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A TRANSFORMATION THAT PRESERVES PRINCIPAL MINORS OF SKEW-SYMMETRIC MATRICES[*]

ABDERRAHIM BOUSSAÏRI[†] AND BRAHIM CHERGUI[†]

Abstract. It is well known that two $n \times n$ symmetric matrices have equal corresponding principal minors of all orders if and only if they are diagonally similar. This result cannot be extended to arbitrary matrices. The aim of this work is to give a new transformation that preserves principal minors of skew-symmetric matrices.

Key words. Skew-symmetric matrix, Principal minor, Diagonal similarity, Graph, Digraph, Orientation.

AMS subject classifications. 15A15, 05C50.

1. Introduction. Throughout this paper, all matrices are real or complex. The identity matrix of order n is denoted by I_n and the transpose of a matrix A by A^t. A *minor* of a matrix A is the determinant of a square submatrix of A, and the determinant of a principal submatrix is a *principal minor*. The *order* of a minor is k if it is the determinant of a $k \times k$ submatrix.

In this work, we consider the following problem.

Problem 1.1. *What is the relationship between two matrices having equal corresponding principal minors of all orders?*

For symmetric matrices, this problem has been solved by Engel and Schneider [4]. More precisely, it follows from their work (see [4, Theorem 3.5]) that two symmetric matrices A, B have equal corresponding principal minors of all orders if and only if there exists a $\{-1, 1\}$-diagonal matrix D such that $B = D^{-1}AD$.

Consider now two arbitrary $n \times n$ matrices A and B. We say that A, B are *diagonally similar up to transposition* if there exists a nonsingular diagonal matrix D such that $B = D^{-1}AD$ or $B^t = D^{-1}AD$. Clearly, diagonal similarity up to transposition preserves all principal minors. But, as observed in [4] and [5] (see Remark 1.2 below), this is not, in general, the unique way to construct a pair of matrices having equal principal minors.

Remark 1.2. Consider the following skew-symmetric matrices:

$$A := \begin{pmatrix} R & -T^t \\ T & S \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} R^t & -T^t \\ T & S \end{pmatrix},$$

where R, S are square matrices.

We will see in Corollary 2.7 that if rank $T \leq 1$, then A and B have equal corresponding principal minors of all orders. However, these matrices are not always diagonally similar up to transposition.

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Hartfiel and Loewy [5], and then Loewy [6] considered a class of matrices excluding the situation of the previous remark. Their work concerns irreducible matrices with an additional condition. In order to state the main theorem of Loewy [6], we need the following definitions and notations. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let X, Y be two nonempty subsets of $[n]$ (where $[n] := \{1, \ldots, n\}$). We denote by $A[X, Y]$ the submatrix of A having row indices in X and column indices in Y. If $X = Y$, then $A[X, X]$ is a *principal submatrix* of A and we abbreviate this to $A[X]$. A square matrix A is *irreducible* if there exists no permutation matrix P, so that A can be reduced to the form $PAP^t = \begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix}$, where X and Y are square matrices.

The main theorem of Loewy [6] is stated as follows.

Theorem 1.3. *Let A, B be two $n \times n$ matrices. Suppose $n \geq 4$, A irreducible and for every partition of $[n]$ into two subsets X, Y with $|X| \geq 2$, $|Y| \geq 2$, either rank $A[X, Y] \geq 2$ or rank $A[Y, X] \geq 2$. If A and B have equal corresponding principal minors of all orders, then they are diagonally similar up to transposition.*

For skew-symmetric matrices with no zeros off the diagonal, we have improved this theorem in [1] by considering only the principal minors of order at most 4.

