

2017

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Recommended Citation

Hai, Guojun and Cvetkovic-Ilic, Dragana S.. (2017), "Generalized left and right Weyl spectra of upper triangular operator matrices", *Electronic Journal of Linear Algebra*, Volume 32, pp. 41-50.
DOI: <https://doi.org/10.13001/1081-3810.3373>

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GENERALIZED LEFT AND RIGHT WEYL SPECTRA OF UPPER TRIANGULAR OPERATOR MATRICES*

GUOJUN HAI[†] AND DRAGANA S. CVETKOVIĆ-ILIC[‡]

Abstract. In this paper, for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, the sets of all $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is generalized Weyl and generalized left (right) Weyl, are completely described. Furthermore, the following intersections and unions of the generalized left Weyl spectra

$$\bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{lw}^g(M_C) \quad \text{and} \quad \bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{lw}^g(M_C)$$

are also described, and necessary and sufficient conditions which two operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ have to satisfy in order for M_C to be a generalized left Weyl operator for each $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, are presented.

Key words. Operator matrix, Generalized left(right) Weyl, Spectrum.

AMS subject classifications. 47A10, 47A53, 47A55.

1. Introduction. Let \mathcal{H}, \mathcal{K} be infinite dimensional complex separable Hilbert spaces, and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the set of all bounded linear operators from \mathcal{H} to \mathcal{K} . For simplicity, we also write $\mathcal{B}(\mathcal{H}, \mathcal{H})$ as $\mathcal{B}(\mathcal{H})$. By $\mathcal{F}(\mathcal{H}, \mathcal{K})$ we denote the set of all operators from $\mathcal{B}(\mathcal{H}, \mathcal{K})$ with a finite dimensional range. For a given $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A , respectively. Let $n(A) = \dim \mathcal{N}(A)$, $\beta(A) = \text{codim} \mathcal{R}(A)$, and $d(A) = \dim \mathcal{R}(A)^\perp$.

If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is such that $\mathcal{R}(A)$ is closed and $n(A) < \infty$, then A is said to be a upper semi-Fredholm operator. If $\beta(A) < \infty$, then A is called a lower semi-Fredholm operator. A semi-Fredholm operator is one which is either upper semi-Fredholm or lower semi-Fredholm. An operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is called Fredholm if it is both lower semi-Fredholm and upper semi-Fredholm. The subset of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ consisting of all Fredholm operators is denoted by $\Phi(\mathcal{H}, \mathcal{K})$. By $\Phi_+(\mathcal{H}, \mathcal{K})$ ($\Phi_-(\mathcal{H}, \mathcal{K})$) we denote the set of all upper (lower) semi-Fredholm operators from $\mathcal{B}(\mathcal{H}, \mathcal{K})$.

If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is such that $\mathcal{R}(A)$ is closed and $n(A) \leq d(A)$, then A is a generalized left Weyl operator. If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is such that $\mathcal{R}(A)$ is closed and $d(A) \leq n(A)$, then A is a generalized right Weyl operator. Notice that in the cases of generalized left (right) Weyl operators, $n(A)$ and $d(A)$ are allowed to be infinity. An operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a generalized Weyl operator if it is both generalized right Weyl and generalized left Weyl. The set of all generalized Weyl operators from $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is denoted by $W^g(\mathcal{H}, \mathcal{K})$.

Let $W_{gl}(\mathcal{H}, \mathcal{K})$ ($W_{gr}(\mathcal{H}, \mathcal{K})$) denote the subset of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ consisting of all generalized left (right) Weyl operators. For an operator $C \in \mathcal{B}(\mathcal{H})$, the generalized left (right) Weyl spectrum $\sigma_{lw}^g(C)$ ($\sigma_{rw}^g(C)$) is defined by

$$\sigma_{lw}^g(C)(\sigma_{rw}^g(C)) = \{\lambda \in \mathbb{C} : C - \lambda I \text{ is not generalized left (right) Weyl}\}.$$

*Received by the editors on August 10, 2016. Accepted for publication on January 11, 2017. Handling Editor: Torsten Ehrhardt.

