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GENERALIZED LEFT AND RIGHT WEYL SPECTRA OF UPPER TRIANGULAR OPERATOR MATRICES

GUOJUN HAI† AND DRAGANA S. CVETKOVIĆ-ILIĆ‡

Abstract. In this paper, for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, the sets of all $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is generalized Weyl and generalized left (right) Weyl, are completely described. Furthermore, the following intersections and unions of the generalized left Weyl spectra

$$
\bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{lw}^g(M_C) \quad \text{and} \quad \bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{lw}^g(M_C)
$$

are also described, and necessary and sufficient conditions which two operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ have to satisfy in order for $M_C$ to be a generalized left Weyl operator for each $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, are presented.

Key words. Operator matrix, Generalized left(right) Weyl, Spectrum.

AMS subject classifications. 47A10, 47A53, 47A55.

1. Introduction. Let $\mathcal{H}, \mathcal{K}$ be infinite dimensional complex separable Hilbert spaces, and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For simplicity, we also write $\mathcal{B}(\mathcal{H})$ as $\mathcal{B}(\mathcal{H})$. By $\mathcal{J}(\mathcal{H}, \mathcal{K})$ we denote the set of all operators from $\mathcal{B}(\mathcal{H}, \mathcal{K})$ with a finite dimensional range. For a given $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of $A$, respectively. Let $n(A) = \dim \mathcal{N}(A)$, $\beta(A) = \text{codim} \mathcal{R}(A)$, and $d(A) = \dim \mathcal{R}(A)^\perp$.

If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is such that $\mathcal{R}(A)$ is closed and $n(A) < \infty$, then $A$ is said to be a upper semi-Fredholm operator. If $\beta(A) < \infty$, then $A$ is called a lower semi-Fredholm operator. A semi-Fredholm operator is one which is either upper semi-Fredholm or lower semi-Fredholm. An operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is called Fredholm if it is both lower semi-Fredholm and upper semi-Fredholm. The subset of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ consisting of all Fredholm operators is denoted by $\Phi(\mathcal{H}, \mathcal{K})$. By $\Phi_+(\mathcal{H}, \mathcal{K})$ ($\Phi_-(\mathcal{H}, \mathcal{K})$) we denote the set of all upper (lower) semi-Fredholm operators from $\mathcal{B}(\mathcal{H}, \mathcal{K})$.

If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is such that $\mathcal{R}(A)$ is closed and $n(A) \leq d(A)$, then $A$ is a generalized left Weyl operator. If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is such that $\mathcal{R}(A)$ is closed and $d(A) \leq n(A)$, then $A$ is a generalized right Weyl operator. Notice that in the cases of generalized left (right) Weyl operators, $n(A)$ and $d(A)$ are allowed to be infinity. An operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a generalized Weyl operator if it is both generalized right Weyl and generalized left Weyl. The set of all generalized Weyl operators from $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is denoted by $W^g(\mathcal{H}, \mathcal{K})$.

Let $W_{gl}(\mathcal{H}, \mathcal{K})$ ($W_{gr}(\mathcal{H}, \mathcal{K})$) denote the subset of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ consisting of all generalized left (right) Weyl operators. For an operator $C \in \mathcal{B}(\mathcal{H})$, the generalized left (right) Weyl spectrum $\sigma_{lw}^g(C)$ ($\sigma_{rw}^g(C)$) is defined by

$$
\sigma_{lw}^g(C)(\sigma_{rw}^g(C)) = \{ \lambda \in \mathbb{C} : C - \lambda I \text{ is not generalized left (right) Weyl} \}.
$$