We will describe now another way to construct a pair of skew-symmetric matrices having equal corresponding principal minors of all orders. Let $A = [a_{ij}]$ be a $n \times n$ matrix. Following [1], a subset X of $[n]$ is a *HL-clan* of A if both of matrices $A\left[X, \overline{X}\right]$ and $A\left[\overline{X}, X\right]$ have rank at most 1 (where $\overline{X} := [n] \setminus X$). By definition, $\emptyset$, $[n]$ and singletons are HL-clans. Consider now the particular case when A is skew-symmetric and let X be a subset of $[n]$. We denote by $Inv(X, A) := [t_{ij}]$ the matrix obtained from A as follows. For any $i, j \in [n]$, $t_{ij} = -a_{ij}$ if $i, j \in X$ and $t_{ij} = a_{ij}$, otherwise. As we have mentioned in Remark 1.2, if X is an HL-clan of A, then $Inv(X, A)$ and A have equal corresponding principal minors of all orders. More generally, let A and B be two skew-symmetric matrices and assume that there exists a sequence $A_0 = A, \ldots, A_m = B$ of $n \times n$ skew-symmetric matrices such that for $k = 0, \ldots, m-1$, $A_{k+1} = Inv(X_k, A_k)$, where X_k is a HL-clan of A_k. It is easy to see that A and B have equal corresponding principal minors. Two matrices A, B obtained in this way are called *HL-clan-reversal-equivalent.* This defines an equivalence relation between $n \times n$ skew-symmetric matrices which preserves principal minors. In the converse direction, we propose the following conjecture.

Conjecture 1.4. *Two $n \times n$ skew-symmetric real matrices have equal corresponding principal minors of all orders if and only if they are HL-clan-reversal-equivalent.*

We will restrict ourselves to the class $\mathcal{M}_n$ of $n \times n$ skew-symmetric matrices with entries from $\{-1, 0, 1\}$ and such that all off-diagonal entries of the first row are nonzero. We obtain the following theorem, which is a partial answer to the conjecture above.

Theorem 1.5. *Let $A, B \in \mathcal{M}_n$. Then, the following statements are equivalent:*

i) A and B have equal corresponding principal minors of order at most 4;
ii) A and B have equal corresponding principal minors of all orders;
iii) A and B are HL-clan-reversal-equivalent.

2. HL-clan-reversal-equivalence. In this section, we present some properties of HL-clan-reversal-equivalence. We start with the following basic facts. Let $A = [a_{ij}]$ be a skew-symmetric $n \times n$ matrix.

Fact 2.1. *If $D = [d_{ij}]$ is a nonsingular diagonal matrix, then A and $D^{-1}AD$ have the same HL-clans.*

Proof. Let X be a subset of $[n]$. We have the following equalities:

$$\begin{aligned}(D^{-1}AD)\left[\overline{X},X\right] &= (D^{-1}\left[\overline{X}\right])(A\left[\overline{X},X\right])(D\left[X\right]),\\ (D^{-1}AD)\left[X,\overline{X}\right] &= (D^{-1}\left[X\right])(A\left[X,\overline{X}\right])(D\left[\overline{X}\right]).\end{aligned}$$

But, if the matrices $D\left[X\right]$ and $D\left[\overline{X}\right]$ are nonsingular, then $(D^{-1}AD)\left[\overline{X},X\right]$ and $A\left[\overline{X},X\right]$ (respectively $(D^{-1}AD)\left[X,\overline{X}\right]$ and $A\left[X,\overline{X}\right]$) have the same rank. Therefore, A and $D^{-1}AD$ have the same HL-clans. ∎

FACT 2.2. *If C is an HL-clan of A, then it is an HL-clan of $Inv(C,A)$.*

It suffices to see that

$$\begin{aligned}A\left[C,\overline{C}\right] &= Inv(C,A)\left[C,\overline{C}\right],\\ A\left[\overline{C},C\right] &= Inv(C,A)\left[\overline{C},C\right].\end{aligned}$$

FACT 2.3. *If C is an HL-clan of A and X is a subset of $[n]$, then $C\cap X$ is an HL-clan of $A[X]$ and $Inv(C,A)[X] = Inv(C\cap X, A[X])$.*

Proof. We have rank $(A\left[C\cap X, X\setminus(C\cap X)\right]) \leq$ rank $(A\left[C,\overline{C}\right]) \leq 1$ because $A\left[C\cap X, X\setminus(C\cap X)\right]$ is a submatrix of $A\left[C,\overline{C}\right]$ and C is an HL-clan of A. Analogously, we have rank $(A\left[X\setminus(C\cap X), C\cap X\right]) \leq$ rank $(A\left[\overline{C},C\right]) \leq 1$. It follows that $C\cap X$ is an HL-clan of $A[X]$. The second statement is trivial. ∎

The next proposition states that HL-clan-reversal-equivalence generalizes diagonal similarity up to transposition.