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The generalized Weyl spectrum is defined by

$$\sigma_w^g(C) = \{\lambda \in \mathbb{C} : C - \lambda I \text{ is not generalized Weyl}\}.$$

In this paper, we address the question for which operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, there exists an operator $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that an upper-triangular operator matrix

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix},$$

is generalized left (right) Weyl. There are many papers which consider some types of invertibility, regularity and some other properties of an upper-triangular operator matrix M_C (see [1]–[17] and references therein) as well as various types of spectra of M_C . This paper is a continuation of the work presented in [10], where the sets $\bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_w^g(M_C)$ and $\bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_w^g(M_C)$ are described and some necessary and sufficient conditions for the existence of $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized Weyl are given, but the set of all such operators C is not described. As a corollary of our main results we obtain a description of all $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized Weyl, and we denote this set by $S_{GW}(A, B)$. The sets $\bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{lw}^g(M_C)$ and $\bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{lw}^g(M_C)$ are described for given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ as well as the set of all $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized left Weyl which is denoted by $S_{GLW}(A, B)$. In an analogous way, similar results can be provided for $\bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{rw}^g(M_C)$ and $\bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{rw}^g(M_C)$.

2. Results. In this section, by \mathcal{H}, \mathcal{K} we denote complex separable Hilbert spaces. For given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, by M_C we denote

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix},$$

where $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Evidently, for given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, arbitrary $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ can be represented by

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \rightarrow \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \end{pmatrix}. \quad (2.1)$$

First, we will state some auxiliary lemmas which will be used in the proof of the main result.

LEMMA 2.1. *If $A \in \mathcal{B}(\mathcal{H})$ and $D \in \mathcal{F}(\mathcal{H})$, then $\mathcal{R}(A+D)$ is closed if and only if $\mathcal{R}(A)$ is closed.*

LEMMA 2.2. *Let $S \in \mathcal{B}(\mathcal{H})$, $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $R \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be given operators.*

(i) *If $\mathcal{R}(S)$ is non-closed and $\mathcal{R}(\begin{pmatrix} S & T \end{pmatrix})$ is closed, then $n(\begin{pmatrix} S & T \end{pmatrix}) = \infty$.*

(ii) *If $\mathcal{R}(S)$ is non-closed and $\mathcal{R}(\begin{pmatrix} S \\ R \end{pmatrix})$ is closed, then $d(\begin{pmatrix} S \\ R \end{pmatrix}) = \infty$.*

Proof. (i) Suppose that $\mathcal{R}(S)$ is non-closed, $\mathcal{R}(\begin{pmatrix} S & T \end{pmatrix})$ is closed and $n(\begin{pmatrix} S & T \end{pmatrix}) < \infty$. Then $\begin{pmatrix} S & T \end{pmatrix}$ is a left Fredholm operator which implies that there exists an operator $\begin{pmatrix} X \\ Y \end{pmatrix} : \mathcal{H} \rightarrow \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix}$ such that

$$\begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} S & T \end{pmatrix} = I + K,$$

for some compact operator $K \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$. Hence, $XS = I_{\mathcal{H}} + K_1$, for some compact operator $K_1 \in \mathcal{B}(\mathcal{H})$ which implies that S is left Fredholm and so $\mathcal{R}(S)$ is closed, which is a contradiction.

(ii) The proof follows by taking adjoints in (i). \square

In the following theorem, for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, we present necessary and sufficient conditions for the existence of $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is a generalized left Weyl operator, and we completely describe the set of all such $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

THEOREM 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. There exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized left Weyl if and only if one of the following conditions is satisfied:*

(i) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed and $n(A) + n(B) \leq d(A) + d(B)$. In this case,

$$S_{GLW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C \text{ is given by (2.1), } C_3 \text{ has closed range, } \right. \\ \left. n(A) + n(C_3) \leq d(C_3) + d(B) \right\}.$$

(ii) $\mathcal{R}(A)$ is closed, $\mathcal{R}(B)$ is non-closed and $d(A) = \infty$. In this case,

$$S_{GLW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(B^*) + \mathcal{R}(C^* P_{\mathcal{R}(A)^\perp}) \text{ is closed} \right\}.$$

(iii) $\mathcal{R}(A)$ is non-closed, $\mathcal{R}(B)$ is closed and $n(B) = d(A) + d(B) = \infty$. In this case,

$$S_{GLW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(A) + \mathcal{R}(CP_{\mathcal{N}(B)}) \text{ is closed, } \right. \\ \left. d(B) + \text{codim}(\mathcal{R}(A) + \mathcal{R}(CP_{\mathcal{N}(B)})) = \infty \right\}.$$

(iv) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are non-closed and $n(B) = d(A) = \infty$. In this case,

$$S_{GLW}(A, B) = \{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(M_C) \text{ is closed} \}.$$

For simplicity, we will divide the statement of this theorem into four propositions and prove each of them separately.

