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APPLICATION OF AN IDENTITY FOR SUBTREES WITH A GIVEN EIGENVALUE∗

KENJI TOYONAGA† AND CHARLES R. JOHNSON‡

Abstract. For an Hermitian matrix whose graph is a tree and for a given eigenvalue having Parter vertices, the possibilities for the multiplicity are considered. If \( V = \{v_1, \ldots, v_k\} \) is a fragmenting Parter set in a tree relative to the eigenvalue \( \lambda \), and \( T_{i+1} \) is the component of \( T - \{v_1, v_2, \ldots, v_i\} \) in which \( v_{i+1} \) lies, it is shown that \( \sum_{i} N_i = m_A(\lambda) + 2k - 1 \), in which \( N_i \) is the number of components of \( T_i - v_i \) in which \( \lambda \) is an eigenvalue. This identity is applied to make several observations, including about when a set of strong Parter vertices leaves only 3 components with \( \lambda \) and about multiplicities in binary trees. Furthermore, it is shown that one can construct an Hermitian matrix whose graph is a tree that has a strong Parter set \( V \) such that \( |V| = k \) for each \( k \) in \( 1 \leq k \leq m - 1 \) for given multiplicity \( m \geq 2 \) of an eigenvalue \( \lambda \). Finally, some examples are given, in which the notion of a fragmenting Parter set is used.

Key words. Tree, Eigenvalues, Hermitian matrices, Multiplicity, Parter vertex.

AMS subject classifications. 05C05, 15A18, 15A57, 13H15, 05C50.

1. Introduction. For an undirected graph \( G \) on \( n \) vertices, denote by \( \mathcal{H}(G) \) the set of all \( n \)-by-\( n \) Hermitian matrices with graph \( G \). No requirement, other than reality, is placed upon the diagonal entries of \( A \in \mathcal{H}(G) \). Let \( \sigma(A) \) denote the eigenvalues of \( A \), including multiplicities. For \( \lambda \in \sigma(A) \), we denote the multiplicity of \( \lambda \), as an eigenvalue of \( A \), by \( m_A(\lambda) \). When there is an identified \( A \in \mathcal{H}(G) \), we often speak interchangeably about the graph and the matrix, for convenience.

Our interest here is in the case in which \( G \) is a tree \( T \) and \( A \in \mathcal{H}(T) \) is an Hermitian matrix. In that event, when \( A \in \mathcal{H}(T) \) and \( m_A(\lambda) \geq 2 \), there is remarkable structure present [2, 7, 10], and there may be such structure even when \( m_A(\lambda) < 2 \) [2]. The multiplicities of eigenvalues of an Hermitian matrix whose graph is a tree have been studied in many papers. Parter sets or P-sets for an eigenvalue have been studied in several papers [1, 2, 4, 5, 6] as well.
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For a vertex $u$ of $T$, we denote the $(n-1)$-by-$(n-1)$ principal submatrix of $A \in \mathcal{H}(T)$, resulting from deletion of the row and column corresponding to $u$, by $A(u)$; its graph is $T-u$. A vertex $u$ of $T$ is called a Parter vertex [2] if $m_{A(u)}(\lambda) = m_A(\lambda)+1$. If $m_A(\lambda) \geq 2$, there is always at least one Parter vertex in $T$, and there may be Parter vertices even when $m_A(\lambda) < 2$. Furthermore, if $m_A(\lambda) \geq 2$, then there will be a Parter vertex $u$ such that $\lambda$ occurs as an eigenvalue in at least 3 principal submatrices of $A$, corresponding to branches of $T$ at the Parter vertex $u$ of $T$ [2,7]. In this case, $u$ is called a strong Parter vertex [2]. In general, vertex $u$ is Parter if and only if it has a neighbor $w$ in $T$ such that $m_{A[T-u,w]}(\lambda) = m_{A[T_u]}(\lambda) - 1$ in which $A[S]$ denotes the principal submatrix of $A$ corresponding to the subgraph $S$ of $T$, $T_w$ is the branch of $T$ at $u$ and containing $w$, and $T_{w-w}$ is the subtree of $T_w$ induced by deletion of $w$. Such a vertex in a tree is called a downer vertex and such a branch a downer branch [2].