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The generalized Weyl spectrum is defined by
\[ \sigma^G_{\text{gl}}(C) = \{ \lambda \in \mathbb{C} : C - \lambda I \text{ is not generalized Weyl} \} . \]
In this paper, we address the question for which operators \( A \in \mathcal{B}(\mathcal{H}) \) and \( B \in \mathcal{B}(\mathcal{K}) \), there exists an operator \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) such that an upper-triangular operator matrix
\[ M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \left( \begin{array}{c} \mathcal{H} \\ \mathcal{K} \end{array} \right) \to \left( \begin{array}{c} \mathcal{H} \\ \mathcal{K} \end{array} \right) , \]
is generalized left (right) Weyl. There are many papers which consider some types of invertibility, regularity and some other properties of an upper-triangular operator matrix \( M_C \) (see [11–12] and references therein) as well as various types of spectra of \( M_C \). This paper is a continuation of the work presented in [10], where the sets \( \bigcup_{\mathcal{C} \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{\text{gl}}^{c}(M_C) \) and \( \bigcap_{\mathcal{C} \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{\text{gl}}^{c}(M_C) \) are described and some necessary and sufficient conditions for the existence of \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) such that \( M_C \) is generalized Weyl are given, but the set of all such operators \( C \) is not described. As a corollary of our main results we obtain a description of all \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) such that \( M_C \) is generalized Weyl, and we denote this set by \( S_{\text{GLW}}(A, B) \). The sets \( \bigcup_{\mathcal{C} \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{\text{tw}}^{c}(M_C) \) and \( \bigcap_{\mathcal{C} \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{\text{tw}}^{c}(M_C) \) are described for given \( A \in \mathcal{B}(\mathcal{H}) \) and \( B \in \mathcal{B}(\mathcal{K}) \) as well as the set of all \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) such that \( M_C \) is generalized left Weyl which is denoted by \( S_{\text{GLW}}(A, B) \). In an analogous way, similar results can be provided for \( \bigcup_{\mathcal{C} \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{\text{tw}}^{c}(M_C) \) and \( \bigcap_{\mathcal{C} \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{\text{tw}}^{c}(M_C) \).

2. Results. In this section, by \( \mathcal{H}, \mathcal{K} \) we denote complex separable Hilbert spaces. For given operators \( A \in \mathcal{B}(\mathcal{H}) \) and \( B \in \mathcal{B}(\mathcal{K}) \), by \( M_C \) we denote
\[ M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \left( \begin{array}{c} \mathcal{H} \\ \mathcal{K} \end{array} \right) \to \left( \begin{array}{c} \mathcal{H} \\ \mathcal{K} \end{array} \right) , \]
where \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \). Evidently, for given \( A \in \mathcal{B}(\mathcal{H}) \) and \( B \in \mathcal{B}(\mathcal{K}) \), arbitrary \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) can be represented by
\[ C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} : \left( \begin{array}{c} \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{array} \right) \to \left( \begin{array}{c} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{array} \right) , \]
(2.1)

First, we will state some auxiliary lemmas which will be used in the proof of the main result.

**Lemma 2.1.** If \( A \in \mathcal{B}(\mathcal{H}) \) and \( D \in \mathcal{J}(\mathcal{H}) \), then \( \mathcal{R}(A + D) \) is closed if and only if \( \mathcal{R}(A) \) is closed.

**Lemma 2.2.** Let \( S \in \mathcal{B}(\mathcal{H}), T \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) and \( R \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) be given operators.

(i) If \( \mathcal{R}(S) \) is non-closed and \( \mathcal{R}(\left( \begin{array}{cc} S & T \end{array} \right) ) \) is closed, then \( n(\left( \begin{array}{cc} S & T \end{array} \right) ) = \infty \).

(ii) If \( \mathcal{R}(S) \) is non-closed and \( \mathcal{R}(\left( \begin{array}{c} S \\ R \end{array} \right) ) \) is closed, then \( d(\left( \begin{array}{c} S \\ R \end{array} \right) ) = \infty \).

**Proof.** (i) Suppose that \( \mathcal{R}(S) \) is non-closed, \( \mathcal{R}(\left( \begin{array}{cc} S & T \end{array} \right) ) \) is closed and \( n(\left( \begin{array}{cc} S & T \end{array} \right) ) < \infty \). Then \( ( \begin{array}{cc} S & T \end{array} ) \) is a left Fredholm operator which implies that there exists an operator \( \left( \begin{array}{c} X \\ Y \end{array} \right) : \mathcal{H} \to \left( \begin{array}{c} \mathcal{H} \\ \mathcal{K} \end{array} \right) \) such that
\[ \left( \begin{array}{c} X \\ Y \end{array} \right) ( \begin{array}{cc} S & T \end{array} ) = I + K , \]
for some compact operator \( K \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K}) \). Hence, \( XS = I_{\mathcal{H}} + K_1 \), for some compact operator \( K_1 \in \mathcal{B}(\mathcal{H}) \) which implies that \( S \) is left Fredholm and so \( \mathcal{R}(S) \) is closed, which is a contradiction.
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(ii) The proof follows by taking adjoints in (i). □

