(ij) \circ \xi \circ (ij)$ is a cycle of length $p$, by Proposition 5.1, $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi \circ (ij) \circ \alpha^{-1})$ or $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi^{-1} \circ (ij) \circ \alpha^{-1})$. Suppose that $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi^{-1} \circ (ij) \circ \alpha^{-1})$.

Let $S = P((ij) \circ \xi \circ (ij))$. By Proposition 4.4 $S_{1\xi^{-1}(1)} + \cdots + S_{n\xi^{-1}(n)} = S'_{1\rho^{-1}(1)} + \cdots + S'_{n\rho^{-1}(n)}$, where $S' = T(S)$. Since

\[ S_{1\xi^{-1}(1)} + \cdots + S_{n\xi^{-1}(n)} = |\{(a : (ij) \circ \xi^{-1}(a) = \xi^{-1} \circ (ij)(a)\}| = n - 3 \]

and

\[ S'_{1\rho^{-1}(1)} + \cdots + S'_{n\rho^{-1}(n)} = |\{(a : (ij) \circ \xi^{-1}(a) = \xi \circ (ij)(a)\}| = n - p - 1, \]

then $p = 2$ (impossible). So $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi \circ (ij) \circ \alpha^{-1})$. $\blacksquare$

Let $\theta = (a \theta(a) \cdots \theta^{p-1}(a))$ be a cycle of length $p$, $\theta \neq \xi$, where $a \in \{1, \ldots, n\}$ and

\[ \theta^l(a) = \begin{cases} a & \text{if } l = 0, \\ \theta(\theta^{-1}(a)) & \text{if } l > 0. \end{cases} \]
If $\xi(a) = a$, let $t$ be an integer such that $\xi(t) \neq t$. Using Claim 1,
\[ TP((at) \circ \xi \circ (at)) = P(a \circ (at) \circ \xi \circ (at) \circ a^{-1}) \]
and if $\beta = (at) \circ \xi \circ (at)$ then $\beta$ is a cycle of length $p$ verifying $\beta(a) \neq a$. So, we can assume that $\xi(a) \neq a$.

Let $s$ be the smallest positive integer that $\theta^s(a) \neq \theta^{s-1}(a)$. Consequently, $s < p$ and $\theta^u(a) = \xi^u(a)$, for $u = 0, \ldots, s - 1$.

- If $\xi(\theta^s(a)) = \theta^s(a)$, let $\xi(\theta^{s-1}(a)) = r$ (note that we have $\xi(\theta^{s-1}(a)) \neq \theta^{s-1}(a)$ because $\xi(\theta^{s-1}(a)) = \theta^{s-1}(a)$ implies that $\xi(a) = a$). Using Claim 1,
\[ TP((\theta^s(a)r) \circ \xi \circ (\theta^s(a)r)) = P(a \circ (\theta^s(a)r) \circ \xi \circ (\theta^s(a)r) \circ a^{-1}) \]
and if $\beta_1 = (\theta^s(a)r) \circ \xi \circ (\theta^s(a)r)$ then $\beta_1$ is a cycle of length $p$ verifying $\beta_1^u(a) = \theta^u(a)$, for $u = 0, \ldots, s$.

- If $\xi(\theta^s(a)) \neq \theta^s(a)$, let $\xi(\theta^{s-1}(a)) = r$. Since $n \neq p$, let $k$ be an integer such that $\xi(k) = k$. Using Claim 1,
\[ TP((\theta^s(a)k) \circ \xi \circ (\theta^s(a)k)) = P(a \circ (\theta^s(a)k) \circ \xi \circ (\theta^s(a)k) \circ a^{-1}) \]
and if $\beta_2 = (\theta^s(a)k) \circ \xi \circ (\theta^s(a)k)$ then $\beta_2$ is a cycle of length $p$ verifying $\beta_2^u(a) = \theta^u(a)$, for $u = 0, \ldots, s - 1$, $\beta_2(\theta^s(a)) = \theta^s(a)$ and $\beta_2(\theta^{s-1}(a)) = r$. Using what we proved above, we conclude that there is a cycle of length $p$, $\beta_3$, such that $\beta_3^u(a) = \theta^u(a)$, for $u = 0, \ldots, s$ and $TP(\beta_3) = P(\alpha\beta_3\alpha^{-1})$.