PROPOSITION 2.4. *Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be two $n\times n$ skew-symmetric matrices. If A and B are diagonally similar up to transposition, then they are HL-clan-reversal-equivalent.*

Proof. Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be two $n\times n$ skew-symmetric matrices diagonally similar up to transposition. As $B^t=-B=Inv([n],B)$, we can assume that $B=\Delta^{-1}A\Delta$ for some nonsingular diagonal matrix Δ. It is easy to see that $b_{ij}=\pm a_{ij}$ for $i,j\in[n]$ and hence Δ may be chosen to be a $\{-1,1\}$-diagonal matrix. ∎

We conclude by Lemma 2.5 below.

LEMMA 2.5. *Let $A=[a_{ij}]$ be an $n\times n$ skew-symmetric matrix and let D be a $\{-1,1\}$-diagonal matrix. Then A and $D^{-1}AD$ are HL-clan-reversal-equivalent.*

Proof. We denote by $d_1,d_2,\ldots,d_n$ the diagonal entries of D.
Let $U_D:=\{i\in[n]: d_i=-1\}$. We will show by induction on $t:=|U_D|$ that there exists a sequence $A_0=A,\ldots,A_m=D^{-1}AD$ of $n\times n$ skew-symmetric matrices such that for $k=0,\ldots,m-1$, $A_{k+1}=Inv(X_k,A_k)$ where $X_k=\emptyset$, $X_k=[n]$ or $[n]\setminus X_k$ is a singleton. If $t=0$, then $D^{-1}AD=A$ and hence it suffices to take $m=1$, $A_0=A$ and $X_0=\emptyset$. Now assume that $t>0$. Let $j\in U_D$ and consider the diagonal matrix $\Delta^{(j)}=diag(\delta_1,\ldots,\delta_n)$ where $\delta_j=-1$ and $\delta_i=1$ if $i\neq j$. Clearly $|U_{D\Delta^{(j)}}|=t-1$ and then, by induction hypothesis, there exists a sequence $A_0=A,\ldots,A_m=(D\Delta^{(j)})^{-1}AD\Delta^{(j)}$ of $n\times n$ skew-symmetric matrices such that for $k=0,\ldots,m-1$, $A_{k+1}=Inv(X_k,A_k)$ where $X_k=\emptyset$, $X_k=[n]$ or $[n]\setminus X_k$ is a singleton. To prove that A and $D^{-1}AD$ are HL-clan-reversal-equivalent, it suffices to extend the sequence $A_0=A,\ldots,A_m$ by adding two terms, $A_{m+1}:=Inv([n],A_m)$ and $A_{m+2}:=Inv([n]\setminus\{j\},A_{m+1})$. ∎

The following proposition appears in another form in [5, Lemma 5].

PROPOSITION 2.6. *Let $A=[a_{ij}]$ be a skew-symmetric $n\times n$ matrix. If X is an HL-clan of A, then* $\det(Inv(X,A))=\det(A)$.

Proof. Without loss of generality, we can assume that $X = \{1, \dots, p\}$. We will show that A and $Inv(X, A)$ have the same characteristic polynomial. As X is an HL-clan of A, the submatrix $A[\overline{X}, X]$ has rank at most 1 and hence there are two column vectors $\alpha = \begin{pmatrix} \alpha_{p+1} \\ \vdots \\ \alpha_n \end{pmatrix}$ and $\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}$ such that $A[\overline{X}, X] = \alpha\beta^t$.

Let $A[X] := R$ and $A[\overline{X}] := S$. Then $A = \begin{pmatrix} R & -\beta\alpha^t \\ \alpha\beta^t & S \end{pmatrix}$ and $Inv(X, A) = \begin{pmatrix} R^t & -\beta\alpha^t \\ \alpha\beta^t & S \end{pmatrix}$, where $R^t = -R$. We will prove that A and $Inv(X, A)$ have the same characteristic polynomial.