In the following theorem, for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, we present necessary and sufficient conditions for the existence of $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_C$ is a generalized left Weyl operator, and we completely describe the set of all such $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

**THEOREM 2.3.** Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. There exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_C$ is generalized left Weyl if and only if one of the following conditions is satisfied:

(i) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed and $n(A) + n(B) \leq d(A) + d(B)$. In this case,

$$S_{GLW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C \text{ is given by } (2.1) \text{, } C_3 \text{ has closed range}, n(A) + n(C_3) \leq d(C_3) + d(B) \right\}.$$

(ii) $\mathcal{R}(A)$ is closed, $\mathcal{R}(B)$ is non-closed and $d(A) = \infty$. In this case,

$$S_{GLW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(A)^\perp + \mathcal{R}(CP_{\mathcal{K}(B)}) \text{ is closed} \right\}.$$

(iii) $\mathcal{R}(A)$ is non-closed, $\mathcal{R}(B)$ is closed and $n(B) = d(A) + d(B) = \infty$. In this case,

$$S_{GLW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(A) + \mathcal{R}(CP_{\mathcal{K}(B)}) \text{ is closed}, d(B) + \text{codim}(\mathcal{R}(A) + \mathcal{R}(CP_{\mathcal{K}(B)})) = \infty \right\}.$$

(iv) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are non-closed and $n(B) = d(A) = \infty$. In this case,

$$S_{GLW}(A, B) = \{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(M_C) \text{ is closed} \}.$$

For simplicity, we will divide the statement of this theorem into four propositions and prove each of them separately.

**PROPOSITION 2.4.** Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be such that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed. There exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_C$ is generalized left Weyl if and only if $n(A) + n(B) \leq d(A) + d(B)$. In this case,

$$S_{GLW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C \text{ is given by } (2.1) \text{, } C_3 \text{ has closed range}, n(A) + n(C_3) \leq d(C_3) + d(B) \right\}.$$

**Proof.** If $n(A) + n(B) \leq d(A) + d(B)$, then $M_0$ is a generalized left Weyl operator. Conversely, suppose that there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_C$ is a generalized left Weyl operator and that $C$ is given by $(2.1)$. Then $M_C$ has a matrix representation

$$M_C = \begin{pmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{K}(B) \\ \mathcal{K}(B)^\perp \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{pmatrix},$$

where $A_1 : \mathcal{H} \rightarrow \mathcal{R}(A)$ is right invertible and $B_1 : \mathcal{K}(B)^\perp \rightarrow \mathcal{K}$ is left invertible. Evidently, there exists invertible $U, V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that

$$UM_CV = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{K}(B) \\ \mathcal{K}(B)^\perp \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{pmatrix}. \quad (2.2)$$
Hence, $UM_CV$ is a generalized left Weyl which implies that

$$n(A_1) + n(C_3) \leq d(C_3) + d(B_1). \tag{2.3}$$

Since,

$$n(A_1) = n(A), \quad n(B) = n(C_3) + \dim \mathcal{N}(C_3)^\perp,$$

$$d(B_1) = d(B) \quad \text{and} \quad d(A) = d(C_3) + \dim \mathcal{R}(C_3),$$

having in mind that $\dim \mathcal{N}(C_3)^\perp = \dim \mathcal{R}(C_3)$ and \[2.3\], we get

$$n(A) + n(B) \leq d(A) + d(B).$$

To describe the set of all $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_C$ is a generalized left Weyl, notice that for arbitrary $C$ given by \[2.1\], there exists invertible $U, V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that $UM_CV$ is given by \[2.10\]. Hence, $M_C$ is a generalized left Weyl if and only if $UM_CV$ is a generalized left Weyl which is equivalent with the fact that $\mathcal{R}(C_3)$ is closed and that \[2.3\] holds. $\square$

