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SPECTRAL BOUND FOR SEPARATIONS IN EULERIAN DIGRAPHS*

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Abstract. The spectra of digraphs, unlike those of graphs, is a relatively unexplored territory. In a digraph, a separation is a pair of sets of vertices D and Y such that there are no arcs from D and Y. For a subclass of Eulerian digraphs, a bound on the size of a separation is obtained in terms of the eigenvalues of the Laplacian matrix. An infinite family of tournaments, namely, the Paley digraphs, where the bound holds with equality, is also given.

Key words. Algebraic graph theory, Eigenvalues, Directed graphs.

AMS subject classifications. 05C50, 05C20.

1. Introduction. The theory of graph spectra is a rich and interesting field of study. There has been extensive study about the interplay of eigenvalues of a graph and various graph parameters, such as the diameter [4, 12] or the chromatic number [5, 9, 10]; see also [14]. The relationship between symmetries of a graph and its eigenvalues has also been investigated extensively, for example in [3, 16, 17]. The eigenvalues of the Laplacian matrix of a graph determine the number of connected components. For connected graphs, the eigenvalues of the adjacency matrix determine whether the graph is bipartite. There are spectral bounds on the independence number and many other parameters.

For digraphs, in contrast, there are relatively few results. There is a directed analogue of Wilf's bound on chromatic number [13], however other directed analogues are yet to be found. The adjacency matrix of a digraph is usually difficult to work with; it is not always diagonalizable and may have complex eigenvalues. In addition, the interlacing theorem does not hold for the adjacency matrices of digraphs, in general. The digraphs we consider here are also known as *mixed graphs* in the literature.

The interlacing theorem is a powerful tool for studying the eigenvalues of graphs; see [9] for standard applications to graph eigenvalues. We would like to extend its usage to some classes of digraphs, with possibly different choices of matrices. First, we have a few preliminary definitions. The *adjacency matrix* of a digraph D is a matrix $A(D)$, with rows and columns indexed by the vertices of D, such that $A(u, v) = 0$ if uv is an arc and $A(u, v) = 0$ otherwise. Let $\Delta^+(D)$ be the diagonal matrix with diagonal entries equal to the out-degrees of vertices of the digraph. The *Laplacian matrix* of D, is $L(D) = \Delta^+(D) - A(D)$. We prove a result which generalizes the following theorem of Haemers for graphs to the class of digraphs with normal Laplacian matrices.

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THEOREM 1.1 ([9]). *Let D be a connected graph on n vertices and let (X, Y) be a separation in D. Then*

$$\frac{|Y||X|}{(n-|Y|)(n-|X|)} \leq \frac{|\alpha+\sigma_n|^2}{\alpha^2},$$

where $\alpha = -\frac{1}{2}(\sigma_2+\sigma_n)$ and $0=\sigma_1<\sigma_2\leq\cdots\leq\sigma_n$ are the Laplacian eigenvalues of D.

We give the following generalization of this theorem to digraphs whose Laplacian matrix is normal.

THEOREM 1.2. *Let D be a connected digraph on n vertices where $L(D)$ is normal and (X, Y) be a separation in D. Then*

$$\frac{|Y||X|}{(n-|Y|)(n-|X|)} \leq \frac{|\alpha+\theta|^2}{\alpha^2},$$

where

$$\alpha = \begin{cases} -f(\theta)-f(\nu) & \textit{if } \operatorname{Re}(\lambda)\geq\operatorname{Re}(\theta) \textit{ for all } \lambda\notin\{0,\theta\}; \\ -f(\theta)-g(\mu) & \textit{otherwise} \end{cases}$$

and

- $f(\lambda) = \frac{|\lambda|^2}{2\operatorname{Re}(\lambda)}$,
- *$\theta \neq 0$ is the eigenvalue of $L(D)$ which maximizes f amongst non-zero eigenvalues of $L(D)$,*
- *$\nu \neq 0$ is the eigenvalue of $L(D)$ which minimizes f amongst non-zero eigenvalues of $L(D)$,*
- $g(\lambda) = \frac{\operatorname{Re}(\lambda)(f(\theta)-f(\lambda))}{\operatorname{Re}(\theta)-\operatorname{Re}(\lambda)}$, *and*
- *μ is the eigenvalue of $L(D)$ which minimizes g such that $g(\mu)>0$, if such an eigenvalue exists, and $\mu=0$ otherwise.*

In Section 2, we will give a combinatorial description of digraphs whose Laplacian matrices are normal. For example, any Cayley digraph of an abelian group will have a normal Laplacian matrix. In Section 3, we will prove some preliminary facts about the eigenvalues of digraphs with normal Laplacian. We will give the proof of the main theorem, Theorem 5.2, in Section 5, as well as a corollary for tournaments. In Section 6, we show that this bound holds with equality for an infinite family of tournaments, namely the Paley digraphs.