Repeating this argument, we prove the result.

**Case 2.** Let $n = p \neq 4$.

**Claim 2.** If $i, j \in \{1, \ldots, n\}$, with $i \neq j$, verify $\xi(i) = j$, then $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi \circ (ij) \circ \alpha^{-1})$.

**Proof of Claim 2.** Using a similar argument as in Claim 1, suppose that $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi^{-1} \circ (ij) \circ \alpha^{-1})$.

Let $S = P((ij) \circ \xi \circ (ij))$. By Proposition 4.14, $S_{1\xi^{-1}(1)} + \cdots + S_{n\xi^{-1}(n)} = S'_{1p^{-1}(1)} + \cdots + S'_{np^{-1}(n)}$, where $S' = T(S)$. Since
\[ S_{1\xi^{-1}(1)} + \cdots + S_{n\xi^{-1}(n)} = \{|a : (ij) \circ \xi^{-1}(a) = \xi^{-1}(ij)(a)| = n - 3 \}
and
\[ S'_{1p^{-1}(1)} + \cdots + S'_{np^{-1}(n)} = \{|a : (ij) \circ \xi^{-1}(a) = \xi(ij)(a)| = n, \text{ if } p = 3 \}
or
\[ S'_{1p^{-1}(1)} + \cdots + S'_{np^{-1}(n)} = \{|a : (ij) \circ \xi^{-1}(a) = \xi(ij)(a)| = n - p + 1, \text{ if } p > 3 \}
then $p = 4$ (impossible). So $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi \circ (ij) \circ \alpha^{-1})$.

Let $\theta = (a \theta(a) \cdots \theta^{p-1}(a))$ be a cycle of length $p$, $\theta \neq \xi$, where $a \in \{1, \ldots, n\}$ and
\[ \theta^l(a) = \begin{cases} a & \text{if } l = 0, \\ \theta(\theta^{l-1}(a)) & \text{if } l > 0. \end{cases} \]
Since $n = p$ then $\xi(a) \neq a$. Let $s$ be the smallest positive integer that $\theta^s(a) \neq \xi(\theta^{s-1}(a))$. Consequently, $s < p - 1$ and $\theta^s(a) = \xi^u(a)$, for $u = 0, \ldots, s - 1$. Since $n = p$, there is an integer $k$ such that $p - 1 \geq k > s$ and $\xi^k(a) = \theta^u(a)$. Using Claim 2,

$$TP((\xi^k(a)\xi^{k-1}(a)) \circ \xi \circ (\xi^k(a)\xi^{k-1}(a))) = P(\alpha \circ (\xi^k(a)\xi^{k-1}(a)) \circ \xi \circ (\xi^k(a)\xi^{k-1}(a)) \circ \alpha^{-1}),$$

and if $\beta_4 = (\xi^k(a)\xi^{k-1}(a)) \circ \xi \circ (\xi^k(a)\xi^{k-1}(a))$, then $\beta_4$ is a cycle of length $p$ verifying $\beta_4^u(a) = \theta^u(a)$, for $u = 0, \ldots, s - 1$ and $\beta_4^{k-1}(a) = \xi^k(a) = \theta^u(a)$. Using this argument we obtain a cycle of length $p$, $\beta_5$, such that $\beta_5^u(a) = \theta^u(a)$, for $u = 0, \ldots, s$ and $TP(\beta_5) = P(\alpha \beta_5 \alpha^{-1})$.

Repeating this argument, we prove the result.

The proof of part 2) is analogous. $\Box$

For each $i, j \in \{1, \ldots, n\}$ let $U_{i,j}$ be the subset of $\Omega_n$ such that

$$U_{i,j} = \{P \in \Omega_n : P \text{ is a permutation matrix and } P_{i,j} = 1\}.$$ 

These sets are very important for our study.