**Proposition 2.5.** Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be such that $\mathcal{R}(A)$ is closed and $\mathcal{R}(B)$ is non-closed. There exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_C$ is generalized left Weyl if and only if $d(A) = \infty$. In this case,

$$S_{GLW}(A,B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(B^*) + \mathcal{R}(C^*P_{\mathcal{R}(A)^\perp}) \text{ is closed} \right\}.$$

**Proof.** Suppose that $d(A) = \infty$. Then $M_{C_0}$ is a generalized left Weyl operator for $C_0$ given by

$$C_0 = \begin{pmatrix} 0 \\ J \end{pmatrix} : \mathcal{K} \rightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp,$$

where $J : \mathcal{K} \rightarrow \mathcal{R}(A)^\perp$ is unitary. Evidently, $M_{C_0}$ is represented by

$$M_{C_0} = \begin{pmatrix} A_1 & 0 \\ 0 & J \\ 0 & B \end{pmatrix} : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K},$$

where $A_1 : \mathcal{H} \rightarrow \mathcal{R}(A)$ is right invertible. Since $J$ is invertible, there exists an invertible operator $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that

$$UM_{C_0} = \begin{pmatrix} A_1 & 0 \\ 0 & J \\ 0 & 0 \end{pmatrix} : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K}.$$

Now, it is clear that $UM_{C_0}$ is a generalized left Weyl operator, and so $M_{C_0}$ is a generalized left Weyl operator.

Conversely, suppose that there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_C$ is generalized left Weyl. Then $M_C$ has a matrix representation

$$M_C = \begin{pmatrix} A_1 & C_1 \\ 0 & C_2 \\ 0 & B \end{pmatrix} : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K}, \tag{2.4}$$
where $A_1 : \mathcal{H} \rightarrow \mathcal{R}(A)$ is right invertible. Thus, there exists an invertible operator $V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that

$$M_C V = \begin{pmatrix} A_1 & 0 \\ 0 & C_2 \\ 0 & B \end{pmatrix} : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K}. \quad (2.5)$$

Now we will show that $d(A) = \infty$: Indeed, if $d(A) < \infty$, then $\mathcal{R}(C_2^1)$ is finite dimensional. Since $\mathcal{R}(M_C V)$ is closed, we have that $\mathcal{R}\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right)$ is closed, which implies that $\mathcal{R}(B^*) + \mathcal{R}(C_2^1)$ is closed. This, together with $\dim \mathcal{R}(C_2^1) < \infty$, implies that $\mathcal{R}(B)$ is closed. This is a contradiction. Hence, $d(A) = \infty$.

In order to describe the set $S_{GLW}(A, B)$, notice that for arbitrary $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $M_C$ has a form $(2.4)$ and that there exists an invertible operator $V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that $M_C V$ is given by $(2.5)$. Hence, $M_C$ is generalized left Weyl if and only if $C$ is such that $\mathcal{R}\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right)$ is closed and that

$$n(A_1) + n\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right) \leq d(A_1) + d\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right). \quad (2.6)$$

Notice that by Lemma 2.2, we have that for each $C_2 \in \mathcal{B}(\mathcal{K}, \mathcal{R}(A)^\perp)$ such that $\mathcal{R}\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right)$ is closed, it follows that $d\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right) = \infty$. Thus,

$$S_{GLW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(B^*) + \mathcal{R}(C^*P_{\mathcal{R}(A)^\perp}) \text{ is closed} \right\}. \quad \square$$

**Proposition 2.6.** Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be such that $\mathcal{R}(A)$ is non-closed and $\mathcal{R}(B)$ is closed. There exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_C$ is generalized left Weyl if and only if $n(B) = d(A) + d(B) = \infty$. In this case,

$$S_{GLW}(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(A) + \mathcal{R}(C^*P_{\mathcal{R}(B)}) \text{ is closed}, \quad d(B) + \text{codim}(\mathcal{R}(A) + \mathcal{R}(C^*P_{\mathcal{R}(B)})) = \infty \right\}. \quad \square$$

**Proof.** Suppose that $n(B) = d(A) + d(B) = \infty$. Then there exists a left invertible operator $C_1 : \mathcal{N}(B) \rightarrow \mathcal{H}$ such that $\mathcal{R}(C_1) = \mathcal{R}(A)$. We will prove that $M_C$ is a generalized left Weyl operator for $C$ given by

$$C = \begin{pmatrix} C_1 \\ 0 \end{pmatrix} : \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \rightarrow \mathcal{H}.$$