A set $V$ of vertices $\{v_1, \ldots, v_k\}$ is called a Parter set for $\lambda \in \sigma(A), A \in \mathcal{H}(T)$ if $m_{A[T-v]}(\lambda) = m_A(\lambda)+k$. By the interlacing inequalities, it is clear that each vertex of $V$ is Parter in $T$. (The converse is not generally true.) If $T$ has a Parter vertex for $\lambda \in \sigma(A), A \in \mathcal{H}(T)$, its removal from $T$ will leave some components (subtrees) in which $\lambda$ is an eigenvalue of the corresponding principal submatrix of $A$. Some of these components may also include Parter vertices (relative to the component). Repeated removal of such Parter vertices in each component will eventually leave components in each of which $\lambda$ appears no more than once. In this event, the Parter set of removed vertices is called a fragmenting Parter set (f-Parter set, for short).

In the field of validated numerical analysis, it is generally considered that it is difficult to enclose a multiple eigenvalue with large multiplicity in a narrow interval. In [9], a numerical method for validating existence of an eigenvalue with large multiplicity of an Hermitian matrix whose graph is a tree is given; the fragmenting subgraph obtained by removing Parter vertices and software INTLAB [8] is used. INTLAB is the Matlab toolbox for reliable computing and self-validating algorithms. The notion of fragmenting subgraphs can be applied to numerical analysis. So, it is important to study the property of fragmenting subgraphs obtained by deleting Parter vertices in a tree.

In Figure 1.1, we give an example of an f-Parter set for a tree $T$ with an eigenvalue $\lambda$ of multiplicity 3. The numbers in Figure 1.1 represent the multiplicity of an eigenvalue $\lambda$ in the subgraph under the vertex next to the number. We can easily construct an Hermitian matrix $A$ with eigenvalue $\lambda$ whose graph is the tree $T$ in Figure 1.1. If we suppose Parter vertices in $V$ are sequentially removed in the order $v_1, v_2$, then $v_i$ is a strong Parter vertex in $T_i$, $i=1,2$. Therefore, $V = \{v_1, v_2\}$ is a fragmenting strong Parter set (strong f-Parter set, for short) for $\lambda \in \sigma(A), A \in \mathcal{H}(T)$. 


2. Main results. Our purpose here is to prove an identity for the system of subtrees associated with the sequential removal of vertices in an f-Parter set that is interesting by itself (Theorem 2.1) and then to apply the identity in several ways.

Let \( V = \{v_1, \ldots, v_k\} \) be a Parter set for \( \lambda \in \sigma(A), A \in \mathcal{H}(T) \). Let \( T_{i+1} \) be the component of \( T - \{v_1, \ldots, v_i\} \) in which vertex \( v_{i+1} \) lies, \( i = 0, \ldots, k - 1 \). We set \( T_1 = T \). If, further, \( v_{i+1} \) is a strong Parter in \( T_{i+1}, i = 0, \ldots, k - 1 \), we call \( V \) a strong Parter set. When the multiplicity of an eigenvalue \( \lambda \) is given, we consider upper bounds for the cardinality of a strong f-Parter set of the tree for eigenvalue \( \lambda \). Further, in Theorem 2.2, we characterize a certain maximality of strong f-Parter sets. And we consider the number of components in which \( \lambda \) occurs as an eigenvalue exactly once, when we remove the strong Parter vertices in a strong f-Parter set. In Proposition 2.5, when \( T \) is a full binary tree, we give upper bounds for the cardinality of a strong f-Parter set. In Theorem 2.6, we show that given the multiplicity \( m \geq 2 \) of an eigenvalue, there exist Hermitian matrices, whose graph is a tree, that have a strong Parter set \( V \) for the eigenvalue with \( |V| = k \) for each \( k \) in \( 1 \leq k \leq m - 1 \). Finally, we give simple examples to illustrate our results.

We also define \( N_i \) to be the number of components of \( T_i - v_i \) in which \( \lambda \) is an eigenvalue of the corresponding principal submatrix of \( A \). Of course, if \( V \) is a strong Parter set, then \( N_i \geq 3 \), for all \( i \).

