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APPLICATION OF AN IDENTITY FOR SUBTREES WITH A GIVEN EIGENVALUE

KENJI TOYONAGA† AND CHARLES R. JOHNSON‡

Abstract. For an Hermitian matrix whose graph is a tree and for a given eigenvalue having Parter vertices, the possibilities for the multiplicity are considered. If $V = \{v_1, \ldots, v_k\}$ is a fragmenting Parter set in a tree relative to the eigenvalue $\lambda$, and $T_{i+1}$ is the component of $T - \{v_1, v_2, \ldots, v_i\}$ in which $v_{i+1}$ lies, it is shown that $\sum_i N_i = m_A(\lambda) + 2k - 1$, in which $N_i$ is the number of components of $T_i - v_i$ in which $\lambda$ is an eigenvalue. This identity is applied to make several observations, including about when a set of strong Parter vertices leaves only 3 components with $\lambda$ and about multiplicities in binary trees. Furthermore, it is shown that one can construct an Hermitian matrix whose graph is a tree that has a strong Parter set $V$ such that $|V| = k$ for each $k$ in $1 \leq k \leq m - 1$ for given multiplicity $m \geq 2$ of an eigenvalue $\lambda$. Finally, some examples are given, in which the notion of a fragmenting Parter set is used.

Key words. Tree, Eigenvalues, Hermitian matrices, Multiplicity, Parter vertex.

AMS subject classifications. 05C05, 15A18, 15A57, 13H15, 05C50.

1. Introduction. For an undirected graph $G$ on $n$ vertices, denote by $\mathcal{H}(G)$ the set of all $n$-by-$n$ Hermitian matrices with graph $G$. No requirement, other than reality, is placed upon the diagonal entries of $A \in \mathcal{H}(G)$. Let $\sigma(A)$ denote the eigenvalues of $A$, including multiplicities. For $\lambda \in \sigma(A)$, we denote the multiplicity of $\lambda$, as an eigenvalue of $A$, by $m_A(\lambda)$. When there is an identified $A \in \mathcal{H}(G)$, we often speak interchangeably about the graph and the matrix, for convenience.

Our interest here is in the case in which $G$ is a tree $T$ and $A \in \mathcal{H}(T)$ is an Hermitian matrix. In that event, when $A \in \mathcal{H}(T)$ and $A(\lambda) \geq 2$, there is remarkable structure present [2, 7, 10], and there may be such structure even when $m_A(\lambda) < 2$ [2]. The multiplicities of eigenvalues of an Hermitian matrix whose graph is a tree have been studied in many papers. Parter sets or P-sets for an eigenvalue have been studied in several papers [1, 2, 4, 5, 6] as well.
For a vertex $u$ of $T$, we denote the $(n-1)$-by-$(n-1)$ principal submatrix of $A \in \mathcal{H}(T)$, resulting from deletion of the row and column corresponding to $u$, by $A(u)$; its graph is $T - u$. A vertex $u$ of $T$ is called a Parter vertex \cite{2} if $m_{A(u)}(\lambda) = m_A(\lambda) + 1$. If $m_A(\lambda) \geq 2$, there is always at least one Parter vertex in $T$, and there may be Parter vertices even when $m_A(\lambda) < 2$. Furthermore, if $m_A(\lambda) \geq 2$, then there will be a Parter vertex $u$ such that $\lambda$ occurs as an eigenvalue in at least 3 principal submatrices of $A$, corresponding to branches of $T$ at the Parter vertex $u$ of $T$ \cite{2,7}. In this case, $u$ is called a strong Parter vertex \cite{2}. In general, vertex $u$ is Parter if and only if it has a neighbor $w$ in $T$ such that $m_{A[T_u - w]}(\lambda) = m_{A[T_u]}(\lambda) - 1$ in which $A[S]$ denotes the principal submatrix of $A$ corresponding to the subgraph $S$ of $T$, $T_w$ is the branch of $T$ at $u$ and containing $w$, and $T_w - w$ is the subtree of $T_w$ induced by deletion of $w$. Such a vertex in a tree is called a downer vertex and such a branch a downer branch \cite{2}.