2. Directed graphs with normal Laplacian matrix.

We are interested in digraphs whose Laplacian matrix is normal. We will assume our digraphs have no loops or parallel arcs (arcs with the same orientation between the head and tail vertices), but we allow two oppositely oriented arcs between a pair of vertices.

We can easily give a combinatorial description of digraphs whose Laplacian is normal. Let D be a digraph. We denote by $d(u)$ the out-degree of a vertex u. Let $d^+(u,v)$ be the number of common out-neighbours of u and v; that is

$$d^+(u,v) = |\{w \mid uw, vw \in E(D)\}|.$$

Similarly, let $d^-(u,v) = |\{w \mid wu, wv \in E(D)\}|$.

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LEMMA 2.1. *Let D be a digraph. The Laplacian of D is normal if and only if, for every pair of vertices u, v,*

$$d^-(u,v) - d^+(u,v) = \begin{cases} 0 & \text{if } uv, vu \in E(D) \text{ or } uv, vu \notin E(D); \\ d(u) - d(v) & \text{if } uv \in E(D) \text{ and } vu \notin E(D); \\ d(v) - d(u) & \text{if } vu \in E(D) \text{ and } uv \notin E(D). \end{cases}$$

Proof. We will look at the (u,v) entry of the matrices $L^T L$ and LL^T. We have that

$$(L^T L)(u,v) = \sum_{w \in V(D)} L(w,u)L(w,v) = d^-(u,v) + d(u)L(u,v) + d(v)L(v,u).$$

Similarly,

$$(LL^T)(u,v) = \sum_{w \in V(D)} L(u,w)L(v,w) = d^+(u,v) + d(u)L(v,u) + d(v)L(u,v).$$

Note that $(L^T L)(u,u) = (LL^T)(u,u)$ for all $u \in V(D)$. For $u \neq v$, we have that $(L^T L)(u,v) = (LL^T)(u,v)$ if and only if

$$d^-(u,v) - d^+(u,v) = \begin{cases} 0 & \text{if } L(v,u) = L(u,v); \\ d(u) - d(v) & \text{if } L(u,v) = -1 \text{ and } L(v,u) = 0; \\ d(v) - d(u) & \text{if } L(v,u) = -1 \text{ and } L(u,v) = 0. \end{cases}$$

whence the lemma follows. □

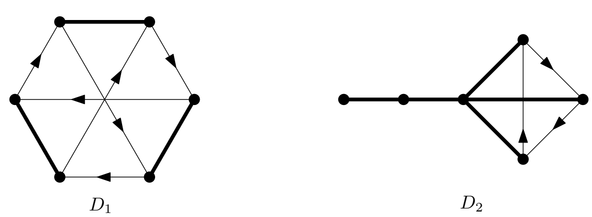

FIGURE 1. *Examples of digraphs D_1 and D_2, where $L(D_1)$ and $L(D_2)$ are normal. Two oppositely oriented arcs between two vertices (called a digon) are shown as a thick edge.*

Figure 1 shows examples of digraphs with normal Laplacian matrices. We will now show the digraphs with normal Laplacian are a subclass of Eulerian digraphs. A digraph D is *weakly connected* if the underlying graph of D is connected. A digraph D is *strongly connected* if, for every pair of vertices x and y, there is a directed walk from x to y in D.