**Proposition 5.3.** Let $\chi$ be an irreducible character of $S_n$ of degree greater than one, $\chi$, and $p$ be the number of boundary boxes of the Young Diagram associated with $\chi$. Let $T$ be a unital semilinear surjective map from $\Omega_n$ into $\Omega_n$ that preserves $d_\chi$. Let $i, j \in \{1, \ldots, n\}$ where $i \neq j$, and $P$ be a permutation matrix, such that $P \in U_{i,j}$. Assume that $\xi$ is a cycle of length $p$, and $T(P(\xi)) = P(\rho)$. Then one of the following conditions must hold:

1. If $\rho = \alpha \circ \xi \circ \alpha^{-1}$, then $T(P) \in U_{\alpha(i),\alpha(j)}$.
2. If $\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}$, then $T(P) \in U_{\alpha(j),\alpha(i)}$.

**Proof.** We will prove (1). Let $\pi \in S_n$ such that $\pi(j) = i$. Therefore, $P(\pi) \neq I$ and $P(\pi) \in U_{i,j}$. By hypothesis, $\rho = \alpha \circ \xi \circ \alpha^{-1}$. We will see that $T(P(\pi)) \in U_{\alpha(i),\alpha(j)}$. Let $P(\theta) = T(P(\pi))$. We shall consider several cases:

**Case 1.** Let $n \geq 5$. If $n \geq 5$, and the number of boundary boxes of the Young diagram associated with $\chi$ is $p$, then $p \geq 4$. Suppose that $T(P(\pi)) = P(\theta) \notin U_{\alpha(i),\alpha(j)}$, i.e.,

$$\alpha^{-1} \circ \theta \circ \alpha(j) \neq i$$

Let $\theta' = \alpha^{-1} \circ \theta \circ \alpha$, then by Corollary 4.5

$$c[P(\zeta), P(\pi)] = c[T(P(\zeta)), P(\theta)],$$

whenever $\zeta$ is a cycle of length $p$.

Since $n \geq 5$, we can choose $a \in \{1, \ldots, n\}$ such that

$$a \neq i, \; a \neq j, \; \pi(a) \neq j,$$

and we can choose $b \in \{1, \ldots, n\}$ such that

$$b \neq i, \; b \neq j, \; b \neq a, \; \theta'(a) \neq b, \; \text{and} \; \theta'(b) \neq j.$$
Let us consider the cycles $\xi_1$ and $\eta$ of length $p$, defined by

$$\xi_1(a) = b, \quad \xi_1(b) = j, \quad \xi_1(j) = i, \quad \eta(a) = j, \quad \eta(j) = b, \quad \eta(b) = i,$$

and $\xi_1(q) = \eta(q)$ for all $q \notin \{a, b, j\}$.

Since $\xi_1(j) = \pi(j)$ and $\eta(q) \neq \pi(q)$ for all $q \in \{a, b, j\}$, then

$$c[P(\xi_1), P(\pi)] > c[P(\eta), P(\pi)],$$

which implies that

$$c[T(P(\xi_1)), P(\theta)] > c[T(P(\eta)), P(\theta)].$$

By Proposition 5.2 we have

$$c[P(\alpha \circ \xi_1 \circ \alpha^{-1}), P(\theta)] > c[P(\alpha \circ \eta \circ \alpha^{-1}), P(\theta)].$$

Since $\xi_1(q) \neq \theta'(q)$ for all $q \in \{a, b, j\}$, then

$$c[P(\alpha \circ \xi_1 \circ \alpha^{-1}), P(\theta)] \leq c[P(\alpha \circ \eta \circ \alpha^{-1}), P(\theta)],$$

which is a contradiction. So $T(P(\pi)) \in U_{\alpha(j), \alpha(j)}$.

Case 2. Let $n = 3$ and $\chi = [2, 1]$. Since $p = 3$, if $\pi$ is a cycle of length 3, then the result is obtained using Proposition 5.2. If $\pi$ is a cycle of length 2, then the result is obtained using Proposition 4.7.

Case 3. Let $n = 4$ and $\chi = [3, 1]$ or $\chi = [2, 1, 1]$. In this case, we can not use Proposition 5.2 since the number of boundary boxes of the Young Diagram associated with $\chi$ is $p = 4$. If $\pi$ is a cycle of length 2, then the result is obtained using Proposition 4.7.

