2016

Graphs with reciprocal eigenvalue properties

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GRAPHS WITH RECIPROCAL EIGENVALUE PROPERTIES[*]

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Abstract. In this paper, only simple graphs are considered. A graph G is nonsingular if its adjacency matrix $A(G)$ is nonsingular. A nonsingular graph G satisfies reciprocal eigenvalue property (*property R*) if the reciprocal of each eigenvalue of the adjacency matrix $A(G)$ is also an eigenvalue of $A(G)$ and G satisfies strong reciprocal eigenvalue property (*property SR*) if the reciprocal of each eigenvalue of the adjacency matrix $A(G)$ is also an eigenvalue of $A(G)$ and they both have the same multiplicities. From the definitions property SR implies property R. Furthermore, for some classes of graphs (for example, trees), it is known that these properties are equivalent. However, the equivalence of these two properties is not yet known for any nonsingular graph. In this article, it is shown that these properties are not equivalent in general.

Key words. Graph, Adjacency matrix, Property R, Property SR.

AMS subject classifications. 05C50, 15A18.

1. Introduction. Let G be a simple, undirected graph on n vertices. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. We use $[i, j]$ to denote an edge between the vertices i and j. By P_n, we denote the path on n vertices. The notation $i \sim j$ means 'i is adjacent to j'. The adjacency matrix $A(G)$ of G is the square symmetric matrix of size n whose (i, j)-th entry a_{ij} is 1 if $[i, j] \in E(G)$ and 0 otherwise.

Let $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ be the eigenvalues of $A(G)$. We say λ is an eigenvalue of G if λ is an eigenvalue of $A(G)$. We use $\sigma(G)$ to denote the spectrum of G. A graph G is *nonsingular* if $A(G)$ is nonsingular. A *perfect matching* is a collection of vertex disjoint edges that span the graph.

It is well-known that a connected graph G is bipartite if and only if $-\lambda \in \sigma(G)$ whenever $\lambda \in \sigma(G)$.

A contrasting question is to characterize nonsingular graphs G for which $\frac{1}{\lambda} \in \sigma(G)$ whenever, $\lambda \in \sigma(G)$. A nonsingular graph G satisfies *property* SR (*strong reciprocal eigenvalue property*) if $\frac{1}{\lambda} \in \sigma(G)$ whenever $\lambda \in \sigma(G)$ and both have the same multiplicity. When the multiplicity condition is relaxed, we say that G satisfies *property* R (*reciprocal eigenvalue property*).

The study of property SR began in 1978 under different names. The class of nonsin-

gular trees with property SR was independently characterized in [2, 4]. In [2], Cvetkovic, Gutman and Simic used the term 'property C' to mean 'property SR'. In [4], Godsil and Mckay used the term 'symmetric property' to mean 'property SR'.

In 2006, Barik, Neumann and Pati [1] have restated this eigenvalue property and called it property SR. The authors also introduced the property R. In [1], the authors supplied a complete characterization of nonsingular trees with property SR and nonsingular trees with property R. Surprisingly, it turns out that these two properties are equivalent.

In [5], Neumann and Pati extended the notion of reciprocal eigenvalue properties to weighted graphs and proved that these two properties are equivalent for the weighted trees under some restrictions of weight. In [6], Panda and Pati showed that property R is equivalent to property SR for a class of weighted connected bipartite graphs with unique perfect matchings such that the graph obtained by contracting all matching edges is also bipartite. They considered the weight of each matching edge is 1 and the weight of each nonmatching edge is positive real number. It is natural to ask whether property R is equivalent to property SR for any nonsingular graphs.

In this article, we construct a class of graphs that satisfy property R but not property SR.

2. Construction. In this section, we construct a class of graphs that satisfy property R but not property SR. The following is a construction of such class of graphs.

