

2017

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Recommended Citation

Lou, Zhenzhen; Huang, Qiongxiang; and Huang, Xueyi. (2017), "On the construction of Q -controllable graphs", *Electronic Journal of Linear Algebra*, Volume 32, pp. 365-379.

DOI: <https://doi.org/10.13001/1081-3810.3298>

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ON THE CONSTRUCTION OF Q-CONTROLLABLE GRAPHS*

ZHENZHEN LOU[†], QIONGXIANG HUANG[†], AND XUEYI HUANG[†]

Abstract. A connected graph is called Q -controllable if its signless Laplacian eigenvalues are mutually distinct and main. Two graphs G and H are said to be Q -cospectral if they share the same signless Laplacian spectrum. In this paper, infinite families of Q -controllable graphs are constructed, by using the operator of rooted product introduced by Godsil and McKay. In the process, infinitely many non-isomorphic Q -cospectral graphs are also constructed, especially, including those graphs whose signless Laplacian eigenvalues are mutually distinct.

Key words. Q -Spectrum, Q -Controllable graph, Q -Cospectral graph.

AMS subject classifications. 05C50.

1. Introduction. All graphs considered here are simple and undirected. For a graph $G = (V(G), E(G))$ of order n with vertex set $V(G) = \{1, 2, \dots, n\}$, we denote by $A(G)$ and $D(G) = \text{diag}(d_1, d_2, \dots, d_n)$ the adjacency matrix and diagonal degree matrix of G , respectively, where d_i is the degree of the vertex i . Then the matrix $Q(G) = D(G) + A(G)$ is called the *signless Laplacian matrix* (Q -matrix for short) of the graph G . Since $Q(G)$ is positive semidefinite, all its eigenvalues are nonnegative. These eigenvalues are called the *signless Laplacian eigenvalues* (Q -eigenvalues for short) of G . Let $\xi_1 > \xi_2 > \dots > \xi_s \geq 0$ be all the distinct Q -eigenvalues of G with multiplicities m_1, m_2, \dots, m_s ($\sum_{i=1}^s m_i = n$), respectively. The *signless Laplacian spectrum* (Q -spectrum for short) of G is defined to be $\text{Spec}_Q(G) = \{\xi_1^{m_1}, \xi_2^{m_2}, \dots, \xi_s^{m_s}\}$. Two graphs G and H are called Q -cospectral if $\text{Spec}_Q(G) = \text{Spec}_Q(H)$, and a graph G is said to be *determined by its Q -spectrum* (DQS for short) if $G \cong H$ whenever $\text{Spec}_Q(G) = \text{Spec}_Q(H)$ for any graph H .

Given a graph G of order n and a graph H with root vertex u , the *rooted product graph* $G \circ H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and identifying the vertex v_i of G with the vertex u in the i -th copy of H for every $1 \leq i \leq n$ (Godsil and McKay [5]). Let P_s be the path of order s . If we take $H = P_s$ ($s \geq 1$), and the root vertex $u = u_1$ one of pendant vertices of H , then the rooted product graph $G \circ P_s$ is shown in Fig. 1 (see Section 3).

A Q -eigenvalue of G is called a *main Q -eigenvalue* if it has an eigenvector \mathbf{x} such that $\mathbf{j}^T \mathbf{x} \neq 0$ (\mathbf{j} is the $n \times 1$ all-ones vector), and a *non-main Q -eigenvalue* otherwise. Connected graphs whose Q -eigenvalues are mutually distinct and main are called *Q -controllable graphs*. Throughout the paper, we denote by \mathcal{G}^Q (resp., \mathcal{G}_n^Q) the set of connected graphs (resp., with n vertices) whose eigenvalues are mutually distinct, and \mathcal{G}^{Q^*} (resp., $\mathcal{G}_n^{Q^*}$) the set of Q -controllable graphs (resp., with n vertices).

For a graph G on n vertices with adjacency matrix A and diagonal degree matrix D , a *universal adjacency matrix* associated with G is defined to be $U = \gamma_A A + \gamma_D D + \gamma_I I + \gamma_J J$, where I denotes the identity matrix, J denotes the all-ones matrix, and $\gamma_A \neq 0$, γ_D , γ_I and γ_J are constants [6]. Note that $U = Q(G)$ if we take $\gamma_A = \gamma_D = 1$ and $\gamma_I = \gamma_J = 0$. The name “ Q -controllable graph” arised from the concept of U -controllable

*Received by the editors on May 8, 2016. Accepted for publication on February 28, 2017. Handling Editor: Sebastian M. Cioaba.

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graph adopted by A. Farrugia and I. Sciriha [4]. In control theory, a standard system model determined by the differential equation $\dot{\mathbf{x}}(t) = (\gamma_A A + \gamma_D D + \gamma_I I + \gamma_J J)\mathbf{x}(t) = U\mathbf{x}(t) + \mathbf{j}u(t)$ (here the $n \times 1$ vector $\mathbf{x}(t)$ is called the state vector, with given $\mathbf{x}(0)$, and the scalar $u(t)$ is the control input) is controllable if and only if all eigenvalues of U are simple and main [4]. The applications of U -controllable graphs (especially, when $U = A$) in specific control theory problems are considered in [2, 3, 10].

In [4], A. Farrugia and I. Sciriha also proved that each U -controllable graph has trivial automorphism group. However, a graph with trivial automorphism group may not be U -controllable. So they asked which classes of non-regular, asymmetric graphs are U -controllable graphs. In this paper, we give an answer to this question for $U = Q(G)$ by constructing some infinite families of Q -controllable graphs. Concretely, given a graph G in \mathcal{G}^{Q^*} , the infinite families in \mathcal{G}^{Q^*} are constructed from G by using the operation of rooted product recursively, and the spectra of such graphs are also determined by that of G . By the way, we use this method to construct some infinite families of non-isomorphic Q -cospectral graphs, especially, including the graphs in \mathcal{G}^Q . Moreover, the DQS-property of rooted product graphs is also considered.

2. Elementary observations. In this section, we list some basic results that will be useful in the subsequent sections. First of all, we recursively define two sequences of polynomials $\{a_t(q)\}_{t \geq 0}, \{b_t(q)\}_{t \geq 0} \subseteq \mathbb{Z}[q]$:

$$(2.1) \quad a_0(q) = -1, \quad a_1(q) = 1 - q \quad \text{and} \quad a_t(q) = (q - 2)a_{t-1}(q) - a_{t-2}(q) \quad \text{for } t \geq 2.$$

$$(2.2) \quad b_0(q) = -1, \quad b_1(q) = 2 - q \quad \text{and} \quad b_t(q) = (q - 2)b_{t-1}(q) - b_{t-2}(q) \quad \text{for } t \geq 2.$$

By direct computation, $a_2(q) = -q^2 + 3q - 1$, $b_2(q) = -q^2 + 4q - 3$, and so on. Clearly, $a_t(q)$ and $b_t(q)$ can be viewed as an integral coefficient polynomial of q with degree t , respectively. For any $s \geq 1$, we denote by

$$(2.3) \quad f_s(q) = \sum_{t=0}^{s-1} a_t(q).$$

First we give the relation of $a_t(q)$ and $b_t(q)$ for later use.

LEMMA 2.1. *For $t \geq 2$, we have*

$$(2.4) \quad a_t(q) = (q - 1)b_{t-1}(q) - b_{t-2}(q),$$

$$(2.5) \quad q \cdot b_t(q) = (q - 1)a_t(q) - a_{t-1}(q).$$

Proof. First, we will show (2.4) by the way of induction. It is easy to verify that the result holds for $t = 2, 3$. Suppose that the result holds for $t \leq k$, where $k \geq 3$. Then

$$a_{k-1}(q) = (q - 1)b_{k-2}(q) - b_{k-3}(q) \quad \text{and} \quad a_k(q) = (q - 1)b_{k-1}(q) - b_{k-2}(q).$$

For $t = k + 1$, then we have

$$\begin{aligned} a_{k+1}(q) &= (q - 2)a_k(q) - a_{k-1}(q) \\ &= (q - 2)[(q - 1)b_{k-1}(q) - b_{k-2}(q)] - [(q - 1)b_{k-2}(q) - b_{k-3}(q)] \\ &= (q - 1)[(q - 2)b_{k-1}(q) - b_{k-2}(q)] - [(q - 2)b_{k-2}(q) - b_{k-3}(q)] \\ &= (q - 1)b_k(q) - b_{k-1}(q). \end{aligned}$$

Hence, (2.4) holds for any $t \geq 2$.