A set $V$ of vertices $\{v_1, \ldots, v_k\}$ is called a Parter set for $\lambda \in \sigma(A), A \in \mathcal{H}(T)$ if $m_{A[T_{v_1} - v_1]}(\lambda) = m_A(\lambda) + k$. By the interlacing inequalities, it is clear that each vertex of $V$ is Parter in $T$. (The converse is not generally true.) If $T$ has a Parter vertex for $\lambda \in \sigma(A), A \in \mathcal{H}(T)$, its removal from $T$ will leave some components (subtrees) in which $\lambda$ is an eigenvalue of the corresponding principal submatrix of $A$. Some of these components may also include Parter vertices (relative to the component). Repeated removal of such Parter vertices in each component will eventually leave components in each of which $\lambda$ appears no more than once. In this event, the Parter set of removed vertices is called a fragmenting Parter set (f-Parter set, for short).

In the field of validated numerical analysis, it is generally considered that it is difficult to enclose a multiple eigenvalue with large multiplicity in a narrow interval. In \cite{9}, a numerical method for validating existence of an eigenvalue with large multiplicity of an Hermitian matrix whose graph is a tree is given; the fragmenting subgraph obtained by removing Parter vertices and software INTLAB \cite{8} is used. INTLAB is the Matlab toolbox for reliable computing and self-validating algorithms. The notion of fragmenting subgraphs can be applied to numerical analysis. So, it is important to study the property of fragmenting subgraphs obtained by deleting Parter vertices in a tree.

In Figure 1.1, we give an example of an f-Parter set for a tree $T$ with an eigenvalue $\lambda$ of multiplicity 3. The numbers in Figure 1.1 represent the multiplicity of an eigenvalue $\lambda$ in the subgraph under the vertex next to the number. We can easily construct an Hermitian matrix $A$ with eigenvalue $\lambda$ whose graph is the tree $T$ in Figure 1.1. If we suppose Parter vertices in $V$ are sequentially removed in the order $v_1, v_2$, then $v_i$ is a strong Parter vertex in $T_i, i = 1, 2$. Therefore, $V = \{v_1, v_2\}$ is a fragmenting strong Parter set (strong f-Parter set, for short) for $\lambda \in \sigma(A), A \in \mathcal{H}(T)$.
2. Main results. Our purpose here is to prove an identity for the system of subtrees associated with the sequential removal of vertices in an f-Parter set that is interesting by itself (Theorem 2.1) and then to apply the identity in several ways.

Let \( V = \{v_1, \ldots, v_k\} \) be a Parter set for \( \lambda \in \sigma(A), A \in \mathcal{H}(T) \). Let \( T_{i+1} \) be the component of \( T - \{v_1, \ldots, v_i\} \) in which vertex \( v_{i+1} \) lies, \( i = 0, \ldots, k - 1 \). We set \( T_1 = T \). If, further, \( v_{i+1} \) is a strong Parter in \( T_{i+1}, i = 0, \ldots, k - 1 \), we call \( V \) a strong Parter set. When the multiplicity of an eigenvalue \( \lambda \) is given, we consider upper bounds for the cardinality of a strong f-Parter set of the tree for eigenvalue \( \lambda \). Further, in Theorem 2.2, we characterize a certain maximality of strong f-Parter sets. And we consider the number of components in which \( \lambda \) occurs as an eigenvalue exactly once, when we remove the strong Parter vertices in a strong f-Parter set. In Proposition 2.5, when \( T \) is a full binary tree, we give upper bounds for the cardinality of a strong f-Parter set. In Theorem 2.6, we show that given the multiplicity \( m \geq 2 \) of an eigenvalue, there exist Hermitian matrices, whose graph is a tree, that have a strong Parter set \( V \) for the eigenvalue with \( |V| = k \) for each \( k \) in \( 1 \leq k \leq m - 1 \). Finally, we give simple examples to illustrate our results.