DEFINITION 2.1. Let G_k, $k \geq 2$, be a graph obtained by taking k copies of P_4 and one copy of $P_2 = [u, v]$, and then joining the vertex u to every vertex in one copy of P_4 and to every degree two vertex in the remaining $k - 1$ copies. Let $\mathcal{F} = \{G_k \mid k \in \mathbb{N}, k \geq 2\}$.
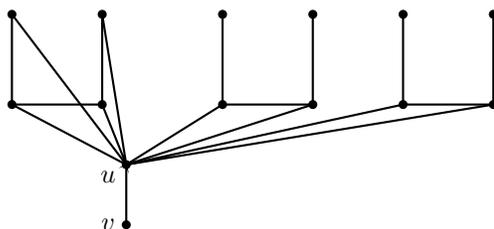

FIG. 2.1. *A graph G_3 in $\mathcal{F}$.*

We shall use the following two results from the literature.

LEMMA 2.2. [3] *Let v be a vertex in the graph G and $C(v)$ be the set of all cycles containing v. Then,*

$$P(G; x) = xP(G - v; x) - \sum_{u \sim v} P(G - u - v; x) - 2 \sum_{Z \in C(v)} P(G - Z; x).$$

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LEMMA 2.3. [3] *Let v be a vertex of degree 1 in the graph G and u be the vertex adjacent to v. Then,*

$$P(G; x) = xP(G - v; x) - P(G - u - v; x).$$

The following result tells us that each graph in $\mathcal{F}$ satisfies property R, but does not satisfy property SR.

THEOREM 2.4. *Let $G_k \in \mathcal{F}$. Then,*

$$P(G_k; x) = (x^2 + x - 1)^{k-1}(x^2 - x - 1)^{k-1}(x^4 - x^3 - (4 + 2k)x^2 - x + 1)(x^2 + x - 1).$$

Hence, G_k satisfies property R but does not satisfy property SR.

Proof. By Lemma 2.3, we see that $P(G_k; x) = xP(G_k - v; x) - P(G_k - v - u)$. In the graph $G_k - v$, all the cycles are containing the vertex u. Then by using Lemma 2.2, we see that

$$
\begin{aligned}
&P(G_k - v; x) \\
&= xP(G_k - v - u; x) - \sum_{s \sim u} P(G_k - v - s - u; x) - 2 \sum_{Z \in C(V)} P(G_k - v - Z; x) \\
&= x[P(P_4; x)]^k - 2kx[P(P_4; x)]^{k-1}P(P_2; x) - 2[P(P_4; x)]^{k-1}P(P_3; x) \\
&\quad -2kx^2[P(P_4; x)]^{k-1} - 4[P(P_4; x)]^{k-1}P(P_2; x) - 4x[P(P_4; x)]^{k-1} - 2[P(P_4; x)]^{k-1}.
\end{aligned}
$$

We know that,

$$P(P_4; x) = (x^2 + x - 1)(x^2 - x - 1); \quad P(P_2; x) = (x^2 - 1); \quad P(P_3; x) = x^3 - 2x.$$

Hence,

$$
\begin{aligned}
P(G_k - v; x) = (x^2 + x - 1)^{k-1}(x^2 - x - 1)^{k-1}&[x(x^2 + x - 1)(x^2 - x - 1) \\
&-2kx(x^2 - 1) - 2(x^3 - 2x) - 2kx^2 - 4(x^2 - 1) - 4x - 2].
\end{aligned}
$$

Therefore,

$$P(G_k; x) = (x^2 + x - 1)^{k-1}(x^2 - x - 1)^{k-1}(x^4 - x^3 - (4 + 2k)x^2 - x + 1)(x^2 + x - 1).$$

Notice that if λ is a zero of the polynomial $(x^2 + x - 1)$, then $\frac{1}{\lambda}$ is a zero of the polynomial $(x^2 - x - 1)$. The polynomial $(x^4 - x^3 - (4 + 2k)x^2 - x + 1)$ is palindromic. Hence, the reciprocal of a zero of this polynomial is also a zero of this polynomial and both have the same multiplicity. Hence, G_k has property R. However, as there is an extra factor $(x^2 + x - 1)$ present in $P(G_k; x)$, we see that the zeros and the reciprocal zeros do not have the same multiplicity. Thus, G_k cannot have property SR. ∎

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