Similarly, (2.5) holds. It completes the proof. □

Now we define two $(s - 1) \times s$ matrices C and D with respect to q :

$$C = \begin{bmatrix} 1 & 2-q & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2-q & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2-q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2-q & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1-q \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{s-1}(q) \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & a_{s-2}(q) \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & a_{s-3}(q) \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_3(q) \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & a_2(q) \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & a_1(q) \end{bmatrix}.$$

Then we have the following result.

LEMMA 2.2. *Let x be a column vector in \mathbb{R}^s . Then $Cx = 0$ if and only if $Dx = 0$.*

Proof. Let us define a matrix as below:

$$P = \begin{bmatrix} 1 & -b_1(q) & -b_2(q) & -b_3(q) & \cdots & b_{s-3}(q) & -b_{s-2}(q) \\ 0 & 1 & -b_1(q) & -b_2(q) & \cdots & -b_{s-4}(q) & -b_{s-3}(q) \\ 0 & 0 & 1 & -b_1(q) & \cdots & -b_{s-5}(q) & -b_{s-4}(q) \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -b_1(q) & -b_2(q), \\ 0 & 0 & 0 & 0 & \cdots & 1 & -b_1(q) \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

where $\{b_t(q)\}_{t \geq 0}$ is defined in (2.2). Combining Lemma 2.1, one can directly verify that

$$PC = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & (q-1)b_{s-2}(q) - b_{s-3}(q) \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & (q-1)b_{s-3}(q) - b_{s-4}(q) \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & (q-1)b_{s-4}(q) - b_{s-5}(q) \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & (q-1)b_2(q) - b_1(q) \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & (q-1)b_1(q) - b_0(q) \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 - q \end{bmatrix} = D,$$

and our result follows because P is invertible. □

The following lemma simplifies the expression of $f_s(q)$.

LEMMA 2.3. *Let $\{a_t(q)\}_{t \geq 0}$ and $\{b_t(q)\}_{t \geq 0}$ be defined in (2.1) and (2.2). Then*

$$f_s(q) = a_0(q) + a_1(q) + \cdots + a_{s-1}(q) = \begin{cases} -q \cdot b_{k-1}^2(q) & \text{if } s = 2k, \quad k \geq 1; \\ -a_k^2(q) & \text{if } s = 2k + 1, \quad k \geq 0. \end{cases}$$

Proof. First, we have $f_1(q) = a_0(q) = -1 = -a_0^2(q)$, $f_2(q) = a_0(q) + a_1(q) = -q = -q \cdot b_0^2(q)$ and $f_3(q) = a_0(q) + a_1(q) + a_2(q) = -(q-1)^2 = -a_1^2(q)$. Thus, the result holds for $s = 1, 2, 3$.

Suppose that our result holds for $s = 2k - 1, 2k, 2k + 1$ ($k \geq 1$), i.e.,

$$(2.6) \quad f_{2k-1}(q) = a_0(q) + a_1(q) + \cdots + a_{2k-2}(q) = -a_{k-1}^2(q),$$

$$(2.7) \quad f_{2k}(q) = a_0(q) + a_1(q) + \cdots + a_{2k-1}(q) = -qb_{k-1}^2(q),$$

$$(2.8) \quad f_{2k+1}(q) = a_0(q) + a_1(q) + \cdots + a_{2k}(q) = -a_k^2(q).$$

It needs to show that the result holds for $s = 2k + 2, 2k + 3$, i.e.,

$$(2.9) \quad f_{2k+2}(q) = a_0(q) + a_1(q) + \cdots + a_{2k+1}(q) = -q \cdot b_k^2(q),$$

$$(2.10) \quad f_{2k+3}(q) = a_0(q) + a_1(q) + \cdots + a_{2k+2}(q) = -a_{k+1}^2(q).$$

From (2.6)–(2.8), we get

$$(2.11) \quad a_{2k-1}(q) = a_{k-1}^2(q) - q \cdot b_{k-1}^2(q) \quad \text{and} \quad a_{2k}(q) = q \cdot b_{k-1}^2(q) - a_k^2(q).$$

Thus, from (2.8), (2.11) and Lemma 2.1, we have

$$\begin{aligned} & a_0(q) + a_1(q) + \cdots + a_{2k}(q) + a_{2k+1}(q) \\ &= -a_k^2(q) + a_{2k+1}(q) \\ &= -a_k^2(q) + (q-2)a_{2k}(q) - a_{2k-1}(q) \\ &= -a_k^2(q) + (q-2)[q \cdot b_{k-1}^2(q) - a_k^2(q)] - [a_{k-1}^2(q) - q \cdot b_{k-1}^2(q)] \\ &= -(q-1)a_k^2(q) + q(q-1)b_{k-1}^2(q) - a_{k-1}^2(q) \\ &= -(q-1)[b_k(q) + b_{k-1}(q)]^2 + q(q-1)b_{k-1}^2(q) - [b_{k-1}(q) + b_{k-2}(q)]^2 \\ &= -(q-1)b_k^2(q) - 2(q-1)b_k(q)b_{k-1}(q) + (qb_{k-1}(q) + b_{k-2}(q)) \cdot b_k(q) \\ &= -(q-1)b_k^2(q) - b_k(q) \cdot [(q-2)b_{k-1}(q) - b_{k-2}(q)] \\ &= -(q-1)b_k^2(q) - b_k^2(q) \\ &= -q \cdot b_k^2(q). \end{aligned}$$

It follows that (2.9) holds. Furthermore, by (2.8) and (2.9), we know that

$$(2.12) \quad a_{2k+1}(q) = a_k^2(q) - q \cdot b_k^2(q).$$

Then, from (2.9), (2.12) and Lemma 2.1, we get

$$\begin{aligned} & a_0(q) + a_1(q) + \cdots + a_{2k+2}(q) \\ &= -q \cdot b_k^2(q) + a_{2k+2}(q) \\ &= -q \cdot b_k^2(q) + (q-2)a_{2k+1}(q) - a_{2k}(q) \\ &= -q \cdot b_k^2(q) + (q-2)[a_k^2(q) - q \cdot b_k^2(q)] - [q \cdot b_{k-1}^2(q) - a_k^2(q)] \\ &= -q(q-1)b_k^2(q) + (q-1)a_k^2(q) - q \cdot b_{k-1}^2(q) \\ &= -q(q-1)\left[\frac{(q-1)a_k(q) - a_{k-1}(q)}{q}\right]^2 + (q-1)a_k^2(q) - q \cdot \left[\frac{(q-1)a_{k-1}(q) - a_{k-2}(q)}{q}\right]^2 \\ &= -q(q-1)\left[\frac{a_{k+1}(q) + a_k(q)}{q}\right]^2 + (q-1)a_k^2(q) - q \cdot \left[\frac{a_k(q) + a_{k-1}(q)}{q}\right]^2 \\ &= \frac{1}{q}[-(q-1)a_{k+1}^2(q) - 2(q-1)a_{k+1}(q)a_k(q) + (qa_k(q) + a_{k-1}(q)) \cdot a_{k+1}(q)] \\ &= \frac{1}{q}[-(q-1)a_{k+1}^2(q) - a_{k+1}(q) \cdot [(q-2)a_k(q) - a_{k-1}(q)]] \\ &= \frac{1}{q}[-(q-1)a_{k+1}^2(q) - a_{k+1}^2(q)] \\ &= -a_{k+1}^2(q). \end{aligned}$$

It follows that (2.10) holds. We complete the proof. \square

3. The graphs with distinct Q-eigenvalues. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and $P_s = u_1 u_2 \dots u_s$ a path of length $s-1$. The rooted product graph of G and P_s , denoted by $\hat{G}_s = G \circ P_s$ ($s \geq 1$), is the graph described in Fig. 1 with vertex set $V(\hat{G}_s) = \{(v_i, u_j) \mid 1 \leq i \leq n, 1 \leq j \leq s\}$, and edge set $E(\hat{G}_s) = \{(v_i, u_1) \sim (v_j, u_1) \mid v_i v_j \in E(G)\} \cup \{(v_i, u_k) \sim (v_i, u_{k+1}) \mid 1 \leq i \leq n, 1 \leq k \leq s-1\}$.