Evidently, $M_C$ is represented by

$$M_C = \begin{pmatrix} A & C_1 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \rightarrow \mathcal{H} \oplus \mathcal{K},$$

where $B_1 : \mathcal{N}(B)^\perp \rightarrow \mathcal{K}$ is left invertible and

$$\mathcal{R}(M_C) = (\mathcal{R}(A) + \mathcal{R}(C_1)) \oplus \mathcal{R}(B_1) = \overline{\mathcal{R}(A)} \oplus \mathcal{R}(B).$$

Thus, $\mathcal{R}(M_C)$ is closed and

$$d(M_C) = d(A) + d(B) = \infty,$$
i.e., $M_C$ is a generalized left Weyl operator.

Conversely, suppose that there exists $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that $M_C$ is generalized left Weyl. It follows that $M_C$ has a matrix representation

$$M_C = \begin{pmatrix} A & C_1 & C_2 \\ 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \to \mathcal{H} \oplus \mathcal{K},$$ (2.7)

where $B_1 : \mathcal{N}(B)^\perp \to \mathcal{K}$ is left invertible and there exists an invertible operator $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that

$$UM_C = \begin{pmatrix} A & C_1 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \to \mathcal{H} \oplus \mathcal{K}.$$ (2.8)

Since $UM_C$ has a closed range, by Lemma 2.1 and the fact that $\mathcal{R}(A)$ is non-closed, we have that $n(B) = \infty$. Also, applying Lemma 2.2, we get that $n\left( \begin{pmatrix} A & C_1 \end{pmatrix} \right) = \infty$ which implies that $d(UM_C) = d(B) + d\left( \begin{pmatrix} A & C_1 \end{pmatrix} \right) = \infty$. Since $d\left( \begin{pmatrix} A & C_1 \end{pmatrix} \right) \leq d(A)$, it follows that $d(A) + d(B) = \infty$.

In order to describe the set $S_{GLW}(A,B)$, notice that for arbitrary $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $M_C$ has a form (2.7) and that there exists an invertible operator $V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that $UM_C$ is given by (2.8). Hence, $M_C$ is generalized left Weyl if and only if $C$ is such that $\mathcal{R}\left( \begin{pmatrix} A & C_1 \end{pmatrix} \right)$ is closed and that

$$n\left( \begin{pmatrix} A & C_1 \end{pmatrix} \right) + n(B_1) \leq d\left( \begin{pmatrix} A & C_1 \end{pmatrix} \right) + d(B_1).$$ (2.9)

Notice that if $\mathcal{R}\left( \begin{pmatrix} A & C_1 \end{pmatrix} \right)$ is closed, then by Lemma 2.2 we have that $n\left( \begin{pmatrix} A & C_1 \end{pmatrix} \right) = \infty$. Hence, $M_C$ is a generalized left Weyl operator for $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ if and only if $\mathcal{R}\left( \begin{pmatrix} A & C_1 \end{pmatrix} \right)$ is closed and $d\left( \begin{pmatrix} A & C_1 \end{pmatrix} \right) + d(B_1) = \infty$. Obviously, $d(B_1) = d(B)$. □

**Proposition 2.7.** Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be such that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are non-closed. There exists $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that $M_C$ is generalized left Weyl if and only if $n(B) = d(A) = \infty$. In this case,

$$S_{GLW}(A,B) = \left\{ C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : R(M_C) \text{ is closed} \right\}.$$
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Now, it is evident that \( \mathcal{R}(M_C) \) is closed if and only if \( \begin{pmatrix} A_1 & C_1 \\ 0 & C_3 \end{pmatrix} \) is closed which is equivalent with the fact that \( \mathcal{R}(A) + R(CP_{N(B)}) \) is closed. Also,
\[
d \begin{pmatrix} A_1 & C_1 \\ 0 & C_3 \end{pmatrix} = n \begin{pmatrix} A_1^* & 0 \\ C_1^* & C_3^* \end{pmatrix} = n(C_3^*) = d(C_3),
\]

since \( A_1^* \) is injective (\( \mathcal{R}(A_1) = \mathcal{R}(A) \)). Hence, in this case, the set \( S_{GLW} \) can also be described by
\[
S_{GLW}(A, B) = \{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C \text{ is given by } (2.1), \mathcal{R}(A) + R(CP_{N(B)}) \text{ is closed, } d(C_3) + d(B) = \infty \}.
\]