When we remove Parter vertices in \( V \) sequentially, the number of components in which \( \lambda \) is an eigenvalue of the corresponding principal submatrix of \( A \) satisfies the next relation.

**Theorem 2.1.** Let \( T \) be a tree and \( A \in \mathcal{H}(T) \). If \( V = \{v_1, v_2, \ldots, v_k\} \) is an f-Parter set for \( \lambda \in \sigma(A) \) with \( m_{A[T_i]}(\lambda) \geq 1, 1 \leq i \leq k \), then

\[
\sum_{i=1}^{k} N_i = m_A(\lambda) + 2k - 1.
\]

**Proof.** We give a proof by induction on \( k \). When \( k = 1 \), the formula holds, because of the definition of a Parter vertex. We suppose that when \( |V| \leq k \), the formula
holds. When $|V| = k + 1$, we denote the vertices in $V$ as $V = \{v_1, v_2, \ldots, v_{k+1}\}$. We denote by $B_1, B_2, \ldots, B_l$, $l \geq 1$, the branches at $v_1$ in which \( \lambda \) is an eigenvalue of the corresponding principal submatrix. Let $k_i$ be the number of Parter vertices in $B_i$ that are contained in $V$. Then $0 \leq k_i \leq k$, and $k_1 + \cdots + k_l = k$. The Parter vertices in $V$ that are contained in $B_i$ must be an $f$-Parter set in $B_i$. By assumption, the next relations hold in each branch $B_i$ respectively, in which $N_{ji}$ denotes $N_i$ in branch $B_j$:

$$
\sum_{i=1}^{k_1} N_{1i} = m_{A[B_1]}(\lambda) + 2k_1 - 1;
$$

$$
\sum_{i=1}^{k_2} N_{2i} = m_{A[B_2]}(\lambda) + 2k_2 - 1;
$$

$$
\vdots
$$

$$
\sum_{i=1}^{k_l} N_{li} = m_{A[B_l]}(\lambda) + 2k_l - 1.
$$

Here, If $k_j = 0$, then $m_{A[B_j]}(\lambda) = 1$, and we set $\sum_{i=1}^{k_j} N_{ji} = 0$. By adding the left hand sides and right hand sides of these equations, we get

$$
\sum_{j=1}^{l} \sum_{i=1}^{k_j} N_{ji} = \sum_{i=1}^{l} m_{A[B_i]}(\lambda) + 2k - l.
$$

Equivalently

$$
\sum_{j=1}^{l} \sum_{i=1}^{k_j} N_{ji} + l = m_A(\lambda) + 2k + 1,
$$

or

$$
\sum_{i=1}^{k+1} N_i = m_A(\lambda) + 2(k + 1) - 1.
$$

The claim thus follows by induction. \( \square \)

In the above theorem, we note that $V$ is an $f$-Parter set, but $V$ does not necessarily need to be a strong $f$-Parter set. Of course, if $V$ is a strong $f$-Parter set,
then Theorem 2.1 still holds. Furthermore, as Example 3.2 shows, we note that the condition \( m_{A[T_i]}(\lambda) \geq 1, 1 \leq i \leq k \) is necessary. Examples illustrating Theorem 2.1 occur in Example 3.2 later.

Next we consider the possible cardinalities of a strong f-Parter set for \( \lambda \in \sigma(A) \) in terms of \( m_A(\lambda) \).

**Theorem 2.2.** If \( T \) is a tree, \( A \in H(T) \) and \( V = \{v_1, \ldots, v_k\} \) is a strong f-Parter set for \( \lambda \in \sigma(A) \), then

\[
|V| \leq m_A(\lambda) - 1,
\]

and \( V \) is maximal, that is \( |V| = m_A(\lambda) - 1 \), if and only if \( N_i = 3 \) for all \( i, 1 \leq i \leq k \).

**Proof.** Since \( V \) is a strong f-Parter set for \( \lambda \in \sigma(A) \), by Theorem 2.1,

\[
3|V| \leq \sum_{i=1}^{k} N_i = m_A(\lambda) + 2|V| - 1.
\]

So, \( |V| \leq m_A(\lambda) - 1 \).