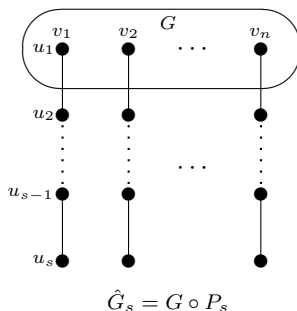


FIGURE 1. The rooted product graph $\hat{G}_s = G \circ P_s$.

If we put $V_j = \{(v_i, u_j) \mid 1 \leq i \leq n\}$ for $1 \leq j \leq s$, then $V(\hat{G}_s) = V_1 \cup V_2 \cup \dots \cup V_s$ is a partition, the adjacency matrix of \hat{G}_s can be written as the block matrix:

$$(3.13) \quad A(\hat{G}_s) = \begin{bmatrix} A(G) & I_n & 0 & \dots & 0 & 0 & 0 \\ I_n & 0 & I_n & \dots & 0 & 0 & 0 \\ 0 & I_n & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & I_n & 0 \\ 0 & 0 & 0 & \dots & I_n & 0 & I_n \\ 0 & 0 & 0 & \dots & 0 & I_n & 0 \end{bmatrix} \begin{matrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{s-2} \\ V_{s-1} \\ V_s \end{matrix}$$

and the corresponding diagonal degree matrix of $A(\hat{G}_s)$ is $D(\hat{G}_s) = \text{diag}(D(G) + I_n, 2I_n, 2I_n, \dots, 2I_n, I_n)$. Hence, we get the Q-matrix of \hat{G}_s as below:

$$(3.14) \quad Q(\hat{G}_s) = \begin{bmatrix} Q(G) + I_n & I_n & 0 & \dots & 0 & 0 & 0 \\ I_n & 2I_n & I_n & \dots & 0 & 0 & 0 \\ 0 & I_n & 2I_n & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2I_n & I_n & 0 \\ 0 & 0 & 0 & \dots & I_n & 2I_n & I_n \\ 0 & 0 & 0 & \dots & 0 & I_n & I_n \end{bmatrix} \begin{matrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{s-2} \\ V_{s-1} \\ V_s \end{matrix}$$

LEMMA 3.1. Let $\hat{G}_s = G \circ P_s$ with the Q-matrix in (3.14). If q is a Q-eigenvalue of \hat{G}_s with the corresponding eigenvector

$$\hat{\mathbf{x}} = [\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_s^T]^T,$$

where the entries of \mathbf{x}_t correspond to the vertices in V_t for $t = 1, 2, \dots, s$, then $\xi = \frac{a_s(q)}{a_{s-1}(q)} + 1$ is a Q-eigenvalue of G associated with eigenvector \mathbf{x}_1 , and $\mathbf{x}_t = \frac{a_{s-t}(q)}{a_{s-1}(q)} \mathbf{x}_1$ for $t = 2, 3, \dots, s$, where $a_t(q)$ is defined in (2.1).

Proof. Since q is an eigenvalue of $Q_s = Q(\hat{G}_s)$ with corresponding eigenvector $\hat{\mathbf{x}}$, we have $Q_s \hat{\mathbf{x}} = q \hat{\mathbf{x}}$. It follows (3.15) and (3.16) from (3.14)

$$(3.15) \quad (Q(G) + I_n)\mathbf{x}_1 + \mathbf{x}_2 = q\mathbf{x}_1,$$

$$(3.16) \quad \begin{cases} \mathbf{x}_1 + 2\mathbf{x}_2 + \mathbf{x}_3 = q\mathbf{x}_2, \\ \mathbf{x}_2 + 2\mathbf{x}_3 + \mathbf{x}_4 = q\mathbf{x}_3, \\ \vdots \\ \mathbf{x}_{s-3} + 2\mathbf{x}_{s-2} + \mathbf{x}_{s-1} = q\mathbf{x}_{s-2}, \\ \mathbf{x}_{s-2} + 2\mathbf{x}_{s-1} + \mathbf{x}_s = q\mathbf{x}_{s-1}, \\ \mathbf{x}_{s-1} + \mathbf{x}_s = q\mathbf{x}_s. \end{cases}$$

Note that (3.15) and (3.16) are equivalent to

$$(3.17) \quad Q(G)\mathbf{x}_1 + \mathbf{x}_2 = (q-1)\mathbf{x}_1,$$

$$(3.18) \quad \begin{cases} \mathbf{x}_1 + \mathbf{x}_3 = (q-2)\mathbf{x}_2, \\ \mathbf{x}_2 + \mathbf{x}_4 = (q-2)\mathbf{x}_3, \\ \vdots \\ \mathbf{x}_{s-3} + \mathbf{x}_{s-1} = (q-2)\mathbf{x}_{s-2}, \\ \mathbf{x}_{s-2} + \mathbf{x}_s = (q-2)\mathbf{x}_{s-1}, \\ \mathbf{x}_{s-1} = (q-1)\mathbf{x}_s. \end{cases}$$

Let C and D be the two matrices defined in Section 2. Clearly, (3.18) is equivalent to $C\hat{\mathbf{x}} = 0$. By Lemma 2.2, we get $D\hat{\mathbf{x}} = 0$, that is,

$$(3.19) \quad D\hat{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{s-1}(q) \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & a_{s-2}(q) \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & a_{s-3}(q) \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_3(q) \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & a_2(q) \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & a_1(q) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \vdots \\ \mathbf{x}_{s-2} \\ \mathbf{x}_{s-1} \\ \mathbf{x}_s \end{bmatrix} = 0$$

We claim that $a_{s-1}(q) \neq 0$. Since otherwise, we have $\mathbf{x}_1 = 0$ from (3.19), and then obtain $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_s = 0$ by (3.17) and (3.18). Thus, according to (3.19), we get

$$\begin{cases} \mathbf{x}_1 = -a_{s-1}(q)\mathbf{x}_s, \\ \mathbf{x}_2 = -a_{s-2}(q)\mathbf{x}_s, \\ \mathbf{x}_3 = -a_{s-3}(q)\mathbf{x}_s, \\ \vdots \\ \mathbf{x}_{s-3} = -a_3(q)\mathbf{x}_s, \\ \mathbf{x}_{s-2} = -a_2(q)\mathbf{x}_s, \\ \mathbf{x}_{s-1} = -a_1(q)\mathbf{x}_s, \end{cases} \iff \begin{cases} \mathbf{x}_2 = \frac{a_{s-2}(q)}{a_{s-1}(q)}\mathbf{x}_1, \\ \mathbf{x}_3 = \frac{a_{s-3}(q)}{a_{s-1}(q)}\mathbf{x}_1, \\ \mathbf{x}_4 = \frac{a_{s-4}(q)}{a_{s-1}(q)}\mathbf{x}_1, \\ \vdots \\ \mathbf{x}_{s-2} = \frac{a_2(q)}{a_{s-1}(q)}\mathbf{x}_1, \\ \mathbf{x}_{s-1} = \frac{a_1(q)}{a_{s-1}(q)}\mathbf{x}_1, \\ \mathbf{x}_s = \frac{a_0(q)}{a_{s-1}(q)}\mathbf{x}_1. \end{cases}$$