We also define \( N_i \) to be the number of components of \( T_i - v_i \) in which \( \lambda \) is an eigenvalue of the corresponding principal submatrix of \( A \). Of course, if \( V \) is a strong Parter set, then \( N_i \geq 3 \), for all \( i \).

When we remove Parter vertices in \( V \) sequentially, the number of components in which \( \lambda \) is an eigenvalue of the corresponding principal submatrix of \( A \) satisfies the next relation.

**Theorem 2.1.** Let \( T \) be a tree and \( A \in \mathcal{H}(T) \). If \( V = \{v_1, v_2, \ldots, v_k\} \) is an f-Parter set for \( \lambda \in \sigma(A) \) with \( m_{A(T_i)}(\lambda) \geq 1, 1 \leq i \leq k \), then

\[
\sum_{i=1}^{k} N_i = m_A(\lambda) + 2k - 1.
\]

**Proof.** We give a proof by induction on \( k \). When \( k = 1 \), the formula holds, because of the definition of a Parter vertex. We suppose that when \( |V| \leq k \), the formula
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holds. When $|V| = k + 1$, we denote the vertices in $V$ as $V = \{v_1, v_2, \ldots, v_{k+1}\}$. We denote by $B_1, B_2, \ldots, B_l, l \geq 1$, the branches at $v_1$ in which $\lambda$ is an eigenvalue of the corresponding principal submatrix. Let $k_i$ be the number of Parter vertices in $B_i$ that are contained in $V$. Then $0 \leq k_i \leq k$, and $k_1 + \cdots + k_l = k$. The Parter vertices in $V$ that are contained in $B_i$ must be an $f$-Parter set in $B_i$. By assumption, the next relations hold in each branch $B_i$ respectively, in which $N_{ji}$ denotes $N_i$ in branch $B_j$:

$$\sum_{i=1}^{k_1} N_{1i} = m_{A[B_1]}(\lambda) + 2k_1 - 1;$$

$$\sum_{i=1}^{k_2} N_{2i} = m_{A[B_2]}(\lambda) + 2k_2 - 1;$$

$$\vdots$$

$$\sum_{i=1}^{k_l} N_{li} = m_{A[B_l]}(\lambda) + 2k_l - 1.$$

Here, if $k_j = 0$, then $m_{A[B_j]}(\lambda) = 1$, and we set $\sum_{i=1}^{k_j} N_{ji} = 0$. By adding the left hand sides and right hand sides of these equations, we get

$$\sum_{j=1}^{l} \sum_{i=1}^{k_j} N_{ji} = \sum_{j=1}^{l} m_{A[B_j]}(\lambda) + 2k - l.$$

Equivalently

$$\sum_{j=1}^{l} \sum_{i=1}^{k_j} N_{ji} + l = m_{A}(\lambda) + 2k + 1,$$

or

$$\sum_{i=1}^{k+1} N_i = m_{A}(\lambda) + 2(k + 1) - 1.$$

The claim thus follows by induction.

In the above theorem, we note that $V$ is an $f$-Parter set, but $V$ does not necessarily need to be a strong $f$-Parter set. Of course, if $V$ is a strong $f$-Parter set,
then Theorem 2.1 still holds. Furthermore, as Example 3.2 shows, we note that the condition $m_{A[T_i]}(\lambda) \geq 1, 1 \leq i \leq k$ is necessary. Examples illustrating Theorem 2.1 occur in Example 3.2 later.

Next we consider the possible cardinalities of a strong f-Parter set for $\lambda \in \sigma(A)$ in terms of $m_A(\lambda)$.

**Theorem 2.2.** If $T$ is a tree, $A \in H(T)$ and $V = \{v_1, \ldots, v_k\}$ is a strong f-Parter set for $\lambda \in \sigma(A)$, then

$$|V| \leq m_A(\lambda) - 1,$$

and $V$ is maximal, that is $|V| = m_A(\lambda) - 1$, if and only if $N_i = 3$ for all $i, 1 \leq i \leq k$.