From Theorem 2.1, we have \( \sum_{i=1}^{k} (N_i - 3) = m_A(\lambda) - (|V| + 1) \). Since \( V \) is a strong f-Parter set, \( N_i \geq 3 \). So, \( |V| = m_A(\lambda) - 1 \) if and only if \( N_i = 3 \) for all \( i, 1 \leq i \leq k \).

The above formula gives an upper bound for \( V \) in terms of \( m_A(\lambda) \) and a lower bound for \( m_A(\lambda) \) in terms of \( |V| \).

**Remark.** If \( V \) is a strong f-Parter set for \( \lambda \in \sigma(A) \), \( A \in H(T) \), \( T \) a tree, then \( T - V \) has \( m_A(\lambda) + |V| \) components with \( \lambda \) as an eigenvalue, each of multiplicity 1.

From Theorem 2.2, given a positive integer \( n \), we can construct an Hermitian matrix, whose graph is a tree, that has a maximal strong Parter set \( V \) with \( |V| = n - 1 \) in which \( N_i = 3, 1 \leq i \leq n - 1 \).

**Corollary 2.3.** Let \( T \) be a tree and \( A \in H(T) \). If \( V = \{v_1, \ldots, v_k\} \) is an f-Parter set for \( \lambda \in \sigma(A) \) with \( m_{A[T_i]}(\lambda) \geq 1, 1 \leq i \leq k \), then of the system of \( \sum_{i=1}^{k} N_i \) subtrees of \( T \) counted in Theorem 2.1, \( k - 1 \) subtrees have \( \lambda \) with multiplicity \( > 1 \).

**Proof.** From Theorem 2.1,

\[
\sum_{i=1}^{k} N_i - (m_A(\lambda) + k) = \sum_{i=1}^{k} N_i - (m_A(\lambda) + |V|) = k - 1.
\]

From the above remark, \( k - 1 \) subtrees at Parter vertices have \( \lambda \) with multiplicity \( > 1 \).
Corollary 2.4. If $V$ is a strong f-Parter set for $\lambda \in \sigma(A)$, $A \in \mathcal{H}(T)$, $T$ a tree, then the number $c$ of components of $T - V$ in which $\lambda$ occurs as an eigenvalue exactly once satisfies

$$m_A(\lambda) + 1 \leq c \leq 2m_A(\lambda) - 1.$$ 

Proof. Theorem 2.2 implies that $1 \leq |V| \leq m_A(\lambda) - 1$. Thus,

$$m_A(\lambda) + 1 \leq |V| + m_A(\lambda) \leq 2m_A(\lambda) - 1.$$ 

Since $V$ is a strong f-Parter set, $c = m_A(\lambda) + |V|$. \qed

A tree $T$ is binary if the degree of each vertex is at most 3. If there are no vertices of degree 2, we call it a full binary tree.

In a full binary tree $T$ on $v$ vertices, if $V$ is a strong f-Parter set for $\lambda \in \sigma(A)$, $A \in \mathcal{H}(T)$, we can get an upper bound for $|V|$ and $m_A(\lambda)$ in terms of $v$.

Proposition 2.5. If $V$ is a strong f-Parter set for $\lambda \in \sigma(A)$, $A \in \mathcal{H}(T)$, $T$ a full binary tree on $v$ vertices then

$$|V| \leq \lceil (v - 1)/3 \rceil$$

and

$$m_A(\lambda) \leq \lceil (v + 2)/3 \rceil.$$ 

Proof. Each strong Parter vertex in $V$ has three incident edges in $T$, $1 \leq i \leq |V|$. Since $T$ has $v - 1$ edges, it is clear that $|V| \leq \lfloor (v - 1)/3 \rfloor$. Since $T$ is a full binary tree, every strong Parter vertex in $V$ has three branches in which $\lambda$ occurs as an eigenvalue. So, $|V| = m_A(\lambda) - 1$ from Theorem 2.2. Therefore, we get an upper bound for $m_A(\lambda)$ in terms of $v$ from the upper bound of $V$. \qed

Theorem 2.6. Given multiplicity $m_A(\lambda) = m \geq 2$ and $1 \leq k \leq m - 1$, there exists an Hermitian matrix $A$ whose graph is a tree, that has a strong Parter set $V$ of size $k$ for $\lambda \in \sigma(A)$.