As a corollary of the previous theorem, we get the description of the set \( \bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{lw}^2(M_C) \):

COROLLARY 2.8. Let \( A \in \mathcal{B}(\mathcal{H}) \) and \( B \in \mathcal{B}(\mathcal{K}) \) be given operators. Then
\[
\bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{lw}^2(M_C) = \{ \lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I) \text{ is not closed, } n(B - \lambda I) < \infty \}
\]
\[
\cup \{ \lambda \in \mathbb{C} : \mathcal{R}(B - \lambda I) \text{ is not closed, } d(A - \lambda I) < \infty \}
\]
\[
\cup \{ \lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I) \text{ is not closed, } \mathcal{R}(B - \lambda I) \text{ is closed, } d(A - \lambda I) + d(B - \lambda I) < \infty \}
\]
\[
\cup \{ \lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I), \mathcal{R}(B - \lambda I) \text{ are closed, } n(A - \lambda I) + n(B - \lambda I) > d(A - \lambda I) + d(B - \lambda I) \}.
\]

Using Theorem 2.3, Remark 1 and the fact that \( A \) is generalized left Weyl if and only if \( A^* \) is generalized right Weyl, we can give the description of the set \( S_{GW}(A, B) \) which consists of all \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) such that \( M_C \) is generalized Weyl. Notice that necessary and sufficient conditions for the existence of \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) such that \( M_C \) is generalized Weyl are given in [10].

THEOREM 2.9. Let \( A \in \mathcal{B}(\mathcal{H}) \) and \( B \in \mathcal{B}(\mathcal{K}) \) be given operators. There exists \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) such that \( M_C \) is generalized Weyl if and only if one of the following conditions is satisfied:

(i) \( \mathcal{R}(A) \) and \( \mathcal{R}(B) \) are closed and \( n(A) + n(B) = d(A) + d(B) \). In this case,
\[
S_{GW}(A, B) = \{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C \text{ is given by } (2.1), C_3 \text{ has closed range, } n(A) + n(C_3) = d(C_3) + d(B) \}.
\]

(ii) \( \mathcal{R}(A) \) is closed, \( \mathcal{R}(B) \) is non-closed and \( d(A) + n(B) = \infty \). In this case,
\[
S_{GW}(A, B) = \{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C \text{ is given by } (2.1), n(A) + n(C_3) = \infty, \mathcal{R}(B^*) + R(C^*P_{\mathcal{R}(A)^\perp}) \text{ is closed} \}.
\]

(iii) \( \mathcal{R}(A) \) is non-closed, \( \mathcal{R}(B) \) is closed and \( n(B) = d(A) + d(B) = \infty \). In this case,
\[
S_{GW}(A, B) = \{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C \text{ is given by } (2.1), d(B) + d(C_3) = \infty, \mathcal{R}(A) + \mathcal{R}(CP_{N(B)}) \text{ is closed} \}.
\]
(iv) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are non-closed and $n(B) = d(A) = \infty$. In this case,

$$S_{GW}(A, B) = \{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(M_C) \text{ is closed}\}.$$ 

**Proof.** Since necessary and sufficient conditions for the existence of $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_C$ is generalized Weyl are given in [10], we need only prove that the set $S_{GW}(A, B)$ is given as claimed in each of the four possible cases appearing above.