**Proof.** Since $V$ is a strong f-Parter set for $\lambda \in \sigma(A)$, by Theorem 2.1,

$$3|V| \leq \sum_{i=1}^{k} N_i = m_A(\lambda) + 2|V| - 1.$$

So, $|V| \leq m_A(\lambda) - 1$.

From Theorem 2.1, we have $\sum_{i=1}^{k} (N_i - 3) = m_A(\lambda) - (|V| + 1)$. Since $V$ is a strong f-Parter set, $N_i \geq 3$. So, $|V| = m_A(\lambda) - 1$ if and only if $N_i = 3$ for all $i, 1 \leq i \leq k$.

The above formula gives an upper bound for $V$ in terms of $m_A(\lambda)$ and a lower bound for $m_A(\lambda)$ in terms of $|V|$.

**Remark.** If $V$ is a strong f-Parter set for $\lambda \in \sigma(A), A \in H(T)$, $T$ a tree, then $T - V$ has $m_A(\lambda) + |V|$ components with $\lambda$ as an eigenvalue, each of multiplicity 1.

From Theorem 2.2, given a positive integer $n$, we can construct an Hermitian matrix, whose graph is a tree, that has a maximal strong Parter set $V$ with $|V| = n - 1$ in which $N_i = 3, 1 \leq i \leq n - 1$.

**Corollary 2.3.** Let $T$ be a tree and $A \in H(T)$. If $V = \{v_1, \ldots, v_k\}$ is an f-Parter set for $\lambda \in \sigma(A)$ with $m_{A[T_i]}(\lambda) \geq 1, 1 \leq i \leq k$, then of the system of $\sum_{i=1}^{k} N_i$ subtrees of $T$ counted in Theorem 2.1, $k - 1$ subtrees have $\lambda$ with multiplicity $> 1$.

**Proof.** From Theorem 2.1,

$$\sum_{i=1}^{k} N_i - (m_A(\lambda) + k) = \sum_{i=1}^{k} N_i - (m_A(\lambda) + |V|) = k - 1.$$

From the above remark, $k - 1$ subtrees at Parter vertices have $\lambda$ with multiplicity $> 1$. 

Corollary 2.4. If $V$ is a strong $f$-Parter set for $\lambda \in \sigma(A)$, $A \in \mathcal{H}(T)$, $T$ a tree, then the number $c$ of components of $T - V$ in which $\lambda$ occurs as an eigenvalue exactly once satisfies
\[ m_A(\lambda) + 1 \leq c \leq 2 m_A(\lambda) - 1. \]

Proof. Theorem 2.2 implies that $1 \leq |V| \leq m_A(\lambda) - 1$. Thus,
\[ m_A(\lambda) + 1 \leq |V| + m_A(\lambda) \leq 2 m_A(\lambda) - 1. \]
Since $V$ is a strong $f$-Parter set, $c = m_A(\lambda) + |V|$. \(\Box\)

A tree $T$ is binary if the degree of each vertex is at most 3. If there are no vertices of degree 2, we call it a full binary tree.

In a full binary tree $T$ on $v$ vertices, if $V$ is a strong $f$-Parter set for $\lambda \in \sigma(A)$, $A \in \mathcal{H}(T)$, we can get an upper bound for $|V|$ and $m_A(\lambda)$ in terms of $v$.

Proposition 2.5. If $V$ is a strong $f$-Parter set for $\lambda \in \sigma(A)$, $A \in \mathcal{H}(T)$, $T$ a full binary tree on $v$ vertices then
\[ |V| \leq \lfloor (v - 1)/3 \rfloor \]
and
\[ m_A(\lambda) \leq \lfloor (v + 2)/3 \rfloor. \]

Proof. Each strong Parter vertex in $V$ has three incident edges in $T_i$, $1 \leq i \leq |V|$. Since $T$ has $v - 1$ edges, it is clear that $|V| \leq \lfloor (v - 1)/3 \rfloor$. Since $T$ is a full binary tree, every strong Parter vertex in $V$ has three branches in which $\lambda$ occurs as an eigenvalue. So, $|V| = m_A(\lambda) - 1$ from Theorem 2.2. Therefore, we get an upper bound for $m_A(\lambda)$ in terms of $v$ from the upper bound of $V$. \(\Box\)

Theorem 2.6. Given multiplicity $m_A(\lambda) = m \geq 2$ and $1 \leq k \leq m - 1$, there exists an Hermitian matrix $A$ whose graph is a tree, that has a strong Parter set $V$ of size $k$ for $\lambda \in \sigma(A)$.