Returning $\mathbf{x}_2 = \frac{a_{s-2}(q)}{a_{s-1}(q)}\mathbf{x}_1$ into (3.17), we get

$$(3.20) \quad Q(G)\mathbf{x}_1 = \left(q - 1 - \frac{a_{s-2}(q)}{a_{s-1}(q)} \right) \mathbf{x}_1 = \left(\frac{a_s(q)}{a_{s-1}(q)} + 1 \right) \mathbf{x}_1.$$

Hence, $\xi = \frac{a_s(q)}{a_{s-1}(q)} + 1$ is a Q -eigenvalue of G associated with eigenvector \mathbf{x}_1 . □

From Lemma 3.1, we know that each Q -eigenvalue q of \hat{G}_s satisfies the equation $a_s(q) + (1 - \xi)a_{s-1}(q) = 0$ for some Q -eigenvalue ξ of G . For this reason, let $\text{Spec}_Q(G) = \{\xi_1^{m_1}, \xi_2^{m_2}, \dots, \xi_d^{m_d}\}$ be the Q -spectrum of G . For $\xi_i \in \text{Spec}_Q(G)$, we define a polynomial of q with degree s as below:

$$(3.21) \quad p_i^{(s)}(q) = a_s(q) + (1 - \xi_i)a_{s-1}(q).$$

Denote by $q_{1,i}, q_{2,i}, \dots, q_{s,i}$ all the roots of $p_i^{(s)}(q)$ and put $\hat{S}(\xi_i) = \{q_{1,i}, q_{2,i}, \dots, q_{s,i}\}$ which we will use frequently in what follows.

Suppose that $g(x)$ and $r(x)$ are real polynomials with real, simple, and disjoint, zeros, and that $\deg(g(x)) > \deg(r(x))$. We say that the zeros of $g(x)$ and $r(x)$ *interlace* if each zero of $r(x)$ lies between two zeros of $g(x)$, and there is at most one zero of $r(x)$ between any two adjacent zeros of $g(x)$.

LEMMA 3.2. (See [1], p. 249) *Suppose that $\{g_n(x)\}_{n \geq 0}$ is a sequence of polynomials defined by a three-term recurrence relation of the form*

$$g_{n+2}(x) = (x - \alpha_{n+1})g_{n+1}(x) - \beta_n g_n(x) \quad \text{with } g_0(x) = 1, g_1(x) = x - \alpha_0,$$

where $\alpha_n \in \mathbb{R}$ and $\beta_n > 0$ for $n = 0, 1, 2, \dots$. Then, for $n \geq 1$, $g_n(x)$ has n real, distinct roots, and the roots of $g_n(x)$ and $g_{n+1}(x)$ interlace.

The following result due to G. Szego (see [9], p. 46, Theorem 3.3.4) guarantees all roots in $\hat{S}(\xi_i)$ are real and simple. Here we would like to rewrite the proof in detail for self-contained.

LEMMA 3.3. *Suppose that $\{g_n(x)\}_{n \geq 0}$ is a sequence of polynomials defined by a three-term recurrence relation of the form*

$$(3.22) \quad g_{n+1}(x) = (x - \alpha)g_n(x) - g_{n-1}(x) \quad \text{with } g_0(x) = 1, g_1(x) = x - 1,$$

where $\alpha \in \mathbb{R}$. Let c be an arbitrary real constant. Then the polynomial $g_{n+1}(x) - cg_n(x)$ has $n + 1$ distinct real zeros.

Proof. The recurrence formula (3.22) is valid for $n = 1$ if we write $g_{-1}(x) = -1$. By the recurrence formula (3.22), we have

$$\frac{g_{n+1}(x)g_n(y) - g_n(x)g_{n+1}(y)}{x - y} = g_n(x)g_n(y) + \frac{g_n(x)g_{n-1}(y) - g_{n-1}(x)g_n(y)}{x - y}.$$

On replacing n by $0, 1, 2, \dots, n$ and adding, we obtain

$$\frac{g_{n+1}(x)g_n(y) - g_n(x)g_{n+1}(y)}{x - y} = \sum_{i=0}^n g_i(x)g_i(y).$$

Taking $y \rightarrow x$, we get $g'_{n+1}(x)g_n(x) - g_{n+1}(x)g'_n(x) = \sum_{i=0}^n g_i^2(x)$, which implies that

$$\left(\frac{g_{n+1}(x)}{g_n(x)} \right)' = \frac{g'_{n+1}(x)g_n(x) - g_{n+1}(x)g'_n(x)}{g_n^2(x)} = \frac{\sum_{i=0}^n g_i^2(x)}{g_n^2(x)} > 0.$$

By Lemma 3.2, we can assume that $g_n(x) = (x - v_1)(x - v_2) \cdots (x - v_n)$, where $v_i \neq v_j$ for $i \neq j$. Then $\frac{g_{n+1}(x)}{g_n(x)}$ has v_1, v_2, \dots, v_n as its poles. Also, from (3.22) we see that $\frac{g_{n+1}(x)}{g_n(x)}$ is asymptotic to $h(x) = x - \alpha$ as $|x| \rightarrow +\infty$. Therefore, $\frac{g_{n+1}(x)}{g_n(x)}$ is strictly increasing from $-\infty$ to $+\infty$ on each of the intervals

$$(-\infty, v_1), (v_1, v_2), \dots, (v_{n-1}, v_n), (v_n, +\infty),$$

which implies $\frac{g_{n+1}(x)}{g_n(x)}$ and $l(x) = c$ have $n + 1$ real, distinct intersection because c is a real constant. Hence, the polynomial $g_{n+1}(x) - cg_n(x)$ has $n + 1$ distinct real zeros. \square

By applying Lemma 3.3 to the sequence of polynomials $\{-a_t(q)\}_{t \geq 0}$, we have the following theorem.

THEOREM 3.4. *Let $\{a_t(q)\}_{t \geq 0}$ be a sequence of polynomials defined in (2.1). Then all the roots of $p_i^{(s)}(q) = a_s(q) + (1 - \xi_i)a_{s-1}(q)$ are real and simple, where $\xi_i \in \text{Spec}_Q(G)$.*

Theorem 3.4 implies that the polynomial $p_i^{(s)}(q)$ has s distinct real roots. In order to give the corresponding relation between the Q -spectra of G and \hat{G}_s , we also need the following lemma.

LEMMA 3.5. *Let q_0 be a root of $p_i^{(s)}(q) = a_s(q) + (1 - \xi_i)a_{s-1}(q)$, where $s \geq 1$ and $\xi_i \in \text{Spec}_Q(G)$. Then $a_{s-1}(q_0) \neq 0$.*

Proof. By contradiction, we assume that $a_{s-1}(q_0) = 0$. Then $a_s(q_0) = 0$ because $p_i^{(s)}(q_0) = 0$, and so $a_{s-2}(q_0) = q_0 a_{s-1}(q_0) - a_s(q_0) = 0$. Successively, we have $a_{s-3}(q_0) = \cdots = a_1(q_0) = a_0(q_0) = 0$, which is a impossible because $a_0(q_0) = -1$. \square

LEMMA 3.6. *For $\xi_i \neq \xi_j \in \text{Spec}_Q(G)$, we have $\hat{S}(\xi_i) \cap \hat{S}(\xi_j) = \emptyset$, where $\hat{S}(\xi_i)$ (resp., $\hat{S}(\xi_j)$) denotes the set of roots of the polynomial $p_i^{(s)}(q)$ (resp., $p_j^{(s)}(q)$) defined in (3.21).*

Proof. By the way of contradiction, suppose that $\hat{S}(\xi_i) \cap \hat{S}(\xi_j) \neq \emptyset$, then $p_i^{(s)}(q)$ and $p_j^{(s)}(q)$ have a common root q_0 for $\xi_i \neq \xi_j$. Then $0 = p_i^{(s)}(q_0) - p_j^{(s)}(q_0) = (\xi_j - \xi_i)a_{s-1}(q_0)$. It follows that $a_{s-1}(q_0) = 0$. However, we know that $a_{s-1}(q_0) \neq 0$ by Lemma 3.5, which is a contradiction. \square

REMARK 3.7. In fact, Theorem 3.4, Lemma 3.5 and Lemma 3.6 always hold when ξ_i and ξ_j are arbitrary real numbers.