LEMMA 2.2. *For a weakly connected digraph D, if $L(D)$ is normal, then D is Eulerian.*

Proof. We will expand $LL^T - L^T L$ to obtain

$$LL^T - L^T L = \Delta^+(A - A^T) + (A^T - A)\Delta^+ + AA^T - A^T A =: M. \tag{2.1}$$

For a square matrix N, let $\text{diag}(N)$ denote the vector consisting of the diagonal entries of N (more precisely, we mean that $(\text{diag}(N))_u = N(u,u)$). Observe that Δ^+ is a diagonal matrix and both A and A^T have zero

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diagonal. Thus,

$$\mathrm{diag}(\Delta^+(A - A^T)) = \mathrm{diag}((A^T - A)\Delta^+) = 0.$$

Since L is normal, $M = 0$ and considering the diagonal of (2.1) gives that $\mathrm{diag}(AA^T) = \mathrm{diag}(A^TA)$. Combinatorially, $(A^TA)_{u,u}$ counts the number of in-neighbours of u and $(AA^T)_{u,u}$ counts the number of out-neighbours of u. Thus, the in-degree and out-degree are equal for every vertex and we obtain that D is Eulerian by a standard theorem, see [1]. ∎

If a weakly connected digraph is Eulerian, then it is also strongly connected. Since there is no confusion, we may say that such graphs are *connected*. We observe that, if the digraph is regular, then the Laplacian matrix is normal if and only if the adjacency matrix is normal. We say that a digraph is *normal* if it has a normal Laplacian matrix and a normal adjacency matrix.

Theorem 2.3. *Every Cayley digraph on an abelian group is normal.*

Proof. Consider a Cayley digraph $D = \mathrm{Cay}(G, C)$ where G is abelian and let A be the adjacency matrix of D. Since D is regular, we need only check that A is normal.

Let u, v be vertices of D and suppose they have a common in-neighbour w. Let $a_1 \neq a_2$ be elements of the connection set C such that $u = a_1w$ and $v = a_2w$. It is clear that a_2u and a_1v are out-neighbour of u and v, respectively. Since the group is abelian, we have that $a_2u = a_2a_1w = a_1a_2w = a_1v$, and so a_2u is a common out-neighbour of u and v. Let $\phi_{u,v}(w)$ be a mapping taking w, a common in-neighbour of u and v, to the uniquely defined common out-neighbour of u and v by the process above. We see that $\phi_{u,v}$ is injective onto the set of common out-neighbours of u and v.

Conversely, following the same argument, we can find an injective mapping from common out-neighbours of u and v to the common in-neighbour of u and v. We have shown combinatorially that $(A^TA)_{u,v} = (AA^T)_{u,v}$. ∎

In general, the adjacency matrix being normal does not have to coincide with the Laplacian matrix being normal. We present some data on all digraph on 4, 5 and 6 vertices. Note that if D has a normal adjacency matrix or a normal Laplacian, it must be Eulerian.

Table 1
Small digraphs with normal Laplacian and adjacency matrices.

	4 vertices	5 vertices	6 vertices
number of digraphs	218	9608	1540944
Eulerian	17	107	2269
regular	5	10	52
normal Laplacian	14	43	194
normal adjacency matrix	14	45	212
normal	14	43	190
connected and Eulerian	12	90	2162
undirected	10	31	43

3. Eigenvalues of normal Laplacian matrices. To prove the main results, we need the following lemma. Lemma 3.1 can be proved by considering $L + L^T$ as the Laplacian of a multi-graph and appealing

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to known results about Laplacians of graphs, so we will omit the proof here. Parts (a) and (b) of Lemma 3.1 follow from well-known results on M-matrices, see [2] and part (c) is true for all normal matrices, see, for example, [11].

LEMMA 3.1. *If D is a connected digraph and $L(D)$ is normal, then*

(a) $L(D)$ has eigenvalue 0 with multiplicity 1;
(b) $\mathrm{Re}(\lambda) > 0$ for $\lambda \neq 0$ an eigenvalue of $L(D)$;
(c) if $L(D)\mathbf{v} = \lambda\mathbf{v}$, then $L(D)^T\mathbf{v} = \bar{\lambda}\mathbf{v}$.

Observe that if D is not connected, then we may consider the spectrum for each connected component of D to obtain the following corollary.

COROLLARY 3.2. *If D is a digraph and $L(D)$ is normal, then $\mathrm{Re}(\lambda) > 0$ for all eigenvalues λ of $L(D)$ except $\lambda = 0$.*

4. Interlacing. Since we make use of the interlacing theorem, we will state it here. If $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_m$ are real numbers with $m < n$, then we say that $(\mu_i)_{i=1}^m$ *interlaces* $(\lambda_i)_{i=1}^n$ if

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$$

for $i = 1, \ldots, m$.