Let λ satisfying $|\lambda| > \lambda_0$ where λ_0 is the spectral radius of R. Then $R - \lambda I_p$ is nonsingular and hence, by using the Schur complement, we have

$$\begin{aligned}
\det(A - \lambda I_n) &= \det(R - \lambda I_p)\det(S - \lambda I_{n-p} + \alpha\beta^t(R - \lambda I_p)^{-1}\beta\alpha^t) \\
&= \det(R - \lambda I_p)\det(S - \lambda I_{n-p} + (\beta^t(R - \lambda I_p)^{-1}\beta)\alpha\alpha^t) \\
&= \det((R - \lambda I_p)^t)\det(S - \lambda I_{n-p} + (\beta^t(R - \lambda I_p)^{-1}\beta)^t\alpha\alpha^t) \\
&= \det((R^t - \lambda I_p))\det(S - \lambda I_{n-p} + (\beta^t(R^t - \lambda I_p)^{-1}\beta)\alpha\alpha^t) \\
&= \det(Inv(X, A) - \lambda I_n).
\end{aligned}$$

It follows that A and $Inv(X, A)$ have the same characteristic polynomial and $\det(A) = \det(Inv(X, A))$. □

The following corollary is a direct consequence of the previous proposition and Fact 2.3.

Corollary 2.7. *Let $A = [a_{ij}]$ be a skew-symmetric $n \times n$ matrix. If X is an HL-clan of A, then $Inv(X, A)$ and A have the same principal minors.*

3. Digraphs and orientations of a graphs.

We start with some definitions about digraphs. A *directed graph* or *digraph* Γ consists of a nonempty finite set V of *vertices* together with a (possibly empty) set E of ordered pairs of distinct vertices called *arcs*. Such a digraph is denoted by (V, E). The *converse* of a digraph Γ denoted by Γ^* is the digraph obtained from Γ by reversing the direction of all its arcs.

Let $\Gamma = (V, E)$ be a digraph and let X be a subset of V. The *subdigraph* of Γ *induced* by X is the digraph $\Gamma[X]$ whose vertex set is X and whose arc set consists of all arcs of Γ which have end-vertices in X.

Two digraphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ are said to be *isomorphic* if there is a bijection φ from V onto V' which preserves arcs, that is $(x, y) \in E$ if and only if $(\varphi(x), \varphi(y)) \in E'$. Any such bijection is called an *isomorphism*. We say that Γ and Γ' are *hemimorphic*, if there exists an isomorphism from Γ onto Γ' or from Γ^* onto Γ'.

Let $\Gamma = (V, E)$ be a digraph. Following [3], a subset X of V is a *clan* of Γ if for any $a, b \in X$ and $x \in V \setminus X$, $(a, x) \in E$ (resp. $(x, a) \in E$)) if and only if $(b, x) \in E$ (resp. $(x, b) \in E$). For a subset X of V, we denote by $Inv(X, \Gamma)$ the digraph obtained from Γ by reversing all arcs of $\Gamma[X]$. Clearly, $Inv(X, Inv(X, \Gamma)) = \Gamma$ and moreover, if X is a clan of Γ, then X is a clan of $Inv(X, \Gamma)$.

Let $G = (V, E)$ be a simple graph (without loops and multiple edges). An *orientation* of G is an assignment of a direction to each edge of G in order to obtain a directed graph $\overrightarrow{G}$. For $x \neq y \in V$, $x \overset{\overrightarrow{G}}{\to} y$ means (x, y) is an arc of $\overrightarrow{G}$. For $Y \subseteq V$ and $x \in V \setminus Y$, $x \overset{\overrightarrow{G}}{\to} Y$ means $x \overset{\overrightarrow{G}}{\to} y$ for every $y \in Y$.

Remark 3.1.

i) There are exactly four possible simple graphs with three vertices: the complete graph K_3, the path P_2, the complement of these two graphs, namely $\overline{K_3}$ and $\overline{P_2}$ (see Figure 1);
ii) The path P_2 has two non-hemimorphic orientations Γ_1 and Γ_2 (see Figure 2 (a));
iii) The complete graph K_3 has two non-hemimorphic orientations Γ_3 and Γ_4 (see Figure 2 (b)).

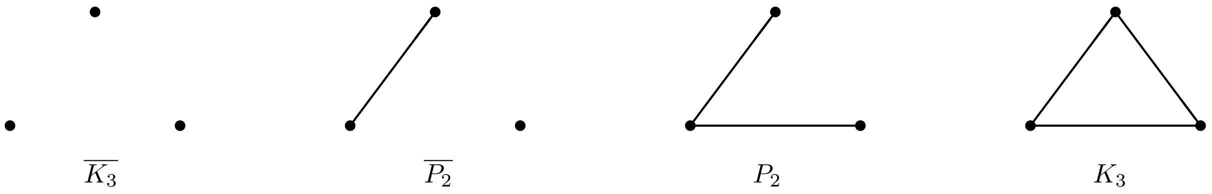

Figure 1.