Now we provide the corresponding relation of the Q -spectra between G and \hat{G}_s in the following theorem.

THEOREM 3.1. *Let $\text{Spec}_Q(G) = \{\xi_1^{m_1}, \xi_2^{m_2}, \dots, \xi_d^{m_d}\}$ be the Q -spectrum of G , and $\hat{S}(\xi_i)$ the set of roots of the polynomial $p_i^{(s)}(q)$ defined in (3.21). Then $\text{Spec}_Q(\hat{G}_s) = m_1 \hat{S}(\xi_1) \cup m_2 \hat{S}(\xi_2) \cup \cdots \cup m_d \hat{S}(\xi_d)$, where $m_i \hat{S}(\xi_i)$ denotes the multiset obtained by the union of m_i copies of $\hat{S}(\xi_i)$, and $\hat{S}(\xi_i) \cap \hat{S}(\xi_j) = \emptyset$ for $1 \leq i \neq j \leq d$.*

Proof. For the graph G , denote by $\mathcal{E}_G(\xi_i) = \langle \mathbf{y}_{1i}, \mathbf{y}_{2i}, \dots, \mathbf{y}_{m_i i} \rangle$ the eigenspace corresponding to ξ_i for each i ($1 \leq i \leq d$). For each $q_{r,i} \in \hat{S}(\xi_i)$ ($1 \leq r \leq s$), we construct a vector

$$\mathbf{b}(q_{r,i}) = \left[1, \frac{a_{s-2}(q_{r,i})}{a_{s-1}(q_{r,i})}, \dots, \frac{a_1(q_{r,i})}{a_{s-1}(q_{r,i})}, \frac{a_0(q_{r,i})}{a_{s-1}(q_{r,i})} \right]^T,$$

where $\{a_t(q_{r,i})\}_{t \geq 0}$ is defined in (2.1) and $a_{s-1}(q_{r,i}) \neq 0$ by Lemma 3.5, and then set

$$\hat{\mathbf{y}}_k(q_{r,i}) = \mathbf{b}(q_{r,i}) \otimes \mathbf{y}_{ki} = \left[\mathbf{y}_{ki}^T, \frac{a_{s-2}(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki}^T, \dots, \frac{a_1(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki}^T, \frac{a_0(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki}^T \right]^T,$$

where $1 \leq k \leq m_i$. Next we will verify that $\hat{\mathbf{y}}_1(q_{r,i}), \dots, \hat{\mathbf{y}}_{m_i}(q_{r,i})$ are linearly independent eigenvectors of $Q(\hat{G}_s)$ with respect to the eigenvalue $q_{r,i}$. In fact, we know that $a_t(q_{r,i}) = (q_{r,i} - 2)a_{t-1}(q_{r,i}) - a_{t-2}(q_{r,i})$

($t = 2, \dots, s$) with the initial condition $a_0(q_{r,i}) = -1$ and $a_1(q_{r,i}) = 1 - q_{r,i}$. Then $a_t(q_{r,i}) + 2a_{t-1}(q_{r,i}) + a_{t-2}(q_{r,i}) = q_{r,i}a_{t-1}(q_{r,i})$, and so we get

$$(3.23) \quad \frac{a_t(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki} + 2 \frac{a_{t-1}(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki} + \frac{a_{t-2}(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki} = q_{r,i} \frac{a_{t-1}(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki}$$

for $t = 2, \dots, s$, because $a_{s-1}(q_{r,i}) \neq 0$ by Lemma 3.5. In particular, if we take $t = s$, combining (3.20) and (3.23), we obtain

$$(3.24) \quad Q(G) \mathbf{y}_{ki} + \mathbf{y}_{ki} + \frac{a_{s-2}(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki} = q_{r,i} \mathbf{y}_{ki}.$$

Moreover, by the initial condition, we have

$$(3.25) \quad a_1(q_{r,i}) + a_0(q_{r,i}) = q_{r,i} \cdot a_0(q_{r,i}).$$

Therefore, from (3.14) and (3.23)–(3.25), we can verify that

$$\begin{aligned} Q(\hat{G}_s) \hat{\mathbf{y}}_k(q_{r,i}) &= \begin{bmatrix} Q(G) + I_n & I_n & 0 & \cdots & 0 & 0 \\ I_n & 2I_n & I_n & \cdots & 0 & 0 \\ 0 & I_n & 2I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2I_n & I_n \\ 0 & 0 & 0 & \cdots & I_n & I_n \end{bmatrix} \begin{bmatrix} \mathbf{y}_{ki} \\ \frac{a_{s-2}(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki} \\ \frac{a_{s-3}(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki} \\ \vdots \\ \frac{a_1(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki} \\ \frac{a_0(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki} \end{bmatrix} \\ &= q_{r,i} \begin{bmatrix} \mathbf{y}_{ki} \\ \frac{a_{s-2}(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki} \\ \frac{a_{s-3}(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki} \\ \vdots \\ \frac{a_1(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki} \\ \frac{a_0(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_{ki} \end{bmatrix} = q_{r,i} \hat{\mathbf{y}}_k(q_{r,i}). \end{aligned}$$

Hence, $\hat{\mathbf{y}}_k(q_{r,i})$ is an eigenvector of $Q(\hat{G}_s)$ with respect $q_{r,i}$ for each $1 \leq k \leq m_i$. Moreover, we see that $\hat{\mathbf{y}}_1(q_{r,i}), \dots, \hat{\mathbf{y}}_{m_i}(q_{r,i})$ are linearly independent because $\mathbf{y}_{1i}, \dots, \mathbf{y}_{m_i i}$ are linearly independent.

By Theorem 3.4, we know that for each ξ_i , the roots $q_{1,i}, \dots, q_{s,i} \in \hat{S}(\xi_i)$ are real and distinct. Furthermore, for any two distinct ξ_i and ξ_j , we have $\hat{S}(\xi_i) \cap \hat{S}(\xi_j) = \emptyset$ by Lemma 3.6, and so $q_{r,i} \neq q_{u,j}$ for $1 \leq r, u \leq s$. Hence, the eigenvectors belonging to $\{\mathbf{y}_k(q_{r,i}) \mid 1 \leq i \leq d, 1 \leq r \leq s, 1 \leq k \leq m_i\}$ are linearly independent, and we have obtained $\sum_{i=1}^d sm_i = sn = |V(\hat{G}_s)|$ such eigenvectors of $Q(\hat{G}_s)$. Therefore, the Q -eigenvalue $q_{r,i}$ of \hat{G}_s has multiplicity exactly m_i for $1 \leq i \leq d$ and $1 \leq r \leq s$.

This completes the proof. □

Recall that \mathcal{G}^Q (resp., \mathcal{G}_n^Q) denotes the set of connected graphs (resp. with n vertices) whose Q -eigenvalues are mutually distinct. We have the following result immediately.