Let A be a Hermitian matrix. Let $\mathcal{P}$ be a partition of the rows of A which induces a partitioning of A into block matrices as follows:

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix},$$

where the A_{ij} are block matrices and the corresponding partition of the rows and columns is $P = \{P_1, \ldots, P_m\}$, as a partition of $[n]$. Note that since we are interested in the eigenvalues of A, which are preserved under permuting the rows and columns simultaneously, we may assume all partitions of the rows of A are of this form. The *quotient matrix of A with respect to partition $\mathcal{P}$* is a $m \times m$ matrix B where the ijth entry is the average row sums of the blocks of A.

THEOREM 4.1 ([9]). *Let A be a Hermitian matrix and B be its partition matrix with respect to partition $\mathcal{P}$. The eigenvalues of B interlace those of A.*

We note that the quotient matrix B is not symmetric, in general, but is diagonalizable with real eigenvalues.

5. Interlacing with the Laplacian. We would like to use interlacing to find bounds of combinatorial parameters of digraph D using eigenvalues of the Laplacian matrix. In particular, we would like to use the same method as the proof of Lemma 6.1 in [9], which is stated here as Theorem 1.1. We prove a result which generalizes the original lemma of Haemers to a sub-class of digraphs. However, as this matrix is not symmetric like in the case for graphs, we need to restrict to digraphs D whose the Laplacian matrices are normal and we also need a few technical lemmas in order to prove Theorem 5.2, the main result.

We denote by $\sigma(L)$ the multiset of eigenvalues of L. If L is a normal matrix, then so is $\alpha I + L$ for any $\alpha \in \mathbb{R}$. For L, we see that the singular values of $L^T L$ are $\{|\lambda| : \lambda \in \sigma(L)\}$. Thus, the singular values of $\alpha I + L$ are $\{|\alpha + \lambda| : \lambda \in \sigma(L)\}$.

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Lemma 5.1. *Let D be a digraph such that $L(D)$ is normal. Let*

$$f(\lambda) = \frac{|\lambda|^2}{2\operatorname{Re}(\lambda)},$$

and let $\theta \neq 0$ be the eigenvalue of $L(D)$ which maximizes f amongst non-zero eigenvalues of $L(D)$. Let

$$g(\lambda) = \frac{\operatorname{Re}(\lambda)(f(\theta) - f(\lambda))}{\operatorname{Re}(\theta) - \operatorname{Re}(\lambda)},$$

and let μ be the eigenvalue of $L(D)$ which minimizes g such that $g(\mu) \geq 0$, if such an eigenvalue exists in the domain of g, and $\mu = 0$ otherwise. Let $\widetilde{L} = \alpha I + L(D)$, where $\alpha \leq -f(\theta)$. Then $|\alpha|$ is the largest singular value of $\widetilde{L}$. Further,

(a) if D is connected and $\operatorname{Re}(\lambda) \geq \operatorname{Re}(\theta)$ for all $\lambda \notin \{0, \theta\}$, then $|\alpha + \theta|$ is the second largest singular value of $\widetilde{L}$;
(b) if D is connected and $-f(\theta) - g(\mu) \leq \alpha$, then $|\alpha + \theta|$ is the second largest singular value of $\widetilde{L}$.

Proof. Note that f is well-defined for non-zero eigenvalues λ of L, since $\operatorname{Re}(\lambda) > 0$ by Lemma 3.1. Observe also that f is positive real-valued and $\alpha \in \mathbb{R}$. The function g is well-defined for λ when $\operatorname{Re}(\lambda) \neq \operatorname{Re}(\theta)$. If there exists an eigenvalue $\lambda \notin \{0, \theta\}$, such that $\operatorname{Re}(\lambda) < \operatorname{Re}(\theta)$, then, we can see that $g(\lambda) \geq 0$, and so μ is non-zero. Also, the range for α in part (b), $[-f(\theta) - g(\mu), -f(\theta)]$, is non-empty.

Since 0 is an eigenvalue of L, we have that $|\alpha|$ is a singular value of $\widetilde{L}$. The singular values of $\widetilde{L}$ are of form $|\alpha + \lambda|$ where λ is an eigenvalue of L. Let λ be a non-zero eigenvalue of L. Consider

$$|\alpha|^2 - |\alpha + \lambda|^2 = \alpha^2 - (\alpha + \lambda)(\overline{\alpha + \lambda}) = \alpha^2 - (\alpha + \lambda)(\alpha + \bar{\lambda}) = -2\alpha \operatorname{Re}(\lambda) - |\lambda|^2. \tag{5.2}$$

By definition of α, we have that

$$-\alpha \geq f(\theta) \geq f(\lambda) = \frac{|\lambda|^2}{2\operatorname{Re}(\lambda)} \tag{5.3}$$

for all nonzero $\lambda \in \sigma(L)$. From (5.2) and (5.3), we obtain: $|\alpha|^2 - |\alpha + \lambda|^2 \geq 0$ and we have shown that $|\alpha|$ is the largest singular value of $\widetilde{L}$.