Let $\pi = (ij) \circ (kl)$ with $i, j, k, l$ distinct on pairs, then by Corollary 4.5 (in this case, if $\sigma$ is a transposition then $\chi(\sigma) = 1$ or $-1$),

$$c[P(ij), P(\pi)] = 2 = c[P(\alpha(i)\alpha(j)), T(P(\pi))].$$

Since $c[I, P(\pi)] = 0$ then $c[I, T(P(\pi))] = 0$. So, $\theta(\alpha(i)) = \alpha(j)$ and $\theta(\alpha(j)) = \alpha(i)$. Therefore, $P(\theta) \in T_{\alpha(j), \alpha(j)}$.

Let $i, j, k$ distinct on pairs. If $\pi = (ijk)$, using Lemma 4.6, $T(P(ijk)) = P(abc)$, where $a, b, c$ are distinct on pairs. Since $\chi(ij) \neq 0$ (in this case, $\chi(ij) = 1$ or $\chi(ij) = -1$), by Corollary 4.5 we have

$$c[P(ij), P(\pi)] = 2 = c[P(\alpha(i)\alpha(j)), T(P(\pi))].$$

Since $c[I, P(\pi)] = 1$ then $c[I, T(P(\pi))] = 1$. So,

$$(abc)(\alpha(i)) = \alpha(j) \quad \text{or} \quad (abc)(\alpha(j)) = \alpha(i),$$

(only one of these conditions because $(abc)$ is not a transposition).

In the same way, using the transposition $(ik)$,

$$(abc)(\alpha(i)) = \alpha(k) \quad \text{or} \quad (abc)(\alpha(k)) = \alpha(i)$$

and using the transposition $(kj)$,

$$(abc)(\alpha(k)) = \alpha(j) \quad \text{or} \quad (abc)(\alpha(j)) = \alpha(k).$$
Consequently,

\[(abc) = (\alpha(i)\alpha(j)\alpha(k)) \quad \text{or} \quad (abc) = (\alpha(j)\alpha(i)\alpha(k)).\]

Since \(\chi\) is a cycle of length 4, then \(\chi\) is one of the following permutations

\[(jikl) \quad \text{or} \quad (jik) \quad \text{or} \quad (jlik) \quad \text{(5.1)}\]

or

\[(jiki) \quad \text{or} \quad (jikl) \quad \text{or} \quad (jikli), \quad \text{(5.2)}\]

with \(l \in \{1, 2, 3, 4\} \setminus \{j, i, k\}\).

If \(\chi\) is equal to a permutation of \(\{5.2\}\), then \(c[P(\chi), P(\pi)] = 0\). Using Corollary \(4.5\) (recall that \(\chi(\chi) \neq 0\)), \(c[P(\alpha \circ \chi \circ \alpha^{-1}), P(abc)] = 0\). Since \(\alpha \circ \chi \circ \alpha^{-1}(\alpha(i)) = \alpha(j)\) or \(\alpha \circ \chi \circ \alpha^{-1}(\alpha(j)) = \alpha(k)\), we conclude that \((abc) = (\alpha(j)\alpha(i)\alpha(k))\).

If \(\chi\) is equal to a permutation of \(\{5.1\}\), then \(c[P(\chi), P(\pi)] = 2\). Using Corollary \(4.5\) \(c[P(\alpha \circ \chi \circ \alpha^{-1}), P(abc)] = 2\). Since \((abc)(\alpha(l)) = \alpha(l)\), we conclude that \((abc) = (\alpha(j)\alpha(i)\alpha(k))\). Therefore, \(P(\theta) = T(P(jik)) = P(\alpha(j)\alpha(i)\alpha(k)) \in U_{\alpha(i),\alpha(j)}\).