PROPOSITION 2.4. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be such that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed. There exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized left Weyl if and only if $n(A) + n(B) \leq d(A) + d(B)$. In this case,*

$$S_{GLW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C \text{ is given by (2.1), } C_3 \text{ has closed range, } \right. \\ \left. n(A) + n(C_3) \leq d(C_3) + d(B) \right\}.$$

Proof. If $n(A) + n(B) \leq d(A) + d(B)$, then M_0 is a generalized left Weyl operator. Conversely, suppose that there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is a generalized left Weyl operator and that C is given by (2.1). Then M_C has a matrix representation

$$M_C = \begin{pmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{pmatrix},$$

where $A_1 : \mathcal{H} \longrightarrow \mathcal{R}(A)$ is right invertible and $B_1 : \mathcal{N}(B)^\perp \longrightarrow \mathcal{K}$ is left invertible. Evidently, there exists invertible $U, V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that

$$UM_CV = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{pmatrix}. \tag{2.2}$$

Hence, $UM_C V$ is a generalized left Weyl which implies that

$$n(A_1) + n(C_3) \leq d(C_3) + d(B_1). \quad (2.3)$$

Since,

$$n(A_1) = n(A), \quad n(B) = n(C_3) + \dim \mathcal{N}(C_3)^\perp,$$

$$d(B_1) = d(B) \quad \text{and} \quad d(A) = d(C_3) + \dim \mathcal{R}(C_3),$$

having in mind that $\dim \mathcal{N}(C_3)^\perp = \dim \mathcal{R}(C_3)$ and (2.3), we get

$$n(A) + n(B) \leq d(A) + d(B).$$

To describe the set of all $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is a generalized left Weyl, notice that for arbitrary C given by (2.1), there exists invertible $U, V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that $UM_C V$ is given by (2.10). Hence, M_C is a generalized left Weyl if and only if $UM_C V$ is a generalized left Weyl which is equivalent with the fact that $\mathcal{R}(C_3)$ is closed and that (2.3) holds. \square

PROPOSITION 2.5. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be such that $\mathcal{R}(A)$ is closed and $\mathcal{R}(B)$ is non-closed. There exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized left Weyl if and only if $d(A) = \infty$. In this case,*

$$S_{GLW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(B^*) + \mathcal{R}(C^* P_{\mathcal{R}(A)^\perp}) \text{ is closed} \right\}.$$

Proof. Suppose that $d(A) = \infty$. Then M_{C_0} is a generalized left Weyl operator for C_0 given by

$$C_0 = \begin{pmatrix} 0 \\ J \end{pmatrix} : \mathcal{K} \longrightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp,$$

where $J : \mathcal{K} \longrightarrow \mathcal{R}(A)^\perp$ is unitary. Evidently, M_{C_0} is represented by

$$M_{C_0} = \begin{pmatrix} A_1 & 0 \\ 0 & J \\ 0 & B \end{pmatrix} : \mathcal{H} \oplus \mathcal{K} \longrightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K},$$

where $A_1 : \mathcal{H} \longrightarrow \mathcal{R}(A)$ is right invertible. Since J is invertible, there exists an invertible operator $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that

$$UM_{C_0} = \begin{pmatrix} A_1 & 0 \\ 0 & J \\ 0 & 0 \end{pmatrix} : \mathcal{H} \oplus \mathcal{K} \longrightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K}.$$

Now, it is clear that UM_{C_0} is a generalized left Weyl operator, and so M_{C_0} is a generalized left Weyl operator.

Conversely, suppose that there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized left Weyl. Then M_C has a matrix representation

$$M_C = \begin{pmatrix} A_1 & C_1 \\ 0 & C_2 \\ 0 & B \end{pmatrix} : \mathcal{H} \oplus \mathcal{K} \longrightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K}, \quad (2.4)$$

where $A_1 : \mathcal{H} \rightarrow \mathcal{R}(A)$ is right invertible. Thus, there exists an invertible operator $V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that

$$M_C V = \begin{pmatrix} A_1 & 0 \\ 0 & C_2 \\ 0 & B \end{pmatrix} : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K}. \quad (2.5)$$

Now we will show that $d(A) = \infty$: Indeed, if $d(A) < \infty$, then $\mathcal{R}(C_2^*)$ is finite dimensional. Since $\mathcal{R}(M_C V)$ is closed, we have that $\mathcal{R}\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right)$ is closed, which implies that $\mathcal{R}(B^*) + \mathcal{R}(C_2^*)$ is closed. This, together with $\dim \mathcal{R}(C_2^*) < \infty$, implies that $\mathcal{R}(B)$ is closed. This is a contradiction. Hence, $d(A) = \infty$.