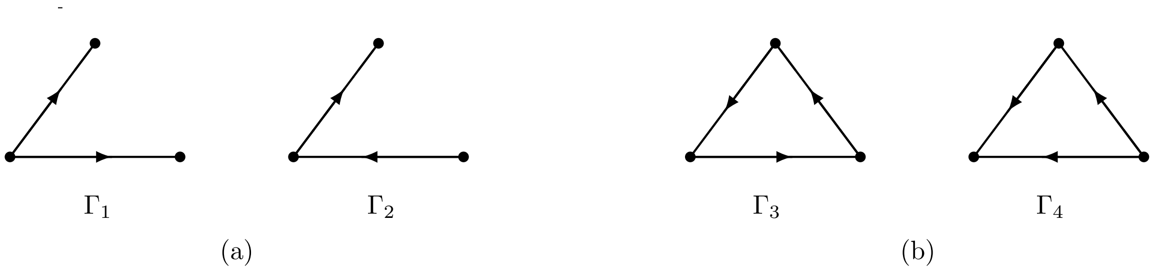

Figure 2.

The proof of our main theorem is based on a result of Boussaïri et al. [2] about the relationship between hemimorphy and clan decomposition of digraphs (see [2, Theorem 3]). Proposition 3.2 below is a special case of this result.

Proposition 3.2. *Let $G = (V, E)$ be a finite simple graph and let G^σ, G^τ be two orientations of G. Then the following statements are equivalent:*

i) $G^\sigma[X]$ and $G^\tau[X]$ are hemimorphic, for any subset X of V of size 3;
ii) There exists a sequence $\sigma_0 = \sigma, \ldots, \sigma_m = \tau$ of orientations of G such that for $i = 0, \ldots, m-1$, $G^{\sigma_{i+1}} = Inv(X_i, G^{\sigma_i})$ where X_i is a clan of G^{σ_i}.

4. Proof of main theorem. Let $G = (V, E)$ be a graph whose vertices are $v_1, v_2, \ldots, v_n$. An *orientation* of G can be seen as a skew-symmetric map σ from $V \times V$ to the set $\{-1, 0, 1\}$ such that $\sigma(i, j) = 1$ if an only if (v_i, v_j) is an arc. Such orientation is denoted by G^σ.

Let G^σ be an orientation of G. The *skew-adjacency* matrix of G^σ is the real skew-symmetric matrix $S(G^\sigma) = [s_{i,j}]$ where $s_{i,j} = 1$ and $s_{j,i} = -1$ if (v_i, v_j) is an arc of G^σ, otherwise $s_{i,j} = s_{j,i} = 0$. Clearly, the entries of $S(G^\sigma)$ depend on the ordering of vertices. But the value of the determinant, $\det(S(G^\sigma))$, is independent of this ordering. So, we can write $\det(G^\sigma)$ instead of $\det(S(G^\sigma))$.

Consider now an $n \times n$ skew-symmetric $\{-1, 0, 1\}$-matrix A. We associate to A its *underlying graph* G with vertex set $[n]$ and such that $\{i, j\}$ is an edge of G if and only if $a_{ij} \neq 0$. Let σ be the map from $[n] \times [n]$ to the set $\{-1, 0, 1\}$ such that $\sigma(i, j) = a_{ij}$. Clearly, G^σ is the unique orientation of G such that $S(G^\sigma) = A$.

Remark 4.1. Let $G = ([n], E)$ be a graph and let G^σ be an orientation of G. Then:

i) For every subset X of $[n]$, we have $S(Inv(X, G^\sigma)) = Inv(X, S(G^\sigma))$;
ii) $Inv([n], G^\sigma) = (G^\sigma)^* = G^{-\sigma}$;
iii) Every clan of G^σ is an HL-clan of $S(G^\sigma)$.

In addition to Corollary 2.7, the proof of our main theorem requires the following lemma.