(i) Suppose that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed and $n(A) + n(B) = d(A) + d(B)$. Using Theorem 2.3, we have that $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is such that $M_C$ is generalized left Weyl if and only if $C_3$ has closed range and

$$n(A) + n(C_3) \leq d(C_3) + d(B).$$

Since we are looking for $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_C$ is generalized Weyl, we are asking for which $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying the previously mentioned condition, $M_C$ is generalized right Weyl i.e. $(M_C)^*$ is generalized left Weyl. Since,

$$(M_C)^* = \begin{pmatrix} B^* & C^* \\ 0 & A^* \end{pmatrix} : \begin{pmatrix} \mathcal{K} \\ \mathcal{H} \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{K} \\ \mathcal{H} \end{pmatrix}$$

and for $C$ given by (2.1), $C^*$ is given by

$$C^* = \begin{pmatrix} C_4^* \\ C_3^* \\ C_1^* \end{pmatrix} : \begin{pmatrix} \mathcal{K} \\ \mathcal{H} \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{K} \\ \mathcal{H} \end{pmatrix},$$

(2.11)

applying Theorem 2.3, we get that $(M_C)^*$ is a generalized left Weyl operator if and only if $\mathcal{R}(C_4^*)$ is closed and

$$n(B^*) + n(C_3^*) \leq d(C_3^*) + d(A^*)$$

which is equivalent with $\mathcal{R}(C_3^*)$ being closed and the inequality $d(C_3) + d(B) \leq n(A) + n(C_3)$. Hence, $M_C$ is a generalized Weyl operator if and only if $C$ is given by (2.1), where $C_3$ has closed range and $n(A) + n(C_3) = d(C_3) + d(B)$.

(ii) Suppose that $\mathcal{R}(A)$ is closed, $\mathcal{R}(B)$ is non-closed and $d(A) = n(A) + n(B) = \infty$. Using Theorem 2.3, we have that $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is such that $M_C$ is generalized left Weyl if and only if $\mathcal{R}(B^*) + \mathcal{R}(C^* P_{\mathcal{R}(A)^{\perp}})$ is closed. By item (iii) of Theorem 2.3, using the representations of $(M_C)^*$ given above, we get that $(M_C)^*$ is a generalized left Weyl operator if and only if $\mathcal{R}(B^*) + \mathcal{R}(C^* P_{\mathcal{R}(A)^{\perp}})$ is closed and $d(A^*) + \text{codim}(\mathcal{R}(B^*) + \mathcal{R}(C^* P_{\mathcal{R}(A)^{\perp}})) = \infty$. By Remark 1, we have that the last condition is equivalent with $d(C_3^*) + d(A^*) = \infty$, i.e., $n(A) + n(C_3) = \infty$, where $C_3$ is the block operator in the representation (2.1) of $C$.

Hence, $M_C$ is a generalized Weyl operator if and only if $C$ is given by (2.1), where $\mathcal{R}(B^*) + \mathcal{R}(C^* P_{\mathcal{R}(A)^{\perp}})$ is closed and $n(A) + n(C_3) = \infty$.

Items (iii) and (iv) can be proved in a similar manner. □

In the next theorem, we present necessary and sufficient conditions which two operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ have to satisfy in order for $M_C$ to be a generalized left Weyl operator for each $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

**Theorem 2.10.** Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then $M_C$ is a generalized left Weyl operator for each $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed and one of the following conditions is satisfied:

1. $d(A) < \infty$, $n(B) = \infty$, $d(B) = \infty$, 

   + 

   2. $d(A) = \infty$, $n(B) = n(B) = \infty$, 

   + 

   3. $d(A) = n(B) = d(B) = \infty$, 

   + 

   4. $d(A) = d(B) = \infty$, $n(B) = \infty$. 

   + 

   5. $d(A) = n(B) = \infty$, $d(B) = \infty$. 

   +
(2) \( d(A) = \infty, \ n(B) < \infty, \)

(3) \( d(A), n(B) < \infty, \ n(A) + n(B) \leq d(A) + d(B). \)

**Proof.** Suppose that \( M_C \) is a generalized left Weyl operator for each \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}). \) If at least one of \( \mathcal{R}(A) \) and \( \mathcal{R}(B) \) is not closed, we have that \( M_0 \) is not a generalized left Weyl operator since its range is not closed. So, it follows that \( \mathcal{R}(A) \) and \( \mathcal{R}(B) \) are closed subspaces.