Proof. In the case of $m = 2$, there exists an Hermitian matrix $A$ whose graph is a star, whose submatrices of $A$ corresponding to three branches at a center of the star have a simple eigenvalue $\lambda$. Then the center is a strong Parter vertex and $|V| = 1$. Next we consider the case of $m \geq 3$. Given multiplicity $m \geq 3$ of an eigenvalue $\lambda$, we can construct an Hermitian matrix $A$ whose graph is a tree $T$ such that $|V| = m - 1$ with $N_i = 3$, $1 \leq i \leq m - 1$, from Theorem 2.2. Then let
V be \{p_1, p_2, \ldots, p_{m-1}\}, and \( T_i \) be a component that contains the Parter vertex \( p_i \) when Parter vertices \( \{p_1, \ldots, p_{i-1}\} \) are removed from \( T \) (\( 2 \leq i \leq m - 1 \)). We set \( T_1 = T \). Then there exists \( k \) such that branches at \( p_k \) in \( T_k \) have the eigenvalue \( \lambda \) with multiplicity at most 1. If \( m_{A[T_k]}(\lambda) = l \), when \( p_k \) is removed from \( T_k \), there exists \( l + 1 \) branches that have the eigenvalue \( \lambda \) with multiplicity 1 in \( T_k \), which we denote by \( T_1, \ldots, T_{i+1} \). When we remove the Parter vertex \( p_k \) from \( T \) and connect \( T_1, \ldots, T_l \) to a Parter vertex that exists in the above position of \( p_k \) in \( T \) by inserting \( l \) new edges. (cf. Example 3.3), then the graph contains \( m - 2 \) strong Parter vertices. By repeating this procedure, we can construct a graph such that \( |V| = k \) for all \( k \), \( 1 \leq k \leq m - 1 \).

3. Examples. We present examples for Theorem 2.2, Theorem 2.1 and Theorem 2.6.

Example 3.1. We give an example for Theorem 2.2 in Figure 3.1. Let \( T \) be a tree, \( A \in \mathcal{H}(T) \) with an eigenvalue \( \lambda \) of multiplicity 4. The maximal cardinality of a strong Parter set for \( \lambda \) relative to \( A \) is 3 from Theorem 2.2. Then there are two patterns of multiplicities as displayed.

The strong \( f \)-Parter set of \( T \) for \( \lambda \) relative to \( A \) is \( V = \{v_1, v_2, v_3\} \), and when Parter vertices in \( V \) are sequentially removed in the order \( v_1, v_2, \) and \( v_3 \). Then \( v_i \) is a strong Parter vertex in \( T_i \). The numbers in the figures represent the multiplicity of the eigenvalue \( \lambda \) relative to the submatrix of \( A \) corresponding to the subgraph under the vertex next to the number.

When Parter vertices are sequentially removed in the order \( v_1, v_2, \) and \( v_3 \), we can see that only three components in \( T_i \), \( 1 \leq i \leq 3 \) have the eigenvalue \( \lambda \) in each subgraph.

![Fig. 3.1.](image)

Example 3.2. Here are examples for Theorem 2.1. Let \( T \) be a tree, \( A \in \mathcal{H}(T) \) with an eigenvalue \( \lambda \). We show the identity \( \sum_{i=1}^{k} N_i = m_A(\lambda) + 2k - 1 \) holds in the cases of multiplicity \( m_{A[T]}(\lambda) = 2, 3 \) and 4, in which \( V \) is an \( f \)-Parter set, \( |V| = k \), and \( N_i \) is the number of components of \( T_i - v_i \) in which \( \lambda \) is an eigenvalue of the corresponding principal submatrix of \( A \). When \( m_{A[T]}(\lambda) = 2 \), the formula in Theorem...
2.1 holds from $|V| = k = 1$ and $N_1 = 3$. So we show the cases of $m_{A[T]}(\lambda) = 3$ and $m_{A[T]}(\lambda) = 4$ in Figure 3.2. The numbers in the figures represent the multiplicity of the eigenvalue $\lambda$ relative to the submatrix of $A$ corresponding to the subgraph under the vertex next to the number. Parter vertices in the f-Parter set are the vertices next to the number 2, 3, and 4. When $m_{A[T]}(\lambda) = 3$ there are two cases in Figure 3.2, then the formula in Theorem 2.1 holds for $k$ and $N_i$, that is, $k = 1$, $N_1 = 4$ at the left graph, and $k = 2$, $N_1 + N_2 = 6$ at the right graph.