To prove statements (a) and (b), we let $\delta = |\alpha + \theta|^2 - |\alpha + \lambda|^2$ for $\lambda \in \sigma(L)$. Since D is connected, L has only one eigenvalue whose real part is equal to 0 by Lemma 3.1. It suffices to show that $\delta(\lambda) \geq 0$ for all nonzero $\lambda \in \sigma(L)$. We expand $\delta(\lambda)$ to obtain

$$\delta(\lambda) = |\theta|^2 - |\lambda|^2 + 2\alpha(\operatorname{Re}(\theta) - \operatorname{Re}(\lambda)).$$

If $\operatorname{Re}(\theta) = \operatorname{Re}(\lambda)$, then $\delta(\lambda) = |\theta|^2 - |\lambda|^2$. In this case,

$$f(\theta) = \frac{|\theta|^2}{2\operatorname{Re}(\theta)} = \frac{|\theta|^2}{2\operatorname{Re}(\lambda)} \geq f(\lambda) = \frac{|\lambda|^2}{2\operatorname{Re}(\lambda)}$$

and, since $\operatorname{Re}(\lambda)$ and $\operatorname{Re}(\theta)$ are positive, $|\theta|^2 \geq |\lambda|^2$ and $\delta(\lambda) \geq 0$. Otherwise, we have that $\operatorname{Re}(\theta) < \operatorname{Re}(\lambda)$, and, recalling that $-\alpha \geq f(\theta) \geq 0$, we may simplify as follows:

$$\begin{aligned}
\delta(\lambda) &\geq |\theta|^2 - |\lambda|^2 + 2f(\theta)(\operatorname{Re}(\lambda) - \operatorname{Re}(\theta)) \\
&= -|\lambda|^2 + \frac{|\theta|^2 \operatorname{Re}(\lambda)}{\operatorname{Re}(\theta)} = 2\operatorname{Re}(\lambda)(f(\theta) - f(\lambda)) \geq 0.
\end{aligned}$$

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We have shown part (a) and also part (b) when $\mathrm{Re}(\lambda) \geq \mathrm{Re}(\theta)$.

For part (b), we need only consider eigenvalues λ such that $\mathrm{Re}(\theta) > \mathrm{Re}(\lambda)$. In this case, we will use that $g(\lambda) \geq g(\mu) \geq 0$ and obtain

$$\begin{aligned}\delta(\lambda) &= |\theta|^2 - |\lambda|^2 + 2\alpha(\mathrm{Re}(\theta) - \mathrm{Re}(\lambda))\\ &\geq |\theta|^2 - |\lambda|^2 + (-2f(\theta) - 2g(\mu))(\mathrm{Re}(\theta) - \mathrm{Re}(\lambda))\\ &= |\theta|^2 - |\lambda|^2 - |\theta|^2 + 2f(\theta)\,\mathrm{Re}(\lambda) - 2g(\mu)(\mathrm{Re}(\theta) - \mathrm{Re}(\lambda)).\end{aligned}$$

Since $\mathrm{Re}(\theta) - \mathrm{Re}(\lambda) > 0$ and $-2g(\mu) \geq -2g(\lambda)$, we obtain

$$\delta(\lambda) \geq -|\lambda|^2 + 2f(\theta)\,\mathrm{Re}(\lambda) - 2g(\lambda)(\mathrm{Re}(\theta) - \mathrm{Re}(\lambda)) = -|\lambda|^2 + 2\,\mathrm{Re}(\lambda)f(\lambda) = 0,$$

and hence, $\delta(\lambda) \geq 0$, as required. ∎

It is worth observing that if we take $\alpha = -f(\theta)$, then

$$|\alpha + \theta|^2 = \alpha^2 + \alpha 2\,\mathrm{Re}(\theta) + |\theta|^2 = |\alpha|^2.$$

We can now prove the main theorem.