In order to describe the set $S_{GLW}(A, B)$, notice that for arbitrary $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, M_C has a form (2.4) and that there exists an invertible operator $V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that $M_C V$ is given by (2.5). Hence, M_C is generalized left Weyl if and only if C is such that $\mathcal{R}\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right)$ is closed and that

$$n(A_1) + n\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right) \leq d(A_1) + d\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right). \quad (2.6)$$

Notice that by Lemma 2.2, we have that for each $C_2 \in \mathcal{B}(\mathcal{K}, \mathcal{R}(A)^\perp)$ such that $\mathcal{R}\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right)$ is closed, it follows that $d\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right) = \infty$. Thus,

$$S_{GLW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(B^*) + \mathcal{R}(C^* P_{\mathcal{R}(A)^\perp}) \text{ is closed} \right\}. \quad \square$$

PROPOSITION 2.6. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be such that $\mathcal{R}(A)$ is non-closed and $\mathcal{R}(B)$ is closed. There exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized left Weyl if and only if $n(B) = d(A) + d(B) = \infty$. In this case,*

$$S_{GLW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(A) + \mathcal{R}(C P_{\mathcal{N}(B)}) \text{ is closed,} \right. \\ \left. d(B) + \text{codim}(\mathcal{R}(A) + \mathcal{R}(C P_{\mathcal{N}(B)})) = \infty \right\}.$$

Proof. Suppose that $n(B) = d(A) + d(B) = \infty$. Then there exists a left invertible operator $C_1 : \mathcal{N}(B) \rightarrow \mathcal{H}$ such that $\mathcal{R}(C_1) = \overline{\mathcal{R}(A)}$. We will prove that M_C is a generalized left Weyl operator for C given by

$$C = \begin{pmatrix} C_1 & 0 \end{pmatrix} : \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \rightarrow \mathcal{H}.$$

Evidently, M_C is represented by

$$M_C = \begin{pmatrix} A & C_1 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \rightarrow \mathcal{H} \oplus \mathcal{K},$$

where $B_1 : \mathcal{N}(B)^\perp \rightarrow \mathcal{K}$ is left invertible and

$$\mathcal{R}(M_C) = (\mathcal{R}(A) + \mathcal{R}(C_1)) \oplus \mathcal{R}(B_1) = \overline{\mathcal{R}(A)} \oplus \mathcal{R}(B).$$

Thus, $\mathcal{R}(M_C)$ is closed and

$$d(M_C) = d(A) + d(B) = \infty,$$

i.e., M_C is a generalized left Weyl operator.

Conversely, suppose that there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized left Weyl. It follows that M_C has a matrix representation

$$M_C = \begin{pmatrix} A & C_1 & C_2 \\ 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \longrightarrow \mathcal{H} \oplus \mathcal{K}, \quad (2.7)$$

where $B_1 : \mathcal{N}(B)^\perp \longrightarrow \mathcal{K}$ is left invertible and there exists an invertible operator $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that

$$UM_C = \begin{pmatrix} A & C_1 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \longrightarrow \mathcal{H} \oplus \mathcal{K}. \quad (2.8)$$

Since UM_C has a closed range, by Lemma 2.1 and the fact that $\mathcal{R}(A)$ is non-closed, we have that $n(B) = \infty$. Also, applying Lemma 2.2, we get that $n\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right) = \infty$ which implies that $d(UM_C) = d(B) + d\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right) = \infty$. Since $d\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right) \leq d(A)$, it follows that $d(A) + d(B) = \infty$.