COROLLARY 3.8. *Let G be a graph of order n . If G has d ($d \leq n$) distinct Q -eigenvalues, then \hat{G}_s has exactly sd distinct Q -eigenvalues. In particular, $\hat{G}_s \in \mathcal{G}_{sn}^Q$ if $G \in \mathcal{G}_n^Q$.*

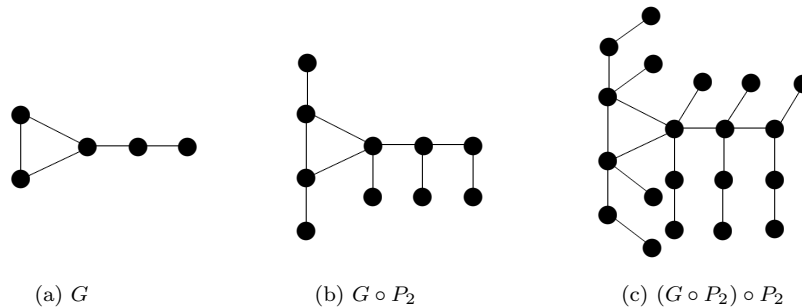


FIGURE 2. *Graphs with distinct Q -eigenvalues.*

In 1974, Harary and Schwenk in [7] posed an interesting problem: “Which graphs have distinct eigenvalues?” As we know there are few results on this problem after 1974. At the same time, there are few results to answer the question “Which graphs have distinct Q -eigenvalues?”. The following remark provides a method to construct infinite families of graphs with distinct Q -eigenvalues. Later, we will give a specific example.

REMARK 3.9. Given a graph $G \in \mathcal{G}_n^Q$. By Corollary 3.8, we obtain an infinite family of graphs in \mathcal{G}^Q , i.e., $\{\hat{G}_{s_i}^i \mid i \geq 0\}$, which are recursively defined by $\hat{G}_{s_0}^0 = G$ and $\hat{G}_{s_i}^i = \hat{G}_{s_{i-1}}^{i-1} \circ P_{s_i}$ for $i \geq 1$ and $s_i \geq 1$.

EXAMPLE 3.10. In Fig. 2(a), $\text{Spec}_Q(G) = \{4.6412, 2.7237, 1.4108, 1.00, 0.2243\}$ that is, $G \in \mathcal{G}_5^Q$. By Theorem 3.1, in Fig. 2(b,c), $\hat{G}_2^1 = G \circ P_2$, $\hat{G}_2^2 = (G \circ P_2) \circ P_2$ are also in \mathcal{G}^Q . In fact, by Remark 3.9, we can construct an infinite family of graphs in \mathcal{G}^Q , that is, $\{\hat{G}_{s_i}^i \mid i \geq 0\}$, which are recursively defined by $\hat{G}_{s_0}^0 = G$ and $\hat{G}_{s_i}^i = \hat{G}_{s_{i-1}}^{i-1} \circ P_{s_i}$ for $i \geq 1$ and $s_i \geq 1$.

4. Main Q -eigenvalue and Q -controllable graphs. Recall that connected graphs whose Q -eigenvalues are mutually distinct and main are called Q -controllable graphs, and \mathcal{G}^{Q*} (resp., \mathcal{G}_n^{Q*}) denotes the set of Q -controllable graphs (resp., with n vertices). In this section, we discuss the relation of main Q -eigenvalues between G and \hat{G}_s , and focus on showing that $\hat{G}_s \in \mathcal{G}_{s_n}^{Q*}$ if $G \in \mathcal{G}_n^{Q*}$ is not bipartite.

Let $\{a_t(q)\}_{t \geq 0}$ and $\{b_t(q)\}_{t \geq 0}$ be the two polynomial sequences defined in (2.1) and (2.2). Now we list two results about these two polynomial sequences.

LEMMA 4.1. *Let q_0 be a root of $p_i^{(s)}(q) = a_s(q) + (1 - \xi_i)a_{s-1}(q)$ and $0 \neq \xi_i \in \text{Spec}_Q(G)$. Then $b_{k-1}(q_0) \neq 0$ if $s = 2k$, and $a_k(q_0) \neq 0$ if $s = 2k + 1$.*

Proof. If $s = 2k$, we shall show that $b_{k-1}(q_0) \neq 0$. By contradiction, assume that $b_{k-1}(q_0) = 0$. Then

$$b_k(q_0) = (q_0 - 2)b_{k-1}(q_0) - b_{k-2}(q_0) = -b_{k-2}(q_0),$$

and so

$$\begin{aligned}
 a_k(q_0) &= (q_0 - 1)b_{k-1}(q_0) - b_{k-2}(q_0) = -b_{k-2}(q_0) = a_0(q_0)b_{k-2}(q_0), \\
 a_{k+1}(q_0) &= (q_0 - 1)b_k(q_0) = -(q_0 - 1)b_{k-2}(q_0) = a_1(q_0)b_{k-2}(q_0), \\
 a_{k+2}(q_0) &= (q_0 - 2)a_{k+1}(q_0) - a_k(q_0) \\
 &= (q_0 - 2)a_1(q_0)b_{k-2}(q_0) - a_0(q_0)b_{k-1}(q_0) = a_2(q_0)b_{k-2}(q_0), \\
 a_{k+3}(q_0) &= (q_0 - 2)a_{k+2}(q_0) - a_{k+1}(q_0) \\
 &= (q_0 - 2)a_2(q_0)b_{k-2}(q_0) - a_1(q_0)b_{k-2}(q_0) = a_3(q_0)b_{k-2}(q_0), \\
 &\vdots \\
 a_{s-2}(q_0) &= a_{2k-2}(q_0) = a_{k-2}(q_0)b_{k-2}(q_0), \\
 a_{s-1}(q_0) &= a_{2k-1}(q_0) = a_{k-1}(q_0)b_{k-2}(q_0).
 \end{aligned}$$

By Lemma 3.5, we know that $a_{s-1}(q_0) \neq 0$. Then we have $a_{k-1}(q_0) \neq 0$ and $b_{k-2}(q_0) \neq 0$. Note that $a_s(q_0) + (1 - \xi_i)a_{s-1}(q_0) = 0$, combining this with Lemma 2.1, we get

$$\xi_i = q_0 - 1 - \frac{a_{s-2}(q_0)}{a_{s-1}(q_0)} = q_0 - 1 - \frac{a_{k-2}(q_0)}{a_{k-1}(q_0)} = \frac{q \cdot b_{k-1}(q_0)}{a_{k-1}(q_0)} = 0,$$

a contradiction. Thus, $b_{k-1}(q_0) \neq 0$.

Similarly, if $s = 2k + 1$ one can easily prove that $a_k(q_0) \neq 0$, and our result follows. □

By Lemma 4.1, we have the following theorem.

THEOREM 4.1. For $\xi_i \in \text{Spec}_Q(G)$, let $\hat{S}(\xi_i)$ be the set of roots of the polynomial $p_i^{(s)}(q)$ defined in (3.21). Then we have

- (1) If ξ_i is a main Q-eigenvalue and $\xi_i \neq 0$, then all the Q-eigenvalues of \hat{G}_s in $\hat{S}(\xi_i)$ are non-zero and main.
- (2) If ξ_i is a non-main Q-eigenvalue, then all the Q-eigenvalues of \hat{G}_s in $\hat{S}(\xi_i)$ are non-main.