THEOREM 5.2. *Let D be a connected digraph on n vertices where $L := L(D)$ is normal. Let f, g, θ and μ be as defined in Lemma 5.1. Also, let $\nu \neq 0$ be the eigenvalue of L which minimizes f amongst non-zero eigenvalues of L. Let Y and X be disjoint vertex sets in D with no arcs from X to Y. Then,*

$$\frac{|Y||X|}{(n-|Y|)(n-|X|)} \leq \frac{|\alpha+\theta|^2}{\alpha^2},$$

where

$$\alpha = \begin{cases} -f(\theta) - f(\nu) & \text{if } \mathrm{Re}(\lambda) \geq \mathrm{Re}(\theta) \text{ for all } \lambda \notin \{0, \theta\};\\ -f(\theta) - g(\mu) & \text{otherwise.}\end{cases}$$

Proof. Let $\alpha = -f(\theta) - g(\mu)$ and let $\widetilde{L} = \alpha I + L$. In L and $\widetilde{L}$, there is an off-diagonal block of 0s, where the rows are indexed by X and columns are indexed by Y. This follows directly from hypothesis that there are no arcs from X to Y. We wish to use interlacing to bound the size of such an off-diagonal block of 0s. Let

$$C = \begin{pmatrix} 0 & \alpha I + L\\ \alpha I + L^T & 0\end{pmatrix} = \begin{pmatrix} 0 & \widetilde{L}\\ \widetilde{L}^T & 0\end{pmatrix}.$$

Note that we use 0 in matrices to represent the zero matrix of the appropriate dimensions. We see that C is symmetric and the eigenvalues of C are $\{\pm|\alpha+\lambda| : \lambda \text{ eigenvalue value of } L\}$. By Lemma 5.1, we see that $|\alpha|$ is the biggest eigenvalue of C and $|\alpha+\theta|$ is the second largest eigenvalue of C.

Since D is Eulerian, each row and column of L sums to 0 and so each row and column of $\widetilde{L}$ sum to α. We may partition the rows of $\widetilde{L}$ into rows indexed by $\{X, V(D) \setminus X\}$ and the columns of $\widetilde{L}$ into columns indexed by $\{V(D) \setminus Y, Y\}$. This partition of $\widetilde{L}$ induces a partition of C where all diagonal blocks are square;

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we can write C as a block matrix with blocks corresponding to the partition,

$$C = \begin{pmatrix} 0 & 0 & \widetilde{L}_{11} & 0 \\ 0 & 0 & \widetilde{L}_{21} & \widetilde{L}_{22} \\ \widetilde{L}_{11}^T & \widetilde{L}_{21}^T & 0 & 0 \\ 0 & \widetilde{L}_{22}^T & 0 & 0 \end{pmatrix}.$$

We let B be the quotient matrix of C with respect to this partition. Recall from Theorem 4.1 that the entries of B are the average row sums of the corresponding blocks of C. We will index the rows and columns of B with $[4]$, for convenience. Since the row and column sums of $\widetilde{L}$ are all equal to α, we see that each row and column sum of $\widetilde{L}_{11}$ and of the matrix $\begin{pmatrix}\widetilde{L}_{21} & \widetilde{L}_{22}\end{pmatrix}$ is equal to α. Then $B(1,3) = B(4,2) = \alpha$ and $B(2,3) + B(2,4) = B(3,1) + B(3,2) = \alpha$.

Observe that $\widetilde{L}_{22}$ is a $n-z \times y$ matrix, where the lower $y \times y$ submatrix is a principal submatrix of $\widetilde{L}$. For $S, T \subseteq V(D)$, let $E(S,T)$ denote the set of edges e such that $t(e) \in S$ and $h(e) \in T$. Let $W = V(D) \backslash (Y \cup X)$. We will find $B(2,4)$ by taking the sum over all of the entries of $\widetilde{L}_{22}$ as follows:

$$(n-2)B(2,4) = \sum_{j=1}^{n} \sum_{\ell=1}^{n} \widetilde{L}_{22}(j,\ell) = y\alpha + \sum_{y \ inY} d^+(y) - |E(Y,Y)| - |E(W,Y)|. \tag{5.4}$$