Proof. In the case of $m = 2$, there exists an Hermitian matrix $A$ whose graph is a star, whose submatrices of $A$ corresponding to three branches at a center of the star have a simple eigenvalue $\lambda$. Then the center is a strong Parter vertex and $|V| = 1$. Next we consider the case of $m \geq 3$. Given multiplicity $m \geq 3$ of an eigenvalue $\lambda$, we can construct an Hermitian matrix $A$ whose graph is a tree $T$ such that $|V| = m - 1$ with $N_i = 3$, $1 \leq i \leq m - 1$, from Theorem 2.2. Then let
V be \( \{p_1, p_2, \ldots, p_{m-1}\} \), and \( T_i \) be a component that contains the Parter vertex \( p_i \) when Parter vertices \( \{p_1, \ldots, p_{i-1}\} \) are removed from \( T \) (\( 2 \leq i \leq m - 1 \)). We set \( T_1 = T \). Then there exists \( k \) such that branches at \( p_k \) in \( T_k \) have the eigenvalue \( \lambda \) with multiplicity at most 1. If \( m_{A[T_k]}(\lambda) = l \), when \( p_k \) is removed from \( T_k \), there exists \( l + 1 \) branches that have the eigenvalue \( \lambda \) with multiplicity 1 in \( T_k \), which we denote by \( T_1, \ldots, T_{l+1} \). When we remove the Parter vertex \( p_k \) from \( T_k \) and connect \( T_1, \ldots, T_l \) to a Parter vertex that exists in the above position of \( p_k \) in \( T \) by inserting \( l \) new edges (cf. Example 3.3), then the graph contains \( m - 2 \) strong Parter vertices. By repeating this procedure, we can construct a graph such that \( |V| = k \) for all \( k \), \( 1 \leq k \leq m - 1 \).

### 3. Examples

We present examples for Theorem 2.2, Theorem 2.1 and Theorem 2.6.

**Example 3.1.** We give an example for Theorem 2.2 in Figure 3.1. Let \( T \) be a tree, \( A \in \mathcal{H}(T) \) with an eigenvalue \( \lambda \). The maximal cardinality of a strong Parter set for \( \lambda \) relative to \( A \) is 3 from Theorem 2.2. Then there are two patterns of multiplicities as displayed.

The strong f-Parter set of \( T \) for \( \lambda \) relative to \( A \) is \( V = \{v_1, v_2, v_3\} \), and when Parter vertices in \( V \) are sequentially removed in the order \( v_1, v_2, \) and \( v_3 \). Then \( v_i \) is a strong Parter vertex in \( T_i \). The numbers in the figures represent the multiplicity of the eigenvalue \( \lambda \) relative to the submatrix of \( A \) corresponding to the subgraph under the vertex next to the number.

When Parter vertices are sequentially removed in the order \( v_1, v_2, \) and \( v_3 \), we can see that only three components in \( T_i \), \( 1 \leq i \leq 3 \) have the eigenvalue \( \lambda \) in each subgraph.