Proof. Suppose that ξ_i is a main Q-eigenvalue of G . By Theorem 3.1, we know that each element of $\hat{S}(\xi_i)$ is a Q-eigenvalue of \hat{G}_s . For each $q_{r,i} \in \hat{S}(\xi_i)$ ($1 \leq r \leq s$), we know that $q_{r,i}$ is a root of $p_i^{(s)}(q) = a_s(q) + (1 - \xi_i)a_{s-1}(q)$, and from (2.1), we have

$$p_i^{(s)}(0) = a_s(0) + (1 - \xi_i)a_{s-1}(0) = \begin{cases} -\xi_i & \text{if } s \text{ is even,} \\ \xi_i & \text{if } s \text{ is odd.} \end{cases}$$

Thus, we conclude that $q_{r,i} \neq 0$ for $1 \leq r \leq s$ since $\xi_i \neq 0$. Let \mathbf{y}_i be an eigenvector of $Q(G)$ with respect to ξ_i such that $\mathbf{j}_n^T \mathbf{y}_i \neq 0$. Then, according to the proof of Theorem 3.1, we know that

$$\hat{\mathbf{y}}(q_{r,i}) = \left[\mathbf{y}_i^T, \frac{a_{s-2}(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_i^T, \dots, \frac{a_1(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_i^T, \frac{a_0(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_i^T \right]^T$$

is an eigenvector of $Q(\hat{G}_s)$ with respect to $q_{r,i}$, and from (2.3) we have

$$\begin{aligned} \mathbf{j}_{sn}^T \hat{\mathbf{y}}(q_{r,i}) &= [\mathbf{j}_n^T, \mathbf{j}_n^T, \dots, \mathbf{j}_n^T] \left[\mathbf{y}_i^T, \frac{a_{s-2}(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_i^T, \dots, \frac{a_1(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_i^T, \frac{a_0(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{y}_i^T \right]^T \\ &= \left(1 + \frac{a_{s-2}(q_{r,i})}{a_{s-1}(q_{r,i})} + \dots + \frac{a_1(q_{r,i})}{a_{s-1}(q_{r,i})} + \frac{a_0(q_{r,i})}{a_{s-1}(q_{r,i})} \right) (\mathbf{j}_n^T \mathbf{y}_i) \\ &= \frac{a_{s-1}(q_{r,i}) + a_{s-2}(q_{r,i}) + \dots + a_1(q_{r,i}) + a_0(q_{r,i})}{a_{s-1}(q_{r,i})} (\mathbf{j}_n^T \mathbf{y}_i) \\ &= \frac{f_s(q_{r,i})}{a_{s-1}(q_{r,i})} (\mathbf{j}_n^T \mathbf{y}_i). \end{aligned}$$

Since $0 \neq \xi_i \in \text{Spec}_Q(G)$ and $q_{r,i}$ is a root of $p_i^{(s)}(q)$. Then $b_{k-1}(q_{r,i}) \neq 0$ for $s = 2k$ and $a_k(q_{r,i}) \neq 0$ for $s = 2k + 1$ by Lemma 4.1. By Lemma 2.3 and $q_{r,i} \neq 0$, we have $f_s(q_{r,i}) \neq 0$, and consequently $\mathbf{j}_{sn}^T \hat{\mathbf{y}}(q_{r,i}) \neq 0$. Hence, $q_{r,i}$ is a main Q -eigenvalue of \hat{G}_s . Thus, (1) follows.

Now suppose that ξ_i is a non-main Q -eigenvalue of G . For each $q_{r,i} \in \hat{S}(\xi_i)$ ($1 \leq r \leq s$), we assume that $\hat{\mathbf{x}} = [\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_{s-1}^T, \mathbf{x}_s^T]^T$ is an eigenvector of $Q(\hat{G}_s)$ corresponding to $q_{r,i}$ as in Lemma 3.1. By Lemma 3.1, \mathbf{x}_1 is an eigenvector of $Q(G)$ with respect to ξ_i and $\mathbf{x}_t = \frac{a_{s-t}(q_{r,i})}{a_{s-1}(q_{r,i})} \mathbf{x}_1$ for $t = 2, 3, \dots, s$. Since ξ_i is non-main, we have $\mathbf{j}_n^T \mathbf{x}_1 = 0$, and so $\mathbf{j}_{sn}^T \hat{\mathbf{x}} = 0$. Thus, (2) follows.

This completes the proof. □

Recall that G is a connected graph throughout this paper. It is known that 0 is a Q -eigenvalue of G if and only if G is bipartite. Now we give the main result of this section.

THEOREM 4.2. *If $G \in \mathcal{G}_n^{Q^*}$ is not a bipartite graph, then $\hat{G}_s \in \mathcal{G}_{sn}^{Q^*}$.*

Proof. By Corollary 3.8, all the Q -eigenvalues of \hat{G}_s are simple, i.e., $\hat{G}_s \in \mathcal{G}_{sn}$. Since G is not a bipartite graph, all the Q -eigenvalues of G is non-zero. Combining this with $G \in \mathcal{G}_n^{Q^*}$, by Lemma 4.1 we may conclude that all the Q -eigenvalues of \hat{G}_s are also main, and so $\hat{G}_s \in \mathcal{G}_{sn}^{Q^*}$. The result follows. □

In fact, if $G \in \mathcal{G}_n^{Q^*}$ is not a bipartite graph, then $\hat{G}_s \in \mathcal{G}_{sn}^{Q^*}$ is also not a bipartite graph. The following remark provides a method to construct infinite families of graphs in \mathcal{G}^{Q^*} . Later, we will give a specific example.

REMARK 4.2. Given a graph $G \in \mathcal{G}_n^{Q^*}$ which is not bipartite. By Theorem 4.2, we have obtained an infinite family of graphs in \mathcal{G}^{Q^*} , i.e., $\{\hat{G}_{s_i}^i \mid i \geq 0\}$, which are recursively defined by $\hat{G}_{s_0}^0 = G$ and $\hat{G}_{s_i}^i = \hat{G}_{s_{i-1}}^{i-1} \circ P_{s_i}$ for $i \geq 1$ and $s_i \geq 1$.

EXAMPLE 4.3. In Fig. 3(a),

$$\text{Spec}_Q(G) = \{6.2422, 3.5496, 2.6524, 2.0000, 1.0855, 0.4703\}$$

and $G \in \mathcal{G}_6^{Q^*}$ is not a bipartite graph. By Theorem 4.2, in Fig. 3(b,c), $\hat{G}_2^1 = G \circ P_2$, $\hat{G}_2^2 = (G \circ P_2) \circ P_2$ are also in \mathcal{G}^{Q^*} . In fact, by Remark 4.2, we can construct an infinite family of graphs in \mathcal{G}^{Q^*} , that is, $\{\hat{G}_{s_i}^i \mid i \geq 0\}$, which are recursively defined by $\hat{G}_{s_0}^0 = G$ and $\hat{G}_{s_i}^i = \hat{G}_{s_{i-1}}^{i-1} \circ P_{s_i}$ for $i \geq 1$ and $s_i \geq 1$.

5. Construction of non-isomorphic Q -cospectral graphs in \mathcal{G}^Q . Theorem 3.1 and Corollary 3.8 provide us a good method to construct the classes of graphs in \mathcal{G}^Q , respectively. In this section, we give some examples. Additionally, the DQS-property of \hat{G}_s is also considered here.

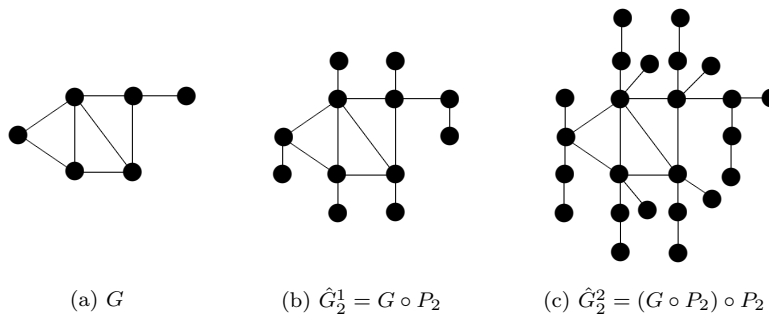


FIGURE 3. Construction of large graphs in G^{Q^*} .

THEOREM 5.1. *Let G and H be two Q -cospectral graphs of order n . Then we have:*

- (1) \hat{G}_s and \hat{H}_s are Q -cospectral for any $s \geq 2$.
- (2) \hat{G}_s and \hat{H}_s are isomorphic if and only if G and H are isomorphic.