Since there are no arcs from X to Y, we have that

$$|E(Y,Y)| + |E(W,Y)| = |E(V(D),Y)| = \sum_{y \in Y} d^-(y). \tag{5.5}$$

Since D is Eulerian, we see that $\sum_{y\in Y} d^-(y) = \sum_{y \in Y} d^+(y)$. Then, from (5.4) and (5.5), we obtain that $B(2,4) = {y\alpha}/{n-z}$, which implies that $B(2,3) = \alpha - {y\alpha}/{n-z}$. By an analoguous argument, we find that $B(3,1) = {z\alpha}/{n-y}$ and $B(3,2) = \alpha - {z\alpha}/{n-y}$. Thus, we have

$$B = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & \alpha - \frac{y\alpha}{n-z} & \frac{y\alpha}{n-z} \\ \frac{z\alpha}{n-y} & \alpha - \frac{z\alpha}{n-y} & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix}.$$

Let the eigenvalues of B be $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n-1} \geq \lambda_{2n}$ be the eigenvalues of C. Observe that C is similar to $-C$ and B is similar to $-B$ by construction of C. Then, we have that $\mu_4 = -\mu_1$, $\mu_3 = -\mu_2$, $\lambda_{2n} = -\lambda_1$ and $\lambda_{2n-1} = -\lambda_2$. Applying the interlacing theorem gives

$$\lambda_1 \geq \mu_1, \ \lambda_2 \geq \mu_2, \ \mu_3 \geq \lambda_{2n-1} \ \text{ and } \ \mu_4 \geq \lambda_{2n}.$$

Recalling that $(\lambda_1, \lambda_2) = (|\alpha|, |\alpha + \theta|)$, we have that

$$\mu_1\mu_2\mu_3\mu_4 = (-1)^2(\mu_1\mu_2)^2 \leq (\lambda_1\lambda_2)^2 = (|\alpha||\alpha+\theta|)^2. \tag{5.6}$$

On the other hand, we see that

$$\mu_1\mu_2\mu_3\mu_4 = \det(B) = \frac{\alpha^2 y}{n-z}\frac{\alpha^2 z}{n-y}. \tag{5.7}$$

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From (5.6) and (5.7), we obtain that

$$\frac{\alpha^2 y}{n-z}\frac{\alpha^2 z}{n-y} \leq (|\alpha||\alpha+\theta|)^2 \;\Leftrightarrow\; \frac{y}{n-z}\frac{z}{n-y} \leq \frac{|\alpha+\theta|^2}{\alpha^2}$$

as claimed. □

Observe that if D is a graph, then the Laplacian is the usual Laplacian matrix of a graph. In this case, the Laplacian is symmetric and hence normal, and so, all eigenvalues are real and non-negative. Thus, for λ an eigenvalue of $L(D)$, we see that $f(\lambda) = g(\lambda) = \lambda/2$. Then, we can recover the original theorem of Haemers [9, Lemma 6.1] for graphs as a corollary of Theorem 5.2.

A tournament has normal adjacency matrix if and only if it is regular (see [6]). Then, the Laplacian matrices of regular tournaments are normal matrices. Let D be a regular tournament on n vertices. In this case, all of the non-zero eigenvalues have real part equal to $n/2$. We see that

$$f(\lambda) = \frac{|\lambda|^2}{2\frac{n}{2}} = \frac{|\lambda|^2}{n}.$$

Then, $\alpha = -|\theta|^2/n - |\nu|^2/n$, where θ and ν are the largest and smallest eigenvalues of $L(D)$ in magnitude. Theorem 5.2 gives the following corollary.

Corollary 5.3. *Let D be a regular tournament on n vertices and (X, Y) be a separation in D. Then*

$$\frac{|Y||X|}{(n-|Y|)(n-|X|)} \leq \frac{|\alpha+\theta|^2}{\alpha^2},$$

where $\alpha = -|\theta|^2/n - |\nu|^2/n$ where θ and ν are the largest and smallest non-zero eigenvalues of $L(D)$ in magnitude.

6. Tight examples. In this section, we will give an infinite family of regular tournaments where the bound in Corollary 5.3 holds with equality. For q a prime power such that $q = 3 \mod 4$, the *Paley digraph* of order q is the digraph whose vertices are the elements of $GF(q)$ and xy is an arc when $y - x$ is a non-zero square in $GF(q)$. Figure 2 shows the Paley digraph of order 7. The Paley digraph is a well-known regular tournament, see [7] and [8].