If \(\pi = (jikl)\) is a cycle of length 4, with \(i, j, k, l\) distinct on pairs, then \(c[I, P(\pi)] = 0 = c[I, P(\theta)]\). Considering the transposition \((ij)\) and using Corollary \(4.5\) we get \(c[P(ij), P(\pi)] = 1 = c[P(\alpha \circ (ij) \circ \alpha^{-1}), P(\theta)]\). Then

\[\theta(\alpha(i)) = \alpha(i) \quad \text{or} \quad \theta(\alpha(i)) = \alpha(j).\]

Suppose that \(\theta(\alpha(i)) = \alpha(j)\). Considering the permutation \((jik)\) and using Corollary \(4.5\) we get \(c[P(jik), P(\pi)] = 2 = c[P(\alpha \circ (jik) \circ \alpha^{-1}), P(\theta)]\). Then

\[\theta(\alpha(i)) = \alpha(k) \quad \text{and} \quad \theta(\alpha(k)) = \alpha(j).\]

So, \(\alpha(k) = \theta(\alpha(i)) = \alpha(j)\). Impossible because \(\theta\) is a permutation. Consequently, \(\theta(\alpha(j)) = \alpha(i)\) and \(P(\theta) = T(P(jikl)) \in U_{\alpha(i),\alpha(j)}\).

Case 4. Let \(n = 4\) and \(\chi = [2, 2]\). Since \(p = 3\), if \(\pi\) is a cycle of length 3, then the result is obtained using Proposition \(5.2\). If \(\pi\) is a cycle of length 2, then the result is obtained using Proposition \(4.7\).

Let \(i, j, k, l\) distinct on pairs. Let \(\pi = (ij) \circ (kl)\) then

\[c[P(ij), P(\pi)] = 2 = c[P(\alpha(i)\alpha(j)), T(P(\pi))]\]

(in this case, \(\chi((ij) \circ (kl)) = 2 \neq 0\)). Since \(c[I, P(\pi)] = 0\) then \(c[I, T(P(\pi))] = 0\). So, \(\theta(\alpha(i)) = \alpha(j)\) and \(\theta(\alpha(j)) = \alpha(i)\). Therefore, \(P(\theta) \in U_{\alpha(i),\alpha(j)}\).

Let \(\pi = (jikl)\) with \(i, j, k, l\) distinct on pairs, then

\[c[P(jik), P(\pi)] = 2 = c[P(\alpha(j)\alpha(i)\alpha(k)), T(P(\pi))]\]

(in this case, \(\chi(jikl) = -1 \neq 0\)). Since \(c[I, P(\pi)] = 0\) then \(c[I, T(P(\pi))] = 0\). So, we must have two of these cases, \(\theta(\alpha(j)) = \alpha(i)\) or \(\theta(\alpha(i)) = \alpha(k)\) or \(\theta(\alpha(k)) = \alpha(j)\). (recall that \(P(\theta) = T(P(\pi))\)). In the
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same way, using \((jil)\) we must have two of these cases, \(\theta(\alpha(i)) = \alpha(l)\) or \(\theta(\alpha(l)) = \alpha(j)\) or \(\theta(\alpha(j)) = \alpha(i)\). If \(\theta(\alpha(j)) \neq \alpha(i)\) then \(\theta(\alpha(i)) = \alpha(k)\), \(\theta(\alpha(k)) = \alpha(j)\) and \(\theta(\alpha(i)) = \alpha(l)\). Impossible because \(\theta\) is a permutation. Consequently, \(\theta(\alpha(j)) = \alpha(i)\).

Therefore, \(P(\theta) = T(P(ijkl)) \in U_{\alpha(i), \alpha(j)}\).

The proof of part 2) is analogous. □

Now we are in conditions to prove the main result of this paper.

\[\text{Proof of Theorem 1.2.}\] If there are \(\sigma, \alpha \in S_n\), with \(\chi(\sigma) = \chi(id)\), such that
\[T(S) = P(\sigma)P(\alpha)SP(\alpha^{-1}),\]
for all \(S \in \Omega_n\), we have that
\[d_\chi(T(S)) = \sum_{\pi \in S_n} \chi(\pi) \prod_{j=1}^{n} T(S)_{j\pi(j)} = \sum_{\rho \in S_n} \chi(\rho \circ \alpha^{-1} \circ \sigma^{-1}) \prod_{j=1}^{n} S_{j\rho(j)}.\]

Since \(\chi(\sigma) = \chi(id)\) then \(\chi(\rho \circ \alpha^{-1} \circ \sigma^{-1}) = \chi(\rho \circ \alpha^{-1}) = \chi(\rho)\) (see Remark 2.1). Consequently, \(d_\chi(T(S)) = \sum_{\rho \in S_n} \chi(\rho) \prod_{j=1}^{n} S_{j\rho(j)} = d_\chi(S)\). Therefore, the map \(T\) preserves \(d_\chi\).