Notice that for arbitrary \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}), \) \( M_C \) is given by

\[
M_C = \begin{pmatrix}
0 & A_1 & C_1 & C_2 \\
0 & 0 & C_3 & C_4 \\
0 & 0 & 0 & B_1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\mathcal{N}(A) \\
\mathcal{N}(A)^\perp \\
\mathcal{N}(B) \\
\mathcal{N}(B)^\perp
\end{pmatrix} \rightarrow \begin{pmatrix}
\mathcal{R}(A) \\
\mathcal{R}(A)^\perp \\
\mathcal{R}(B) \\
\mathcal{R}(B)^\perp
\end{pmatrix},
\]

(2.12)

where \( A_1, B_1 \) are invertible operators and that there exist invertible \( U, V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K}) \) such that

\[
UM_CV = \begin{pmatrix}
0 & A_1 & 0 & 0 \\
0 & 0 & C_3 & 0 \\
0 & 0 & 0 & B_1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\mathcal{N}(A) \\
\mathcal{N}(A)^\perp \\
\mathcal{N}(B) \\
\mathcal{N}(B)^\perp
\end{pmatrix} \rightarrow \begin{pmatrix}
\mathcal{R}(A) \\
\mathcal{R}(A)^\perp \\
\mathcal{R}(B) \\
\mathcal{R}(B)^\perp
\end{pmatrix}.
\]

(2.13)

So, for any \( C_3 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp), \) we have that \( \mathcal{R}(C_3) \) is closed and

\[ n(A) + n(C_3) \leq d(B) + d(C_3). \]

Hence, at least one of \( d(A) \) and \( n(B) \) is finite. So, we will consider all possible cases (there are 3 in total) when at least one of \( d(A) \) and \( n(B) \) is finite.

Suppose first that \( d(A) < \infty \) and \( n(B) = \infty. \) Since for any \( C_3 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp), \) it follows that \( n(C_3) = \infty, \) and since there exists \( C_3 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp) \) such that \( d(C_3) = 0, \) we conclude that \( n(M_C) \leq d(M_C), \) for each \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) if and only if \( d(B) = \infty. \)

If \( d(A) = \infty \) and \( n(B) < \infty \) then for any \( C_3 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp), \) we have that \( d(C_3) = \infty, \) so \( n(M_C) \leq d(M_C) \) is satisfied for any \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}). \)

If \( d(A), n(B) < \infty \) then for any \( C_3 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp), \) we have that \( n(B) - n(C_3) = d(A) - d(C_3), \) so \( n(M_C) \leq d(M_C) \) if and only if \( n(A) + n(B) \leq d(A) + d(B). \)

The converse implication can be proved in the same manner. \( \square \)

As a corollary of the previous theorem, we also get the description of the set \( \bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{lw}^C(M_C): \)

**Corollary 2.11.** For given operators \( A \in \mathcal{B}(\mathcal{H}) \) and \( B \in \mathcal{B}(\mathcal{K}) \) we have

\[
\bigcup_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{lw}^C(M_C) = \left\{ \lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I) \ is \ not \ closed \right\}
\]

\[ \cup \left\{ \lambda \in \mathbb{C} : \mathcal{R}(B - \lambda I) \ is \ not \ closed \right\} \]

\[ \cup \left\{ \lambda \in \mathbb{C} : d(A - \lambda I) = n(B - \lambda I) = \infty \right\} \]

\[ \cup \left\{ \lambda \in \mathbb{C} : d(A - \lambda I), n(B - \lambda I) < \infty, \ n(A - \lambda I) + n(B - \lambda I) > d(A - \lambda I) + d(B - \lambda I) \right\} \]

\[ \cup \left\{ \lambda \in \mathbb{C} : d(B - \lambda I) < n(B - \lambda I) = \infty \right\}. \]
REMARK 2. Throughout the paper, we have used the following fact: For given operators $A \in B(H)$ and $B \in B(K)$ in the each of following three cases:

(i) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed,
(ii) $\mathcal{R}(A)$ is closed, $\mathcal{R}(B)$ is non-closed,
(iii) $\mathcal{R}(A)$ is non-closed, $\mathcal{R}(B)$ is closed,

we have that $\mathcal{R}(MC)$ is closed if and only if the respective condition below is satisfied:

(1) $\mathcal{R}(C)$ is closed,
(2) $\mathcal{R}(B^*) + \mathcal{R}(C^*PR_{A^⊥})$ is closed,
(3) $\mathcal{R}(A) + R(P_{\mathcal{N}(B)})$ is closed.

REFERENCES