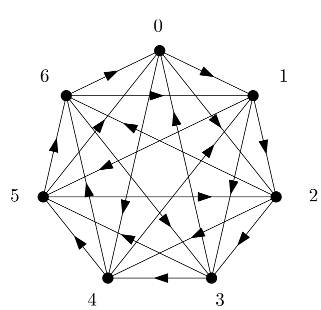

Figure 2. *The Paley digraph of order* 7.

Let $Q = A - A^T$ for a Paley digraph D of order q, where $A := A(D)$. Paley showed that $QQ^T = qI - J$

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in [15]. Since $A^T = J - I - A$, we obtain that

$$A^2 + A + \frac{q+1}{4} I + \frac{-q-1}{4} J = 0.$$

From this we see that A has an eigenvalue $\frac{q-1}{2}$, corresponding to the all ones eigenvector, and the other eigenvalues are $\frac{-1 \pm \sqrt{-q}}{2}$, each with multiplicity $\frac{q-1}{2}$. Thus, Paley digraph of order q have Laplacian eigenvalues 0 with multiplicity 1 and $\frac{q \pm \sqrt{q} i}{2}$ each with multiplicity $\frac{q-1}{2}$.

Let D be the Paley digraph of order q. Corollary 5.3 gives that if (X, Y) is a separation in D, then

$$\frac{|Y||X|}{(n-|Y|)(n-|X|)} \leq \frac{1}{q+1}.$$

If we take X to be a singleton vertex u and Y be the set of the $\frac{q-1}{2}$ out-neighbours of u, we see for this choice of X and Y,

$$\frac{|Y||X|}{(n-|Y|)(n-|X|)} = \frac{(\frac{q-1}{2})}{(\frac{q+1}{2})(q-1)} = \frac{1}{q+1},$$

and the inequality of Corollary 5.3 holds with equality.

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REFERENCES

[1] J. Bang-Jensen and G.Z. Gutin. *Digraphs: Theory, Algorithms and Applications*, second edition. Springer-Verlag, Berlin, 2008.
[2] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences.* Classics in Applied Mathematics, Vol. 9, Society for Industrial and Applied Mathematics, Philadelphia, 1994.
[3] A. Chan and C.D. Godsil. Symmetry and eigenvectors. In: G. Hahn and G. Sabidussi (editors), *Graph Symmetry, Algebraic Methods and Applications*, Kluwer, 497:75–106, 1997.
[4] F.R.K. Chung. Diameters and eigenvalues. *J. Amer. Math. Soc.*, 2:187–196, 1989.
[5] D.M. Cvetković. Chromatic number and the spectrum of a graph. *Publ. Inst. Math. (Beograd) (N.S.)*, 14(28):25–38, 1972.
[6] D. de Caen. The ranks of tournament matrices. *Amer. Math. Monthly*, 98:829–831, 1991.
[7] P. Erdős and A. Rényi. Asymmetric graphs. *Acta Math. Hungar.*, 14:295–315, 1963.
[8] R.L. Graham and J.H. Spencer. A constructive solution to a tournament problem. *Canad. Math. Bull.*, 14:45–48, 1971.
[9] W.H. Haemers. Interlacing eigenvalues and graphs. *Linear Algebra Appl.*, 226–228:593–616, 1995.
[10] A.J. Hoffman. On eigenvalues and colorings of graphs. In: B. Harris (editor), *Graph Theory and its Applications*, Academic Press, New York, 79–91, 1970.
[11] R.A. Horn and C.R. Johnson. *Matrix Analysis*, second edition. Cambridge University Press, Cambridge, 2013.
[12] B. Mohar. Eigenvalues, diameter, and mean distance in graphs. *Graphs Combin.*, 7:53–64, 1991,
[13] B. Mohar. Eigenvalues and colorings of digraphs. *Linear Algebra Appl.*, 432:2273–2277, 2010.
[14] B. Mohar and S. Poljak. Eigenvalues in combinatorial optimization. In: R.A. Brualdi, S. Friedland, and V. Klee (editors), *Combinatorial and Graph-Theoretical Problems in Linear Algebra*, Springer-Verlag, New York, 50:107–151, 1993.
[15] R.E. A.C. Paley. On orthogonal matrices. *Journal of Mathematics and Physics*, 12:311–320, 1933.
[16] P. Rowlinson. The main eigenvalues of a graph: A survey. *Appl. Anal. Discrete Math.*, 1:445–471, 2007.
[17] P. Rowlinson. On graphs with an eigenvalue of maximal multiplicity. *Discrete Math.*, 313:1162–1166, 2013.