In order to describe the set $S_{GLW}(A, B)$, notice that for arbitrary $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, M_C has a form (2.7) and that there exists an invertible operator $V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that UM_C is given by (2.8). Hence, M_C is generalized left Weyl if and only if C is such that $\mathcal{R}\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right)$ is closed and that

$$n\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right) + n(B_1) \leq d\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right) + d(B_1). \quad (2.9)$$

Notice that if $\mathcal{R}\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right)$ is closed, then by Lemma 2.2, we have that $n\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right) = \infty$. Hence, M_C is a generalized left Weyl operator for $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if $\mathcal{R}\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right)$ is closed and $d\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right) + d(B_1) = \infty$. Obviously, $d(B_1) = d(B)$. \square

PROPOSITION 2.7. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be such that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are non-closed. There exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized left Weyl if and only if $n(B) = d(A) = \infty$. In this case,*

$$S_{GLW}(A, B) = \{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : R(M_C) \text{ is closed}\}.$$

Proof. Since $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are non-closed, by Lemma 2.2, we conclude that if $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is such that $\mathcal{R}(M_C)$ is closed, then $n(M_C) = d(M_C) = \infty$. Hence, M_C is generalized left Weyl if and only if $R(M_C)$ is closed. Now, the proof directly follows by Theorem 2.6 of [4]. \square

REMARK 1. It is interesting to notice that the condition $d(B) + \text{codim}(\mathcal{R}(A) + \mathcal{R}(CP_{\mathcal{N}(B)})) = \infty$ from Proposition 2.6, appearing also in item (iii) of Theorem 2.3, can be replaced by the condition $d(C_3) + d(B) = \infty$, where C_3 is the block-operator defined by (2.1). So, if $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are such that $\mathcal{R}(A)$ is non-closed and $\mathcal{R}(B)$ is closed, then

$$M_C = \begin{pmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{pmatrix},$$

where $A_1 : \mathcal{H} \longrightarrow \overline{\mathcal{R}(A)}$ is with a dense range and $B_1 : \mathcal{N}(B)^\perp \longrightarrow \mathcal{K}$ is left invertible. There exists an invertible $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that

$$UM_C = \begin{pmatrix} A_1 & C_1 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{pmatrix}. \quad (2.10)$$

Now, it is evident that $\mathcal{R}(M_C)$ is closed if and only if $\begin{pmatrix} A_1 & C_1 \\ 0 & C_3 \end{pmatrix}$ is closed which is equivalent with the fact that $\mathcal{R}(A) + R(CP_{\mathcal{N}(B)})$ is closed. Also,

$$d\left(\begin{pmatrix} A_1 & C_1 \\ 0 & C_3 \end{pmatrix}\right) = n\left(\begin{pmatrix} A_1^* & 0 \\ C_1^* & C_3^* \end{pmatrix}\right) = n(C_3^*) = d(C_3),$$

since A_1^* is injective ($\mathcal{R}(A_1) = \mathcal{R}(A)$). Hence, in this case, the set S_{GLW} can also be described by

$$S_{GLW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C \text{ is given by (2.1), } \mathcal{R}(A) + R(CP_{\mathcal{N}(B)}) \text{ is closed, } d(C_3) + d(B) = \infty \right\}.$$

As a corollary of the previous theorem, we get the description of the set $\bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{lw}^g(M_C)$:

COROLLARY 2.8. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be given operators. Then*

$$\begin{aligned} \bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{lw}^g(M_C) = & \left\{ \lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I) \text{ is not closed, } n(B - \lambda I) < \infty \right\} \\ & \cup \left\{ \lambda \in \mathbb{C} : \mathcal{R}(B - \lambda I) \text{ is not closed, } d(A - \lambda I) < \infty \right\} \\ & \cup \left\{ \lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I) \text{ is not closed, } \mathcal{R}(B - \lambda I) \text{ is closed,} \right. \\ & \quad \left. d(A - \lambda I) + d(B - \lambda I) < \infty \right\} \\ & \cup \left\{ \lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I), \mathcal{R}(B - \lambda I) \text{ are closed,} \right. \\ & \quad \left. n(A - \lambda I) + n(B - \lambda I) > d(A - \lambda I) + d(B - \lambda I) \right\}. \end{aligned}$$

Using Theorem 2.3, Remark 1 and the fact that A is generalized left Weyl if and only if A^* is generalized right Weyl, we can give the description of the set $S_{GW}(A, B)$ which consists of all $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized Weyl. Notice that necessary and sufficient conditions for the existence of $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized Weyl are given in [10].