Proof. Let $\xi_1, \xi_2, \dots, \xi_n$ be the common Q -eigenvalues (not necessarily different from each other) of G and H . By Theorem 3.1, $\text{Spec}_Q(\hat{G}_s) = \hat{S}(\xi_1) \cup \hat{S}(\xi_2) \cup \dots \cup \hat{S}(\xi_n) = \text{Spec}_Q(\hat{H}_s)$, where $\hat{S}(\xi_i)$ is the root set of $p_i^{(s)}(q) = a_s(q) + (1 - \xi_i)a_{s-1}(q)$. Thus, (1) follows, and (2) is an immediate result by the construction of \hat{G}_s and \hat{H}_s . \square

EXAMPLE 5.1. In Fig. 4, the graphs G and H are a pair of Q -cospectral graphs and $\text{Spec}_Q(G) = \text{Spec}_Q(H) = \{6.3723, 3.0000, 2.0000^2, 0.6722\}$. If $s = 2$, by Theorem 5.1, we obtain that $\hat{G}_2^1 = G \circ P_2$ and $\hat{H}_2^1 = H \circ P_2$ are Q -cospectral but not isomorphic. Actually,

$$\text{Spec}_Q(\hat{G}_2^1) = \text{Spec}_Q(\hat{H}_2^1) = \{7.5255, 4.3028, 3.4142^2, 2.3620, 0.8468, 0.6972, 0.5858^2, 0.2658\}.$$

In fact, for any $s \geq 1$, \hat{G}_s and \hat{H}_s are Q -cospectral and not isomorphic. By Theorem 5.1, we can get infinitely many pairs of non-isomorphic Q -cospectral graphs, i.e., $\hat{G}_{s_i}^i$ and $\hat{H}_{s_i}^i$, which are recursively defined by $\hat{G}_{s_0}^0 = G$, $\hat{H}_{s_0}^0 = H$, and $\hat{G}_{s_i}^i = \hat{G}_{s_{i-1}}^{i-1} \circ P_{s_i}$, $\hat{H}_{s_i}^i = \hat{H}_{s_{i-1}}^{i-1} \circ P_{s_i}$ for $i \geq 1$ and $s_i \geq 1$.

EXAMPLE 5.2. In Fig. 5, the graphs G and H are a pair of Q -cospectral graphs which belong to \mathcal{G}_7^Q , i.e.,

$$\text{Spec}_Q(G) = \text{Spec}_Q(H) = \{4.7757, 3.5892, 2.2763, 2.0000, 1.0000, 0.3588, 0.0000\}$$

If $s = 3$, by Theorem 5.1, we obtain that $\hat{G}_3^1 = G \circ P_3$ and $\hat{H}_3^1 = H \circ P_3$ are Q -cospectral but non-isomorphic.

$$\text{We also verify that } \text{Spec}_Q(\hat{G}_3^1) = \text{Spec}_Q(\hat{H}_3^1) = \left\{ \begin{array}{ccccccc} 6.0363 & 4.9587 & 3.9132 & 3.7321 & 3.2470 & 3.0684 & 3.0 \\ 2.4113 & 2.3182 & 2.0839 & 2.0000 & 1.5550 & 1.1923 & 1.0 \\ 0.3281 & 0.3122 & 0.2791 & 0.2679 & 0.1981 & 0.0981 & 0.0 \end{array} \right\}.$$

In fact, for any $s \geq 1$, \hat{G}_s and \hat{H}_s are Q -cospectral and not isomorphic. By Theorem 5.1, we can get infinitely many pairs of non-isomorphic Q -cospectral graphs belong to \mathcal{G}^Q , i.e., $\hat{G}_{s_i}^i$ and $\hat{H}_{s_i}^i$, which are recursively defined by $\hat{G}_{s_0}^0 = G$, $\hat{H}_{s_0}^0 = H$, and $\hat{G}_{s_i}^i = \hat{G}_{s_{i-1}}^{i-1} \circ P_{s_i}$, $\hat{H}_{s_i}^i = \hat{H}_{s_{i-1}}^{i-1} \circ P_{s_i}$ for $i \geq 1$ and $s_i \geq 1$.

Now, we give the following results about the DQS-problem.

THEOREM 5.2. *If \hat{G}_s is DQS, then G is DQS.*

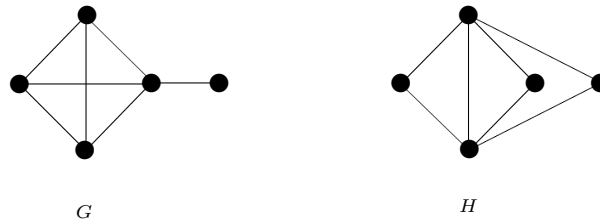


FIGURE 4. A pair of Q -cospectral graphs.

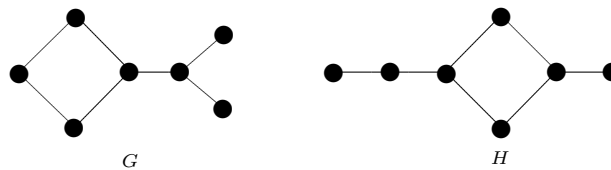


FIGURE 5. A pair of Q -spectral graphs.

Proof. Let H be any graph such that $\text{Spec}_Q(H) = \text{Spec}_Q(G)$. By Theorem 5.1(1) we obtain $\text{Spec}_Q(\hat{H}_s) = \text{Spec}_Q(\hat{G}_s)$, and so $\hat{H}_s \cong \hat{G}_s$ because \hat{G}_s is DQS. Hence, $H \cong G$ by Theorem 5.1(2). \square

Conversely, if G is DQS, we ask whether \hat{G}_s ($s \geq 2$) is also DQS? In [8], M. Mirzakhah and D. Kiani have shown that the sun graph is DQS, that is, if $G = C_n$ (the cycle of order n) and $s = 2$, then $\hat{G}_2 = C_n \circ P_2$ is DQS.

In the following theorem, we give the DQS-property of \hat{G}_s .

THEOREM 5.3. *Let $\hat{G}_s = G \circ P_s$ and $\hat{H}_s = H \circ P_s$ ($s \geq 1$) be two Q -cospectral graphs. Then $\hat{G}_s \cong \hat{H}_s$ if G is DQS.*

Proof. Since $\hat{G}_s = G \circ P_s$ and $\hat{H}_s = H \circ P_s$ are Q -cospectral, we claim that G and H are Q -cospectral. In fact, by the way of contradiction, suppose that $\text{Spec}_Q(G) \neq \text{Spec}_Q(H)$. Let $\text{Spec}_Q(G) = \{\xi_1, \xi_2, \dots, \xi_n\}$ and $\text{Spec}_Q(H) = \{\xi'_1, \xi'_2, \dots, \xi'_n\}$. Then there exists $\xi_i \in \text{Spec}_Q(G)$ such that $\xi_i \neq \xi'_j$ for $1 \leq j \leq n$. Let $\hat{S}(\xi_i)$ be the root set of $p_i^{(s)}(q) = a_s(q) - \xi_i a_{s-1}(q)$ and $\hat{S}(\xi'_j)$ the root set of $p_j^{(s)}(q) = a_s(q) - \xi'_j a_{s-1}(q)$. For each $1 \leq j \leq n$, as Lemma 3.6 and Remark 3.7, we may conclude that $\hat{S}(\xi_i) \cap \hat{S}(\xi'_j) = \emptyset$ because $\xi_i \neq \xi'_j$. Thus, from Theorem 3.1 we know that $\text{Spec}_Q(\hat{G}_s) \neq \text{Spec}_Q(\hat{H}_s) = \hat{S}(\xi'_1) \cup \hat{S}(\xi'_2) \cup \dots \cup \hat{S}(\xi'_n)$, which contracts our assumption. Therefore, $G \cong H$ because G is DQS, and so $\hat{G}_s \cong \hat{H}_s$.

This completes the proof. \square

Finally, we propose a conjecture.

CONJECTURE 1. *If $G \in \mathcal{G}_n^{Q*}$ (with no zero eigenvalue) is DQS, then $\hat{G}_s \in \mathcal{G}_{sn}^{Q*}$ is DQS for any $s \geq 2$.*

Acknowledgment. The authors are grateful to the anonymous referees for their valuable comments and helpful suggestions, which have considerably improved the presentation of this paper. This work is supported by the National Natural Science Foundation of China (grants no. 11671344, 11531011 and 11701492).

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