The proof of the case when \(T(S) = P(\sigma)P(\alpha)STP(\alpha^{-1})\) is similar.

Conversely, suppose that the map \(T\) preserves \(d_\chi\) and is unital.

Let \(p\) be the number of boundary boxes of the Young Diagram associated with \(\chi\) and let \(\alpha \in S_n\) obtained using Proposition 4.7

\[\text{Claim 1.}\] Let \(P\) be a permutation matrix, such that \(P \in U_{ii}\). Then \(T(P) \in U_{\alpha(i), \alpha(i)}\).

\[\text{Proof of Claim 1.}\] Suppose that \(P = P(\pi)\) with \(\pi \in S_n\). Let \(k = c[P, I]\). By Corollary 4.5 \(k = c[T(P), I]\). Let \(i_1, \ldots, i_{n-k}\) be distinct on pairs, such that \(\pi(i_j) \neq i_j\), for all \(j \in \{1, \ldots, n-k\}\).

Assume that \(\xi\) is a cycle of length \(p\), \(T(P(\xi)) = P(\rho)\), with \(\rho = \alpha \circ \xi \circ \alpha^{-1}\) (condition 1) of Proposition 5.3. Since \(P \in U_{\pi(i_j)j}\), then \(T(P) \in U_{\alpha(\pi(i_j))\alpha(i_j)}\), for all \(j \in \{1, \ldots, n-k\}\). As \(k = c[T(P), I]\), then \(T(P) \in U_{rr},\) where \(r \in \{1, \ldots, n\} \\setminus \{\alpha(i_1), \ldots, \alpha(i_{n-k})\}\).

Let us consider \(p_k\), for all \(t \in \{1, \ldots, k\}\), such that \(\alpha(p_k) = r_t\), then \(p_1, \ldots, p_k \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_{n-k}\}\). Since \(P \in U_{i_t}\) then \(\pi(i) = i_t\) and there exists \(p_j \in \{p_1, \ldots, p_k\}\) such that \(p_j = i_t\). Since \(\alpha(i) = \alpha(p_j) = r_j\) then \(T(P) \in U_{\alpha(\alpha(i))\alpha(i)}\).

If we are in the condition 2) of Proposition 5.3 the proof is analogous. □

\[\text{Claim 2.}\] Assume that \(\xi\) is a cycle of length \(p\), and \(T(P(\xi)) = P(\rho)\). Then one of the following conditions must hold:

1. If \(\rho = \alpha \circ \xi \circ \alpha^{-1}\), then \(T(U_{i,j}) = U_{\alpha(i), \alpha(j)}, \forall i, j\).
2. If \(\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}\), then \(T(U_{i,j}) = U_{\alpha(j), \alpha(i)}, \forall i, j\).

\[\text{Proof of Claim 2.}\] By Propositions 5.3 and Claim 1, we know that

1. If \(\rho = \alpha \circ \xi \circ \alpha^{-1}\), then \(T(U_{i,j}) \subseteq U_{\alpha(i), \alpha(j)}, \forall i, j\);
2. If \(\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}\), then \(T(U_{i,j}) \subseteq U_{\alpha(j), \alpha(i)}, \forall i, j\).
Since

$$\varphi : U_{i,j} \rightarrow U_{k,l}$$

$$P \mapsto P(ik)PP(jl)$$

is a bijective map, then

$$|U_{i,j}| = |U_{k,l}|, \ \forall i,j,k,l.$$ So,

1. if $$\rho = \alpha \circ \xi \circ \alpha^{-1}$$, then $$T(U_{i,j}) = U_{\alpha(i),\alpha(j)}$$, $$\forall i,j$$;
2. if $$\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}$$, then $$T(U_{i,j}) = U_{\alpha(j),\alpha(i)}$$, $$\forall i,j$$.