THEOREM 2.9. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be given operators. There exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized Weyl if and only if one of the following conditions is satisfied:*

(i) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed and $n(A) + n(B) = d(A) + d(B)$. In this case,

$$S_{GW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C \text{ is given by (2.1), } C_3 \text{ has closed range, } n(A) + n(C_3) = d(C_3) + d(B) \right\}.$$

(ii) $\mathcal{R}(A)$ is closed, $\mathcal{R}(B)$ is non-closed and $d(A) = n(A) + n(B) = \infty$. In this case,

$$S_{GW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C \text{ is given by (2.1), } n(A) + n(C_3) = \infty, \mathcal{R}(B^*) + R(C^*P_{\mathcal{R}(A)^\perp}) \text{ is closed} \right\}.$$

(iii) $\mathcal{R}(A)$ is non-closed, $\mathcal{R}(B)$ is closed and $n(B) = d(A) + d(B) = \infty$. In this case,

$$S_{GW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C \text{ is given by (2.1), } d(B) + d(C_3) = \infty, \mathcal{R}(A) + \mathcal{R}(CP_{\mathcal{N}(B)}) \text{ is closed} \right\}.$$

(iv) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are non-closed and $n(B) = d(A) = \infty$. In this case,

$$S_{GW}(A, B) = \{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(M_C) \text{ is closed}\}.$$

Proof. Since necessary and sufficient conditions for the existence of $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized Weyl are given in [10], we need only prove that the set $S_{GW}(A, B)$ is given as claimed in each of the four possible cases appearing above.

(i) Suppose that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed and $n(A) + n(B) = d(A) + d(B)$. Using Theorem 2.3, we have that $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is such that M_C is generalized left Weyl if and only if C is given by (2.1), where C_3 has closed range and

$$n(A) + n(C_3) \leq d(C_3) + d(B).$$

Since we are looking for $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized Weyl, we are asking for which $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying the previously mentioned condition, M_C is generalized right Weyl i.e. $(M_C)^*$ is generalized left Weyl. Since,

$$(M_C)^* = \begin{pmatrix} B^* & C^* \\ 0 & A^* \end{pmatrix} : \begin{pmatrix} \mathcal{K} \\ \mathcal{H} \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{K} \\ \mathcal{H} \end{pmatrix}$$

and for C given by (2.1), C^* is given by

$$C^* = \begin{pmatrix} C_4^* & C_2^* \\ C_3^* & C_1^* \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A^*) \\ \mathcal{N}(A^*)^\perp \end{pmatrix} \rightarrow \begin{pmatrix} \overline{\mathcal{R}(B^*)} \\ \mathcal{R}(B^*)^\perp \end{pmatrix}, \quad (2.11)$$

applying Theorem 2.3 we get that $(M_C)^*$ is a generalized left Weyl operator if and only if $\mathcal{R}(C_3^*)$ is closed and

$$n(B^*) + n(C_3^*) \leq d(C_3^*) + d(A^*)$$

which is equivalent with $\mathcal{R}(C_3)$ being closed and the inequality $d(C_3) + d(B) \leq n(A) + n(C_3)$. Hence, M_C is a generalized Weyl operator if and only if C is given by (2.1), where C_3 has closed range and $n(A) + n(C_3) = d(C_3) + d(B)$.

(ii) Suppose that $\mathcal{R}(A)$ is closed, $\mathcal{R}(B)$ is non-closed and $d(A) = n(A) + n(B) = \infty$. Using Theorem 2.3, we have that $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is such that M_C is generalized left Weyl if and only if $\mathcal{R}(B^*) + \mathcal{R}(C^*P_{\mathcal{R}(A)^\perp})$ is closed. By item (iii) of Theorem 2.3, using the representations of $(M_C)^*$ given above, we get that $(M_C)^*$ is a generalized left Weyl operator if and only if $\mathcal{R}(B^*) + \mathcal{R}(C^*P_{\mathcal{N}(A^*)^\perp})$ is closed and $d(A^*) + \text{codim}(\mathcal{R}(B^*) + \mathcal{R}(C^*P_{\mathcal{N}(A^*)^\perp})) = \infty$. By Remark 1, we have that the last condition is equivalent with $d(C_3^*) + d(A^*) = \infty$, i.e., $n(A) + n(C_3) = \infty$, where C_3 is the block operator in the representation (2.1) of C .

Hence, M_C is a generalized Weyl operator if and only if C is given by (2.1), where $\mathcal{R}(B^*) + \mathcal{R}(C^*P_{\mathcal{R}(A)^\perp})$ is closed and $n(A) + n(C_3) = \infty$.