Lemma 4.2. *Given a graph G with four vertices i, j, k, l such that i is adjacent to j, k, l. Let G^σ, G^τ be two orientations of G. If $i \overset{G^\sigma}{\to} \{j, k, l\}$, $i \overset{G^\tau}{\to} \{j, k, l\}$ and $\det(G^\sigma) = \det(G^\tau)$, then $G^\sigma[j, k, l]$ and $G^\tau[j, k, l]$ are hemimorphic.*

Proof. By Remark 3.1, we have four cases to consider.

i) If $G[j, k, l]$ is the empty graph, then $G^\tau[j, k, l] = G^\sigma[j, k, l]$.
ii) If $G[j, k, l]$ is the graph $\overline{P_2}$, then $G^\tau[j, k, l] = G^\sigma[j, k, l]$ or $G^\tau[j, k, l] = (G^\sigma[j, k, l])^*$.
iii) If $G[j, k, l]$ is the path P_2 and $G^\sigma[j, k, l]$ is hemimorphic to Γ_1, then $\det(G^\sigma) = 4$ and $G^\tau[j, k, l]$ is hemimorphic to Γ_1 or Γ_2. The case when $G^\tau[j, k, l]$ is hemimorphic to Γ_2 implies that $\det(G^\tau) = 0$, which is impossible. Analogously, if $G^\sigma[j, k, l]$ is hemimorphic to Γ_2, then $G^\tau[j, k, l]$ must be hemimorphic to Γ_2.
iv) If $G[j, k, l]$ is the complete graph K_3 and $G^\sigma[j, k, l]$ is hemimorphic to Γ_3, then $\det(G^\sigma) = 9$ and $G^\tau[j, k, l]$ is hemimorphic to Γ_3 or Γ_4. As in iii), the case when $G^\tau[j, k, l]$ is hemimorphic to Γ_4 implies that $\det(G^\tau) = 1$, which is impossible. Analogously, if $G^\sigma[j, k, l]$ is hemimorphic to Γ_4, then $G^\tau[j, k, l]$ must be hemimorphic to Γ_4. □

Proof of Theorem 1.5. The implication ii)$\Longrightarrow$i) is obvious. To prove iii)$\Longrightarrow$ii), it suffices to apply Corollary 2.7. Let us prove that i) implies iii). As all off-diagonal entries of the first row in A and B are nonzero, then there are two $\{-1, 1\}$-diagonal matrices D and D' such that the first row of $A' := D^{-1}AD$ (resp $B' := D'^{-1}BD'$) is $(0, 1, 1, \ldots, 1)$. The matrices A' and B' have the same underlying graph G because A and B have equal corresponding principal minors of order 2. Let G^σ (resp. G^τ) be the unique orientation of G such that $S(G^\sigma) = A'$ (resp. $S(G^\tau) = B'$). We will show that i) of Proposition 3.2 holds for G^σ and G^τ. For this, let $X = \{j, k, l\}$ be a subset of $[n]$ of size 3. If $1 \in X$ (for example $j = 1$), then $1 \overset{G^\sigma}{\to} \{k, l\}$, $1 \overset{G^\tau}{\to} \{k, l\}$ and hence $G^\sigma[j, k, l]$ is isomorphic to $G^\tau[j, k, l]$. Assume now that $1 \notin X$ and let $Y := \{1, j, k, l\}$. We have $1 \overset{G^\sigma}{\to} \{j, k, l\}$ and $1 \overset{G^\tau}{\to} \{j, k, l\}$. Moreover, as A and A' (resp. B and B') are diagonally similar, we have $\det(A'[Y]) = \det(A[Y])$, $\det(B'[Y]) = \det(B[Y])$ and hence $\det(A'[Y]) = \det(B'[Y])$ because A and B have equal corresponding principal minors of order 4. By definition, we have $\det(G^\sigma[Y]) = \det(A'[Y])$ and $\det(G^\tau[Y]) = \det(B'[Y])$. It follows that $\det(G^\sigma[Y]) = \det(G^\tau[Y])$ and, then by Lemma 4.2, $G^\sigma[j, k, l]$ and $G^\tau[j, k, l]$ are hemimorphic. Now, from Proposition 3.2, there exists a sequence of orientations $\sigma_0 = \sigma, \ldots, \sigma_m = \tau$ of G such that for $i = 0, \ldots, m-1$, $G^{\sigma_{i+1}} = Inv(X_i, G^{\sigma_i})$ where X_i is a clan of G^{σ_i}. Let $A'_i := S(G^{\sigma_i})$ for $i = 0, \ldots, m$. By Remark 4.1, X_i is is an HL-clan of A'_i and $A'_{i+1} = Inv(X_i, A'_i)$ for $i = 0, \ldots, m-1$. We conclude by applying Proposition 2.4. □

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