Claim 3. Assume that $$\xi$$ is a cycle of length $$p$$, and $$T(P(\xi)) = P(\rho)$$. Then one of the following conditions must hold:

1. if $$\rho = \alpha \circ \xi \circ \alpha^{-1}$$, then $$T(A) = P(\alpha)AP(\alpha^{-1})$$, for all $$A \in \Omega_n$$.
2. if $$\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}$$, then $$T(A) = P(\alpha)A^TP(\alpha^{-1})$$, for all $$A \in \Omega_n$$.

Proof of Claim 3. Since there exist $$\sigma_1, \ldots, \sigma_t \in S_n$$ and $$\lambda_1, \ldots, \lambda_t \in [0,1]$$ with $$\lambda_1 + \cdots + \lambda_t = 1$$ such that $$A = \lambda_1 P(\sigma_1) + \cdots + \lambda_t P(\sigma_t)$$ then

1. if $$\rho = \alpha \circ \xi \circ \alpha^{-1}$$, by Claim 2,

$$T(A) = T(\lambda_1 P(\sigma_1) + \cdots + \lambda_t P(\sigma_t)) = \lambda_1 T(P(\sigma_1)) + \cdots + \lambda_t T(P(\sigma_t))$$

$$= \lambda_1 P(\alpha \circ \sigma_1 \circ \alpha^{-1}) + \cdots + \lambda_t P(\alpha \circ \sigma_t \circ \alpha^{-1})$$

$$= P(\alpha)(\lambda_1 P(\sigma_1) + \cdots + \lambda_t P(\sigma_t))P(\alpha^{-1})$$

$$= P(\alpha)AP(\alpha^{-1}).$$

2. If $$\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}$$, by Claim 2,

$$T(A) = T(\lambda_1 P(\sigma_1) + \cdots + \lambda_t P(\sigma_t)) = \lambda_1 T(P(\sigma_1)) + \cdots + \lambda_t T(P(\sigma_t))$$

$$= \lambda_1 P(\alpha \circ \sigma_1^{-1} \circ \alpha^{-1}) + \cdots + \lambda_t P(\alpha \circ \sigma_t^{-1} \circ \alpha^{-1})$$

$$= P(\alpha)(\lambda_1 P(\sigma_1^{-1}) + \cdots + \lambda_t P(\sigma_t^{-1}))P(\alpha^{-1})$$

$$= P(\alpha)A^TP(\alpha^{-1}).$$

Using Corollary 2.4 we have that if $$\chi \neq [2,2]$$, then $$T(I) = I$$. By Claim 3 and Corollary 4.2, the map $$T$$ must have one of the forms (1) or (2).

If the map $$T$$ is nonunital, then $$T(I) \neq I$$, and in this case, by Corollary 2.4 we must have $$\chi = [2,2]$$. Since $$T(I) = P(\sigma)$$ with $$\chi(\sigma) = \chi(id)$$, we can consider the semilinear map $$\Phi$$ defined by $$\Phi(S) = T(I)^{-1}T(S)$$, since $$T(I)$$ is invertible. The map $$\Phi$$ is unital, and

$$d_\chi(\Phi(S)) = d_\chi(T(I)^{-1}T(S)) = d_\chi(P(\sigma^{-1})T(S)).$$
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Using Remark 2.1 and

\[ d_\chi(P(\sigma^{-1})T(S)) = \sum_{\rho \in S_4} \chi(\rho) \prod_{j=1}^{4} (P(\sigma^{-1})T(S))_{j\rho(j)} = \sum_{\pi \in S_4} \chi(\pi \circ \sigma) \prod_{j=1}^{4} (T(S))_{j\pi(j)} \]

\[ = \sum_{\pi \in S_4} \chi(\pi) \prod_{j=1}^{4} (T(S))_{j\pi(j)} = d_\chi(T(S)) = d_\chi(S), \]

we conclude that \( \Phi \) preserves \( d_\chi \).

By Claim 3 and Corollary 4.2 the result follows.

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