For the case of $m_{A[T]}(\lambda) = 4$, there are five cases. In the last figure, the subgraph with multiplicity 3 has the same two patterns as the case of $m_{A[T]}(\lambda) = 3$. For all the cases, we see that the formula in Theorem 2.1 holds.

We note that the condition $m_{A[T]}(\lambda) \geq 1, 1 \leq i \leq k$ in Theorem 2.1 is indispensable. If this condition is not satisfied, then there is a case such that the formula in Theorem 2.1 does not hold. Let $A$ be an Hermitian matrix, and the corresponding graph be the tree $T$ in Figure 3.3, in which vertex $x_i$ of $T$ corresponds to row $i$ of $A$, $i = 1, \ldots, 7$. The matrix $A$ has an eigenvalue 0 with multiplicity 2, so $m_{A[T]}(0) = 2$. Then $x_1, x_6$ are Parter vertices of $T$ for 0 relative to $A$. Furthermore, $V = \{x_1, x_6\}$ is a Parter set of $T$ for 0 relative to $A$. We denote the subgraph of $T$ induced by vertices
$x_5, x_6$ and $x_7$ by $T_2$. Here we set $V = \{v_1, v_2\}$ by replacing $x_1 = v_1, x_6 = v_2$.

$$A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 3 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 
\end{bmatrix}$$

Let $T = T_1$, and $N_i$ be the number of components of $T_i - v_i$, $i = 1, 2$, in which 0 is an eigenvalue of the principal submatrix of $A$. Since the principal submatrix of $A$ corresponding to $T_2$ does not have eigenvalue 0, and $m_{A[T_2]}(0) = 1$, then $N_1 + N_2 = 4$. Now, since $|V| = k = 2$, $m_A(\lambda) + 2k - 1 = 5$. As a result, we can say that if $m_{A[T_i]}(\lambda) \geq 1$ is not satisfied, then there is a case such that the formula in Theorem 2.1 $\sum_{i=1}^{k} N_i = m_A(\lambda) + 2k - 1$ does not hold. So, the condition $m_{A[T_i]}(\lambda) \geq 1$ in Theorem 2.1 is essential.

**Example 3.3.** Finally, we give examples for Theorem 2.6 in Figure 3.4. For the case of $m_{A[T]}(\lambda) = 4$, we can construct Hermitian matrices whose graphs are trees such that $|V| = 1, 2$ and 3 respectively, with $V$ a strong f-Parter set for $\lambda$ relative to $A \in \mathcal{H}(T)$. The numbers in the figures represent the multiplicity of the eigenvalue $\lambda$ relative to the submatrix of $A$ corresponding to the subgraph under the vertex next to the number.

From Theorem 2.2, we can construct an Hermitian matrix whose graph is a tree $T$ with $|V| = 3$ as the left graph in Figure 3.4, then strong Parter set $V = \{p_1, p_2, p_3\}$. When we remove the vertex $p_2$ from $T$, there exist three branches(vertics) at $p_2$ in $T_2$, which have the eigenvalue with multiplicity 1. Then we pick up two branches(vertics)
from them, and we connect them to the vertex \( p_1 \), then we can get a tree \( T' \) with \(|V| = 2 \) with \( V = \{p_1, p_3\} \). Furthermore, if we remove the vertex \( p_3 \) from \( T' \), and connect two branches(vertices) at \( p_3 \) to the vertex \( p_1 \), then we have a star \( T'' \), where \(|V| = 1 \) with \( V = \{p_1\} \).

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REFERENCES