**Fig. 3.1.**

**Example 3.2.** Here are examples for Theorem 2.1. Let \( T \) be a tree, \( A \in \mathcal{H}(T) \) with an eigenvalue \( \lambda \). We show the identity \( \sum_{i=1}^{k} N_i = m_{A} (\lambda) + 2k - 1 \) holds in the cases of multiplicity \( m_{A[T]}(\lambda) = 2, 3, \) and 4, in which \( V \) is an f-Parter set, \( |V| = k \), and \( N_i \) is the number of components of \( T_i - v_i \) in which \( \lambda \) is an eigenvalue of the corresponding principal submatrix of \( A \). When \( m_{A[T]}(\lambda) = 2 \), the formula in Theorem
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2.1 holds from $|V| = k = 1$ and $N_1 = 3$. So we show the cases of $m_{A[T]}(\lambda) = 3$ and $m_{A[T]}(\lambda) = 4$ in Figure 3.2. The numbers in the figures represent the multiplicity of the eigenvalue $\lambda$ relative to the submatrix of $A$ corresponding to the subgraph under the vertex next to the number. Parter vertices in the $f$-Parter set are the vertices next to the number 2, 3 and 4. When $m_{A[T]}(\lambda) = 3$ there are two cases in Figure 3.2, then the formula in Theorem 2.1 holds for $k$ and $N_i$, that is, $k = 1$, $N_1 = 4$ at the left graph, and $k = 2$, $N_1 + N_2 = 6$ at the right graph.

For the case of $m_{A[T]}(\lambda) = 4$, there are five cases. In the last figure, the subgraph with multiplicity 3 has the same two patterns as the case of $m_{A[T]}(\lambda) = 3$. For all the cases, we see that the formula in Theorem 2.1 holds.

We note that the condition $m_{A[T]}(\lambda) \geq 1$, $1 \leq i \leq k$ in Theorem 2.1 is indispensable. If this condition is not satisfied, then there is a case such that the formula in Theorem 2.1 does not hold. Let $A$ be an Hermitian matrix, and the corresponding graph be the tree $T$ in Figure 3.3, in which vertex $x_i$ of $T$ corresponds to row $i$ of $A$, $i = 1, \ldots, 7$. The matrix $A$ has an eigenvalue 0 with multiplicity 2, so $m_{A[T]}(0) = 2$. Then $x_1, x_6$ are Parter vertices of $T$ for 0 relative to $A$. Furthermore, $V = \{x_1, x_6\}$ is a Parter set of $T$ for 0 relative to $A$. We denote the subgraph of $T$ induced by vertices...
Let $T = T_1$, and $N_i$ be the number of components of $T_i - v_i$, $i = 1, 2$, in which 0 is an eigenvalue of the principal submatrix of $A$. Since the principal submatrix of $A$ corresponding to $T_2$ does not have eigenvalue 0, and $m_{A[T_2 \{v_2\}]}(0) = 1$, then $N_1 + N_2 = 4$. Now, since $|V| = k = 2$, $m_A(\lambda) + 2k - 1 = 5$. As a result, we can say that if $m_{A[T_i]}(\lambda) \geq 1$ is not satisfied, then there is a case such that the formula in Theorem 2.1 is not satisfied. So, the condition $m_{A[T_i]}(\lambda) \geq 1$ in Theorem 2.1 is essential.

**Example 3.3.** Finally, we give examples for Theorem 2.6 in Figure 3.4. For the case of $m_{A[T]}(\lambda) = 4$, we can construct Hermitian matrices whose graphs are trees such that $|V| = 1, 2$ and 3 respectively, with $V$ a strong Parter set for $\lambda$ relative to $A \in H(T)$. The numbers in the figures represent the multiplicity of the eigenvalue $\lambda$ relative to the submatrix of $A$ corresponding to the subgraph under the vertex next to the number.

From Theorem 2.2, we can construct an Hermitian matrix whose graph is a tree $T$ with $|V| = 3$ as the left graph in Figure 3.4, then strong Parter set $V = \{p_1, p_2, p_3\}$. When we remove the vertex $p_2$ from $T$, there exist three branches(vertices) at $p_2$ in $T_2$, which have the eigenvalue with multiplicity 1. Then we pick up two branches(vertices)
from them, and we connect them to the vertex $p_1$, then we can get a tree $T''$ with $|V| = 2$ with $V = \{p_1, p_3\}$. Furthermore, if we remove the vertex $p_3$ from $T''$, and connect two branches (vertices) at $p_3$ to the vertex $p_1$, then we have a star $T'''$, where $|V| = 1$ with $V = \{p_1\}$.

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