Items (iii) and (iv) can be proved in a similar manner. \square

In the next theorem, we present necessary and sufficient conditions which two operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ have to satisfy in order for M_C to be a generalized left Weyl operator for each $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

THEOREM 2.10. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then M_C is a generalized left Weyl operator for each $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed and one of the following conditions is satisfied:*

$$(1) \quad d(A) < \infty, \quad n(B) = \infty, \quad d(B) = \infty,$$

- (2) $d(A) = \infty, n(B) < \infty,$
 (3) $d(A), n(B) < \infty, n(A) + n(B) \leq d(A) + d(B).$

Proof. Suppose that M_C is a generalized left Weyl operator for each $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. If at least one of $\mathcal{R}(A)$ and $\mathcal{R}(B)$ is not closed, we have that M_0 is not a generalized left Weyl operator since its range is not closed. So, it follows that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed subspaces.

Notice that for arbitrary $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, M_C is given by

$$M_C = \begin{pmatrix} 0 & A_1 & C_1 & C_2 \\ 0 & 0 & C_3 & C_4 \\ 0 & 0 & 0 & B_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A) \\ \mathcal{N}(A)^\perp \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{R}(B) \\ \mathcal{R}(B)^\perp \end{pmatrix}, \quad (2.12)$$

where A_1, B_1 are invertible operators and that there exist invertible $U, V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that

$$UM_CV = \begin{pmatrix} 0 & A_1 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & B_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A) \\ \mathcal{N}(A)^\perp \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{R}(B) \\ \mathcal{R}(B)^\perp \end{pmatrix}. \quad (2.13)$$

So, for any $C_3 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp)$, we have that $\mathcal{R}(C_3)$ is closed and

$$n(A) + n(C_3) \leq d(B) + d(C_3).$$

Hence, at least one of $d(A)$ and $n(B)$ is finite. So, we will consider all possible cases (there are 3 in total) when at least one of $d(A)$ and $n(B)$ is finite.

Suppose first that $d(A) < \infty$ and $n(B) = \infty$. Since for any $C_3 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp)$, it follows that $n(C_3) = \infty$, and since there exists $C_3 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp)$ such that $d(C_3) = 0$, we conclude that $n(M_C) \leq d(M_C)$, for each $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if $d(B) = \infty$.

If $d(A) = \infty$ and $n(B) < \infty$ then for any $C_3 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp)$, we have that $d(C_3) = \infty$, so $n(M_C) \leq d(M_C)$ is satisfied for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

If $d(A), n(B) < \infty$ then for any $C_3 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp)$, we have that $n(B) - n(C_3) = d(A) - d(C_3)$, so $n(M_C) \leq d(M_C)$ if and only if $n(A) + n(B) \leq d(A) + d(B)$.

The converse implication can be proved in the same manner. \square

As a corollary of the previous theorem, we also get the description of the set $\bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{lw}^g(M_C)$:

COROLLARY 2.11. For given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ we have

$$\begin{aligned} \bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{lw}^g(M_C) = & \{ \lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I) \text{ is not closed} \} \\ & \cup \{ \lambda \in \mathbb{C} : \mathcal{R}(B - \lambda I) \text{ is not closed} \} \\ & \cup \{ \lambda \in \mathbb{C} : d(A - \lambda I) = n(B - \lambda I) = \infty \} \\ & \cup \{ \lambda \in \mathbb{C} : d(A - \lambda I), n(B - \lambda I) < \infty, \\ & \quad n(A - \lambda I) + n(B - \lambda I) > d(A - \lambda I) + d(B - \lambda I) \} \\ & \cup \{ \lambda \in \mathbb{C} : d(B - \lambda I) < n(B - \lambda I) = \infty \}. \end{aligned}$$

REMARK 2. Throughout the paper, we have used the following fact: For given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ in the each of following three cases:

- (i) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed,
- (ii) $\mathcal{R}(A)$ is closed, $\mathcal{R}(B)$ is non-closed,
- (iii) $\mathcal{R}(A)$ is non-closed, $\mathcal{R}(B)$ is closed,

we have that $\mathcal{R}(M_C)$ is closed if and only if the respective condition below is satisfied:

- (1) $\mathcal{R}(C_3)$ is closed,
- (2) $\mathcal{R}(B^*) + \mathcal{R}(C^*P_{\mathcal{R}(A)^\perp})$ is closed,
- (3) $\mathcal{R}(A) + R(CP_{\mathcal{N}(B)})$ is closed.

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